Every 4×4 biquadratic sum of squares is a sum of eleven squares

Real algebraic geometry · sums of squares on projective varieties

Carles Marín

Statements

Theorem 1 3
Lemma 2 6
Lemma 3 9
Lemma 4 — a regular point 12
Proposition 5 13
Proposition 6 14
Remark 7 15
Proposition 8 15

At a glance

> The exact value **BSR(4,4) = 11**: every biquadratic sum of squares in $x_{1..4}, y_{1..4}$ is a sum of eleven squares of bilinear forms. Qi and Cui gave a form that needs eleven; the previous upper bound was twelve. The proof classifies the Hankel multipliers of rank at most five whose image contains no product (Lemmas 2 and 3) and rules out the two shapes left with second variations built from the alternating forms (Propositions 6 and 8).

🚀 **Start here: Theorem 1**: BSR(4,4) = 11.

Main results:

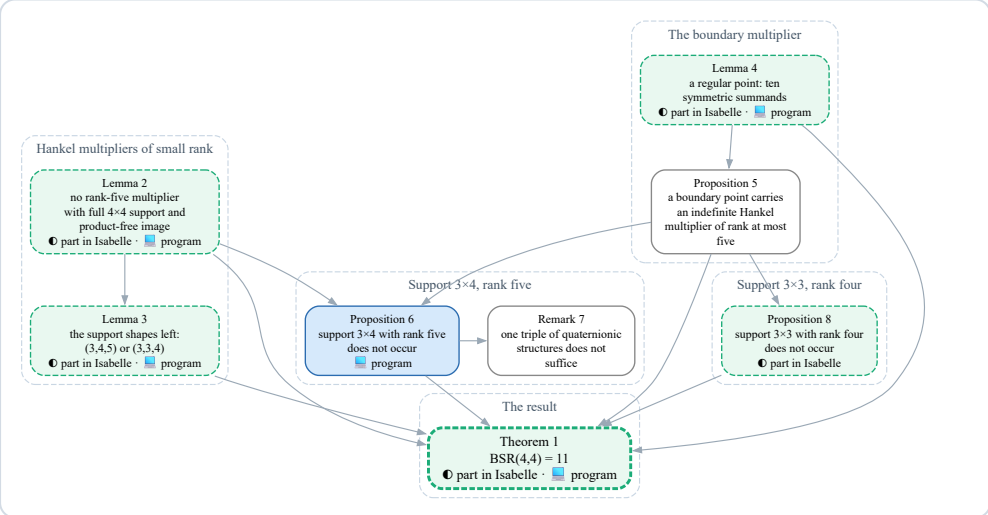
- **Theorem 1**: BSR(4,4) = 11

8 results | ✅ **0** entirely in Lean | ◐ **5** partly in Lean | 💻 **5** checked by a program | written proof only

(a result can carry several tags)

🗺️ Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



📐 Setting (copied from the paper, §§1–2)

Write $x = (x_1, \dots, x_4)$, $y = (y_1, \dots, y_4)$, $V = \mathbb{R}^4 \otimes \mathbb{R}^4$ (the bilinear forms) and R_2 the biquadratic forms, of dimension 100. $\mathrm{BSR}(m, n)$ is the least r such that every biquadratic sum of squares in $m + n$ variables is a sum of r

squares of bilinear forms; $\Phi(h_1,\dots,h_{11}) = h_1^2+\cdots+h_{11}^2$, $\Sigma_{11} = \Phi(V^{11})$ and Σ the cone of sums of squares. A *multiplier* is a symmetric bilinear form H on $A \otimes B$ with the Hankel symmetry $H(x\otimes b, y\otimes c) = H(x\otimes c, y\otimes b)$; its *supports* a, b_0 are the dimensions of the two factors modulo the radicals. Writing $H = T^* J T$ with T onto E, $\dim E = \operatorname{rk} H$, and $T_b x = T(x\otimes b)$, the symmetry reads $\langle T_b x, T_c y\rangle_J = \langle T_c x, T_b y\rangle_J$.

Abstract.

Let $\mathrm{BSR}(m,n)$ be the largest number of squares of real bilinear forms needed to write an $m\times n$ biquadratic sum of squares. We prove $\mathrm{BSR}(4,4) = 11$. The lower bound is Qi and Cui's $\mathrm{BSR}(4,4) \geq z_{2A}(4,4) = 11$; the previous upper bound was 12. The upper bound follows the boundary strategy of Blekherman, Dunbar and Sinn, as in our note on width three: at a hypothetical boundary point of the set of sums of eleven squares inside the cone, a Hankel multiplier of rank at most five satisfies first and second order conditions. A classification of such multipliers without simple tensors in their image leaves, up to exchanging the factors, only two support and rank triples. For support 3×4, alternating variations parametrized by $\mathbb{R}^3\otimes\Lambda^2\mathbb{R}^4$ have a 17-dimensional image in a quadratic space of positive index at most 16. For support 3×3, we use either the width-three trace argument or a syzygy involving the radicals in both factors.

Companion site.

https://karlesmarin.github.io/blueprints/bsr-four-four/: every statement with its dependencies and its checks, and a lab in the browser. This note: doi:10.5281/zenodo.24055900 (Zenodo), with its supplementary material; it continues doi:10.5281/zenodo.24051362.

1 The result

Write $x = (x_1,\dots,x_4)$, $y = (y_1,\dots,y_4)$, $V = \mathbb{R}^4\otimes\mathbb{R}^4$ (identified with the bilinear forms through the monomial basis), and let R_2 be the space of biquadratic forms, of dimension 100. $\mathrm{BSR}(m,n)$ is the least r such that every biquadratic sum of squares in $m+n$ variables is a sum of r squares of bilinear forms. Qi and Cui proved $\mathrm{BSR}(4,4)\geq z_{2A}(4,4) = 11$ [4 , Theorem 3.1] , and $\mathrm{BSR}(4,4)\leq 12$ follows from $\mathrm{BSR}(m,3) = 2m$ [3 , Corollary 7] .

Theorem 1. p. 1

> $\mathrm{BSR}(4,4) = 11$.

Isabelle · ours · part **show**

Only a part is in Isabelle: partly: the regular point, the polar multiplier of Lemma 2 (iii) over the reals, the trace count of Proposition 8, and pieces of Lemma 3 (the support count, the factorisation in the case s = a = r, the case s < a under two hypotheses); outside the formal part: the passage to the surjectivity of dΦ, the extension of the polar multiplier to C, the identification of a simple tensor in im H, the Segre bound of Lemma 3, Lemma 2, Proposition 5 and the second variations

`regular_point_symmetric` · `Isabelle/BSRFourFour/BSR44_Pieces.thy`, line 25 — the regular point (Lemma 4)

```
theorem regular_point_symmetric:
  fixes H :: "nat \<Rightarrow> nat \<Rightarrow> nat \
<Rightarrow> nat \<Rightarrow> real" and n :: nat
  assumes sym: "\<And>a b c d. a < n \<Longrightarrow> b <
n \<Longrightarrow> c < n \<Longrightarrow> d < n \
<Longrightarrow> H a b c d = H c d a b"
    and hankel: "\<And>a b c d. a < n \<Longrightarrow> b <
n \<Longrightarrow> c < n \<Longrightarrow> d < n \
<Longrightarrow> H a b c d = H a d c b"
    and van: "\<And>a b c d. a < n \<Longrightarrow> b < n
\<Longrightarrow> c < n \<Longrightarrow> d < n \
<Longrightarrow> H a b c d + H b a c d = 0"
    and idx: "a < n" "b < n" "c < n" "d < n"
  shows "H a b c d = 0"
```

`support_triples` · `Isabelle/BSRFourFour/BSR44_Pieces.thy`, line 54 — the count closing Lemma 3

```
lemma support_triples:
  fixes a b r :: nat
  assumes "2 \<le> a" "a \<le> 4" "2 \<le> b" "b \<le> 4"
"r \<le> 5" "r \<le> (a - 1) * (b - 1)" "a + 1 \<le> r" "b
+ 1 \<le> r"
  shows "(a, b, r) \<in> {(4, 4, 5), (4, 3, 5), (3, 4, 5),
(3, 3, 4)}"
```

`trace_count` · `Isabelle/BSRFourFour/BSR44_Pieces.thy`, line 92 — the trace count of Proposition 8

```
lemma trace_count:
  fixes lam :: "'i \<Rightarrow> real" and I :: "'i set"
  assumes fin: "finite I" and le1: "\<And>i. i \<in> I \
<Longrightarrow> lam i \<le> 1"
    and rk: "card {i \<in> I. lam i \<noteq> 0} \<le> 4"
and neg: "j \<in> I" "lam j < 0"
  shows "(\<Sum>i\<in>I. lam i) < 3"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box).

Isabelle: 3 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the regular point, the polar multiplier of Lemma 2 (iii) over the reals, the trace count of Proposition 8, and pieces of Lemma 3 (the support count,

> the factorisation in the case s = a = r, the case s < a under two hypotheses); outside the formal part: the passage to the surjectivity of dΦ, the extension of the polar multiplier to C, the identification of a simple tensor in im H, the Segre bound of Lemma 3, Lemma 2, Proposition 5 and the second variations · **Cite:** Theorem 1, p. 1, doi:10.5281/zenodo.24055900

> 🆕 **Classical and new:** the paper: the lower bound 11 is [QC, Theorem 3.1]; the previous upper bound 12 follows from BSR(m,3) = 2m [Mar, Corollary 7].

↳ Not used by another result of the paper.

Take a biquadratic form in $x_1, \dots, x_4$ and $y_1, \dots, y_4$ that is a sum of squares of bilinear forms, however many. The theorem says that eleven squares always suffice, and Qi and Cui gave a form for which ten do not. Before this note the best upper bound was twelve, from width three. The proof does not search for representations: it shows that the set of sums of eleven squares has no boundary inside the cone, because any boundary point would carry a Hankel multiplier of rank at most five that the classification of Lemmas 2–3 and the second variations of Propositions 6 and 8 rule out.

🛡️ **Evidence & scope**

- **Program:** perturbations P + εb² of the Qi–Cui witness: approximate representations with eleven squares in all 312 cases. working script `perturbar_qc.py`, not published

The upper bound is proved in §§2–5. It follows the boundary strategy of Blekherman, Dunbar and Sinn [1 , § § 2–3] in the form of [3 , Proposition 3] . Our argument excludes full support at rank five (Lemma 2), determines the possible support dimensions and ranks (Lemma 3), and rules out both remaining cases by the second-variation arguments in §§4–5 (Propositions 6 and 8); the one in §4 is built from the six alternating forms of $\Lambda^2\mathbb{R}^4$.

2 Hankel multipliers of small rank

Let A, B be finite-dimensional complex vector spaces. A *multiplier* is a symmetric bilinear form H on $A \otimes B$ with

$$H(x \otimes b,\, y \otimes c) = H(x \otimes c,\, y \otimes b); \quad (1) \qquad (1)$$

equivalently, a linear functional on biquadratic forms. Exchanging the roles of the factors, $H(y \otimes b, x \otimes c) = H(x \otimes c, y \otimes b) = H(x \otimes b, y \otimes c)$, so the definition is symmetric in A and B. The *radical* of H in A is $\{x : H(x \otimes B,\ A \otimes B) = 0\}$, and the *support* a is the dimension of A modulo it; likewise b_0 for B. A simple tensor of $(A \otimes B)^*$ is a non-zero

$\alpha \otimes \beta$. If $\operatorname{rk} H = r$, write $H = T^* J T$ with $T : A \otimes B \to E$ onto, $\dim E = r$, J a non-degenerate symmetric form on E, and $T_b x = T(x \otimes b)$. Then $\operatorname{im} H = T^* J(E)$ and (1) reads

$$\langle T_b x, T_c y \rangle_J = \langle T_c x, T_b y \rangle_J. \quad (2) \tag{2}$$

Observation. If $T_c x = \beta(c) e$ for all c, with $x \neq 0$, $\beta \neq 0$, $e \neq 0$, then $\operatorname{im} H$ contains a simple tensor: by (2) the covector $T^* J e$ is $(y \otimes c) \mapsto \beta(c) \langle e, T_d y \rangle_J$ for any d with $\beta(d) = 1$, and it is non-zero because T is onto.

Let $s = \max_c \operatorname{rk} T_c$ and, for generic b, $U = \operatorname{im} T_b$ ($\dim U = s$). Then

$$T_c(\ker T_b) \subseteq U \cap U^{\perp_J} \qquad \text{for all } c: \quad (3) \tag{3}$$

the inclusion in U is the standard fact on maximal rank in a pencil (from the linear terms in t of the minors of order $s+1$ of $T_b + tT_c$ bordering a maximal identity block of T_b), and the orthogonality follows from (2). If $H \neq 0$, the quadratic form $Q(b) = T_b^* J T_b$ is not identically zero (by separate polarization, H is recovered from $q(x, b) = x^\top Q(b) x$), so the generic b can also be taken with $Q(b) \neq 0$. If $\ker T_b \neq 0$, the radical of H in A is zero and $\operatorname{im} H$ has no simple tensor, then (3) and the Observation force $\dim(U \cap U^{\perp_J}) \geq 2$.

Lemma 2. p. 2

> *There is no multiplier on $\mathbb{C}^4 \otimes \mathbb{C}^4$ of rank 5, with supports $a = b_0 = 4$, whose image contains no simple tensor.*

Isabelle · ours · part **show**

Only a part is in Isabelle: only the last step of case (iii), over the reals: X ↦ (tr(X)I + Xᵀ)/2 is injective on real matrices (polar_trace_injective); the paper needs complex matrices, which follows by applying it to the real and imaginary parts — that step is not formalized

`polar_trace_injective` · `Isabelle/BSRFourFour/BSR44_Pieces.thy`, line 67 — case (iii): X ↦ (tr(X) I + Xᵀ)/2 is injective, so the polar of (xᵀb)² has rank 16, not 5

```
lemma polar_trace_injective:
  fixes X :: "nat \<Rightarrow> nat \<Rightarrow> real" and
n :: nat
  assumes zero: "\<And>i j. i < n \<Longrightarrow> j < n \
<Longrightarrow> (if i = j then (\<Sum>k<n. X k k) else 0)
+ X j i = 0"
```

```
    and ij: "i < n" "j < n"
  shows "X i j = 0"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Isabelle: 1 result, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: only the last step of case (iii), over the reals: X ↦ (tr(X)I + X^T)/2 is injective on real matrices (polar_trace_injective); the paper needs complex matrices, which follows by applying it to the real and imaginary parts — that step is not formalized · **Cite:** Lemma 2, p. 2, doi:10.5281/zenodo.24055900

↳ Used in: Theorem 1 (p. 3), Lemma 3 (p. 9), Proposition 6 (p. 14)

Proof

The three cases below are exhaustive: $s = 4$ with some $Q(b)$ invertible, $s = 4$ with every $Q(b)$ singular, and $s \le 3$.

(i) Some $Q(b)$ is invertible. Choosing bases by congruence, $E = \mathbb{C}^4 \oplus \mathbb{C}$ with the standard symmetric product and $T_b x = (x, 0)$. In a basis of a complement of b write $T_i x = (A_i x,\ u_i^\top x)$, $i = 1, 2, 3$. By (2),

$$A_i^\top = A_i, \qquad [A_i, A_j] = u_j u_i^\top - u_i u_j^\top. \tag{4}$$

With $v \odot w = vw^\top + wv^\top$ and $K_{ij} = A_i u_j - A_j u_i$, the Jacobi identity for A_1, A_2, A_3 and (4) give

$$K_{12} \odot u_3 + K_{23} \odot u_1 + K_{31} \odot u_2 = 0. \tag{5}$$

If the u_i span 0, T is not onto. If they span a line, the last coordinate of T is a simple covector of $\operatorname{im} H$. If they span a plane, after a change of basis of the complement $u_3 = 0$ with u_1, u_2 independent; then A_3 commutes with A_1, A_2, and (5) gives $(A_3 u_1) \odot u_2 = (A_3 u_2) \odot u_1$, whence $A_3 u_1 = t u_1$, $A_3 u_2 = t u_2$ for one t (compare coefficients in a basis starting with u_1, u_2). The space $P = \operatorname{im}(A_3 - tI)$ is non-zero (otherwise $T_3 = tT_b$ and B would have a radical, against $b_0 = 4$), stable under all A_i, and orthogonal to u_1, u_2 since A_3 is symmetric; by (4) the A_i commute on P, and a common eigenvector $x \in P$ has $T_c x \in \mathbb{C}(x, 0)$ for all c, so the Observation applies. If the u_i are independent, (5) forces $K_{ij} = t_i u_j - t_j u_i$ for three scalars t_i (complete u_1, u_2, u_3 to a basis: the terms with the fourth vector and the squares u_i^2 force $K_{12} \in \langle u_1, u_2 \rangle$, $K_{23} \in \langle u_2, u_3 \rangle$, $K_{31} \in \langle u_3, u_1 \rangle$, and the

coefficients of u_1u_2, u_2u_3, u_3u_1 give the signs). Then the symmetric matrices

$$\widehat{A}_i = \begin{pmatrix} A_i & u_i \\ u_i^\top & t_i \end{pmatrix}$$

commute (the upper-left block of their commutator vanishes by (4), the upper right one is $K_{ij} + t_j u_i - t_i u_j = 0$, and the rest follows by antisymmetry). A common eigenvector $z \neq 0$ exists over $\mathbb{C}$; if $\widehat{A}_i z = \lambda_i z$, then $\langle z, T_i x\rangle = \lambda_i \langle z, T_b x\rangle$, since $T_i x = \widehat{A}_i T_b x$, so $T^* J z$ is simple, and non-zero because T is onto.

(ii) Some T_b has rank 4 and every $Q(b)$ is singular. Then $U = \operatorname{im} T_b$ is a degenerate hyperplane with a radical of dimension one. In a Witt basis $E = \mathbb{C}^3 \oplus \langle e, f\rangle$, $\langle e, f\rangle = 1$, $\langle e, e\rangle = \langle f, f\rangle = 0$, with $T_b(w, z) = (w, z, 0)$, condition (2) gives each T_c the shape $\left(\begin{smallmatrix} A & p \\ l^\top & t \\ p^\top & q_0 \end{smallmatrix}\right)$ with $A = A^\top$. In $Q(b + \tau c)$ the upper 3×3 block is $I + O(\tau)$, the upper right block is $2\tau p + O(\tau^2)$ and the lower right entry is $2\tau q_0 + \tau^2(p^\top p + 2tq_0)$. Since the determinant vanishes identically, so does the Schur complement as a formal series; its terms of order one and two give $q_0 = 0$ and $-3p^\top p = 0$. Thus $p(c)^\top p(c) = 0$ for every c, and $c \mapsto p(c)$ is linear; *by polarization* $p(c)^\top p(c') = 0$ for all c, c', so $\{p(c)\}$ spans a totally isotropic subspace of $\mathbb{C}^3$, of dimension at most one: $p(c) = \beta(c) p_0$. The covector $T^* J e$, $(w, z; c) \mapsto p(c)^\top w = \beta(c)\, p_0^\top w$, is simple and non-zero.

(iii) $s \leq 3$. By (3) and $Q(b) \neq 0$ generically, $s = 3$ and $\operatorname{rk} Q(b) = 1$ generically: U has a radical of dimension at least two and $Q(b) \neq 0$. So $q(x, b) = x^\top Q(b) x$ is, over $\mathbb{C}(b)$, a scalar times the square of a linear form in x (complete the square from a non-zero diagonal entry). By Gauss' lemma $q = g(b) L(x, b)^2$ in $\mathbb{C}[b][x]$ with L primitive, and the factors of a bihomogeneous polynomial are bihomogeneous. If L has degree 0 in b, the support in A is 1. Otherwise g is constant and $L = x^\top P b$, with P invertible by full support, so after changes of basis and scale $q = (x^\top b)^2$. Its polar multiplier acts on matrices as $X \mapsto (\operatorname{tr}(X) I + X^\top)/2$, which is invertible ($X^\top = -\operatorname{tr}(X) I$ gives $5 \operatorname{tr} X = 0$ and $X = 0$): rank $16 \neq 5$.

Evidence & scope

- **Program:** the Jacobi identity (5), the Schur expansion of case (ii) and the rank 16 of the polar of $(x^\top b)^2$, in exact arithmetic.

supplement.zip: check_symbolic_steps.py

Lemma 3. p. 3

> *Let H be a nonzero multiplier on $\mathbb{C}^4 \otimes \mathbb{C}^4$ of rank $r \leq 5$ whose image contains no simple tensor. Up to exchanging the factors, (a, b_0, r) is either $(3, 4, 5)$ or $(3, 3, 4)$.*

Isabelle · ours · part **show**

Only a part is in Isabelle: pieces: the closing count (support_triples); the case s = a = r (case_s_eq_a), the case s < a with two hypotheses (the inclusion in im T_b of (3) and a one-dimensional radical; case_s_lt_a) and the Observation (observation), each concluding a nonzero FACTORISATION of the functional — in all three, the passage to a nonzero simple tensor α ⊗ β in im H uses the bilinearity of the paper and is not formalized. The Segre bound r ≤ (a−1)(b₀−1), the generic b and the assembly are not formalized

`support_triples` · `Isabelle/BSRFourFour/BSR44_Pieces.thy`, line 54 — the arithmetic that closes Lemma 3

```
lemma support_triples:
  fixes a b r :: nat
  assumes "2 \<le> a" "a \<le> 4" "2 \<le> b" "b \<le> 4"
"r \<le> 5" "r \<le> (a - 1) * (b - 1)" "a + 1 \<le> r" "b
+ 1 \<le> r"
  shows "(a, b, r) \<in> {(4, 4, 5), (4, 3, 5), (3, 4, 5),
(3, 3, 4)}"
```

`case_s_eq_a` · `Isabelle/BSRFourFour/Supports_Lemma.thy`, line 286 — the case s = a = r: the operators $T_b^{-1}T_c$ commute, a common eigenvector gives a simple covector in the image

```
theorem case_s_eq_a:
  fixes J :: "'e::ab_group_add \<Rightarrow> 'e \
<Rightarrow> complex" and T :: "'c \<Rightarrow>
'a::ab_group_add \<Rightarrow> 'e"
    and sm :: "complex \<Rightarrow> 'e \<Rightarrow> 'e"
and smA :: "complex \<Rightarrow> 'a \<Rightarrow> 'a"
  assumes fd: "FinDimVectorSpace smA (UNIV :: 'a set)" and
nz: "\<exists>w::'a. w \<noteq> 0"
    and sym: "\<And>b c x y. J (T b x) (T c y) = J (T c x)
(T b y)"
    and Jadd: "\<And>u v w. J (u + v) w = J u w + J v w"
    and Jscal: "\<And>t u w. J (sm t u) w = t * J u w"
    and nondeg: "\<And>z. (\<And>w. J z w = 0) \
<Longrightarrow> z = 0"
    and Tadd: "\<And>c u v. T c (u + v) = T c u + T c v"
    and Tscal: "\<And>c t u. T c (smA t u) = sm t (T c u)"
    and Tb_inj: "\<And>u. T b u = 0 \<Longrightarrow> u =
0"
    and Tb_onto: "\<And>w. \<exists>y. w = T b y"
  shows "\<exists>e \<beta>. e \<noteq> 0 \<and> \<beta> b
\<noteq> 0 \<and> (\<forall>c y. J e (T c y) = (\<beta> c /
\<beta> b) * J e (T b y))
           \<and> (\<exists>y. J e (T b y) \<noteq> 0)"
```

`case_s_lt_a` · `Isabelle/BSRFourFour/Supports_Lemma.thy`, line 102 —

the case s < a, with the inclusion T_c(ker T_b) ⊆ im T_b and a one-dimensional radical as hypotheses

```
theorem case_s_lt_a:
  fixes J :: "'e::ab_group_add \<Rightarrow> 'e \
<Rightarrow> complex" and T :: "'c \<Rightarrow> 'a \
<Rightarrow> 'e"
    and sm :: "complex \<Rightarrow> 'e \<Rightarrow> 'e"
  assumes sym: "\<And>b c x y. J (T b x) (T c y) = J (T c
x) (T b y)"
    and Jscal: "\<And>t u w. J (sm t u) w = t * J u w"
    and Jadd: "\<And>u v w. J (u + v) w = J u w + J v w"
    and nondeg: "\<And>z. (\<And>c y. J z (T c y) = 0) \
<Longrightarrow> z = 0"
    and x: "T b x = 0" and xrad: "T d0 x \<noteq> 0"
    and inU: "\<And>c. T c x \<in> range (T b)"
    and line: "\<And>u. u \<in> range (T b) \
<Longrightarrow> (\<forall>y. J u (T b y) = 0) \
<Longrightarrow> \<exists>t. u = sm t e"
  shows "\<exists>e' \<beta> d. e' \<noteq> 0 \<and> \
<beta> d \<noteq> 0 \<and> (\<forall>c y. J e' (T c y) = (\
<beta> c / \<beta> d) * J e' (T d y))
           \<and> (\<exists>y. J e' (T d y) \<noteq> 0)"
```

observation · Isabelle/BSRFourFour/Supports_Lemma.thy, line 60 — the Observation of Section 2: T_c x = β(c) e gives a non-zero simple covector

```
theorem observation:
  fixes J :: "'e::ab_group_add \<Rightarrow> 'e \
<Rightarrow> complex" and T :: "'c \<Rightarrow> 'a \
<Rightarrow> 'e"
    and sm :: "complex \<Rightarrow> 'e \<Rightarrow> 'e"
  assumes sym: "\<And>b c x y. J (T b x) (T c y) = J (T c
x) (T b y)"
    and Jscal: "\<And>t u w. J (sm t u) w = t * J u w"
    and nondeg: "\<And>z. (\<And>c y. J z (T c y) = 0) \
<Longrightarrow> z = 0"
    and Tx: "\<And>c. T c x = sm (\<beta> c) e" and d: "\
<beta> d \<noteq> 0" and e: "e \<noteq> 0"
  shows "(\<forall>c y. J e (T c y) = (\<beta> c / \<beta>
d) * J e (T d y)) \<and> (\<exists>y. J e (T d y) \<noteq>
0)"
```

common_eigenvector · Isabelle/BSRFourFour/Supports_Lemma.thy, line 154 — commuting endomorphisms over an algebraically closed field have a common eigenvector (classical; proved here)

```
theorem (in FinDimVectorSpace) common_eigenvector:
  assumes alg_closed: "\<And>p::'a poly. degree p > 0 \
<Longrightarrow> \<exists>c. poly p c = 0"
    and add: "\<And>T u v. T \<in> F \<Longrightarrow> u \
<in> V \<Longrightarrow> v \<in> V \<Longrightarrow> T (u +
v) = T u + T v"
    and hom: "\<And>T a u. T \<in> F \<Longrightarrow> u \
<in> V \<Longrightarrow> T (a \<cdot> u) = a \<cdot> T u"
    and comm: "\<And>S T u. S \<in> F \<Longrightarrow> T \
<in> F \<Longrightarrow> u \<in> V \<Longrightarrow> S (T
u) = T (S u)"
  shows "Subspace W \<Longrightarrow> (\<exists>w\<in>W. w
\<noteq> 0) \<Longrightarrow> (\<And>T. T \<in> F \
<Longrightarrow> T ` W \<subseteq> W)
           \<Longrightarrow> \<exists>u\<in>W. u \<noteq> 0
\<and> (\<forall>T\<in>F. \<exists>c. T u = c \<cdot> u)"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box).
Isabelle: 5 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: pieces: the closing count (support_triples); the case s = a = r (case_s_eq_a), the case s < a with two hypotheses (the inclusion in im T_b of (3) and a one-dimensional radical; case_s_lt_a) and the Observation (observation), each concluding a nonzero FACTORISATION of the functional — in all three, the passage to a nonzero simple tensor α ⊗ β in im H uses the bilinearity of the paper and is not formalized. The Segre bound r ≤ (a−1)(b₀−1), the generic b and the assembly are not formalized · **Cite:** Lemma 3, p. 3, doi:10.5281/zenodo.24055900

↳ Used in: Theorem 1 (p. 3)

Proof

Restrict H to $A' \otimes B'$, with A', B' complements of the two radicals: the rank, the symmetry (1) and the absence of simple tensors are kept (a simple covector of the restricted image extends by zero on the radicals and stays simple and in $\operatorname{im} H$), and now both radicals are zero, with $\dim A' = a$, $\dim B' = b_0$. Since $\operatorname{im} H$ annihilates the radicals, $\mathbb{P}(\operatorname{im} H)$ is a subspace of dimension $r-1$ avoiding the Segre variety of the effective factors, so $r \le (a-1)(b_0-1)$; and $a, b_0 \ge 2$ since $H \neq 0$.

We show $r \ge a+1$, and by the symmetry of (1) in the two factors also $r \ge b_0+1$. Suppose $r \le a$ (≤ 4) and take b generic, with $s = \operatorname{rk} T_b$ maximal and $Q(b) \neq 0$. If $s<a$, then $\ker T_b \neq 0$, and the radical of $U = \operatorname{im} T_b$ has dimension at most $s-1$ (as $Q(b) \neq 0$) and at most $r-s$ (as it lies in $U^{\perp_J}$), so at most $\min(s-1, r-s) \le 1$ because $r \le 4$; this contradicts (3) and the Observation. If $s = a$, then $s = a = r$, $T_b : A' \to E$ is an isomorphism, $Q(b)$ is non-degenerate, and the operators $T_b^{-1}T_c$ are self-adjoint for $Q(b)$ and, by (2), commute; a common eigenvector x has $T_c x \in \mathbb{C}\, T_b x$ for all c, and the Observation gives a simple tensor. So $r \ge \max(a, b_0)+1$.

With $r \le 5$ and $r \le (a-1)(b_0-1)$ the only possibilities are $(a, b_0, r) = (4,4,5), (4,3,5), (3,4,5)$ and $(3,3,4)$, and Lemma 2 excludes $(4,4,5)$.

Evidence & scope

- **Program:** Newton search for rank-five Hankel multipliers with support 3×4 and product-free image: 14 numerical candidates.
  `supplement.zip: search_rank_five_multipliers.py`

3 The boundary multiplier

Let $\Phi(h) = \sum_{i=1}^{11} h_i^2$ on V^{11} and $\Sigma_{11} = \Phi(V^{11})$; it is closed, and $\Sigma \subset R_2$, the cone of sums of squares, is closed, convex, with non-empty interior.

Lemma 4. (a regular point) p. 4

> *Identify both factors with $\mathbb{R}^4$. Let $S \subset V$ be the span of the four forms $x_i y_i$ and the six forms $x_i y_j + x_j y_i$ for $i<j$. If H is a Hankel multiplier and $S \subseteq \ker H$, then $H = 0$. Consequently $d\Phi$ is surjective at the tuple consisting of these ten forms and zero.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the linear algebra; the duality from there to «dΦ onto» is on paper

`regular_point_symmetric` · `Isabelle/BSRFourFour/BSR44_Pieces.thy`, line 25 — a symmetric Hankel multiplier that vanishes on the symmetric tensors in its first argument is zero (in any size n)

```
theorem regular_point_symmetric:
  fixes H :: "nat \<Rightarrow> nat \<Rightarrow> nat \
<Rightarrow> nat \<Rightarrow> real" and n :: nat
  assumes sym: "\<And>a b c d. a < n \<Longrightarrow> b <
n \<Longrightarrow> c < n \<Longrightarrow> d < n \
<Longrightarrow> H a b c d = H c d a b"
    and hankel: "\<And>a b c d. a < n \<Longrightarrow> b <
n \<Longrightarrow> c < n \<Longrightarrow> d < n \
<Longrightarrow> H a b c d = H a d c b"
    and van: "\<And>a b c d. a < n \<Longrightarrow> b < n
\<Longrightarrow> c < n \<Longrightarrow> d < n \
<Longrightarrow> H a b c d + H b a c d = 0"
    and idx: "a < n" "b < n" "c < n" "d < n"
  shows "H a b c d = 0"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box).
Isabelle: 1 result, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the linear algebra; the duality from there to «dΦ onto» is on paper · **Cite:** Lemma 4, p. 4, doi:10.5281/zenodo.24055900

↳ Used in: Theorem 1 (p. 3), Proposition 5 (p. 13)

Proof

> If $S \subseteq \ker H$, then $H(s, v) = H(v, s) = 0$ for all $s \in S$ and $v \in V$, and S is the space of symmetric tensors, so $H(x \otimes y, x \otimes y) = -H(x \otimes y, y \otimes x) = -H(x \otimes x, y \otimes y) = 0$ for all x, y, and $H = 0$ by polarization. Hence the products of the ten summands with V span R_2 (we also checked the rank

100 exactly), and adding an eleventh summand, even zero, gives a point where $d\Phi$ is onto.

Evidence & scope

- **Program:** rank 100 of the products of the ten summands with V, in exact arithmetic; control with the four diagonal summands only: 58. `supplement.zip: check_regular_point.py`

Proposition 5. p. 4

> *If* $\Sigma_{11} \neq \Sigma$, *there are* $F \in \operatorname{int}\Sigma \cap \partial\Sigma_{11}$, *a representation* $F = \sum_{i=1}^{11} h_i^2$ *and an indefinite real multiplier* H *on* V *such that, with* $W = \operatorname{span}(h_i)$: $W \subseteq \ker H$, $\dim W = 11$ *and hence* $\operatorname{rk} H \leq 5$; *the complex image of* H *contains no simple tensor; and* $\sum_i h_i D_i = 0$ *in* R_2 *implies* $\sum_i H(D_i, D_i) \geq 0$.

NEW **Classical and new:** the paper: the argument of [Mar, Proposition 3 and Corollary 4], after [BDS]; only the jet count $100 - 14 + 12 = 98 < 99$ depends on the size.

↳ Used in: Theorem 1 (p. 3), Proposition 6 (p. 14), Proposition 8 (p. 15)

Proof

> We adapt the argument of [3 , Proposition 3 and Corollary 4] to $(m, n) = (4, 4)$, using Lemma 4. The only count that depends on the size is the one excluding forms with a singular point with no real factor. Take a pair of conjugate points distinct in both factors and change coordinates over $\mathbb{C}$ to $(e_1, e_1), (e_2, e_2)$. A first jet has seven coordinates, the value and the six tangent derivatives, and the two first jets use disjoint sets of seven monomial coefficients, so the joint jet map is surjective over $\mathbb{C}$. Conjugation exchanges the two jets, and the restriction to real forms has real rank fourteen. The point varies in real dimension twelve. The incidence therefore has dimension at most $100 - 14 + 12 = 98$, and its projection cannot fill the boundary patch of dimension 99. A singular point with a real factor is impossible at an interior form, by definiteness of the quadratic form in the other factor. Independence of the h_i is as in [3 , Corollary 4] .

By Lemma 3 applied to the complexification of H, the supports of H are, up to exchanging the factors, 3×4 with rank 5, or 3×3 with rank 4.

4 Support 3×4, rank five

Proposition 6. p. 4

> *No triple (F, h, H) satisfying the conclusions of Proposition 5 can have $(a, b_0, r) = (3, 4, 5)$ or $(4, 3, 5)$.*

↳ Used in: Theorem 1 (p. 3), 7 (p. 15)

Proof

Here $\dim \ker H = 16 - 5 = 11$, so $W = \ker H$. Let x_0 span the radical in the first factor and A' be a real complement, $\dim A' = 3$. Then $x_0 \otimes B \subseteq W$, and an orthogonal change of the eleven summands (an orthonormal basis of the preimage of $x_0 \otimes B$, completed) gives $h_i = x_0 \otimes y_i$, $1 \le i \le 4$, for a real basis (y_i) of B. For $z_{ij} = -z_{ji} \in A'$ put

$$D_i = \sum_{j=1}^{4} z_{ij} \otimes y_j \quad (i \le 4), \qquad D_i = 0 \quad (i{>}4).$$

Then $\sum_i h_i D_i = \sum_{i,j} x_0 z_{ij}\, y_i y_j = 0$ by antisymmetry: these are syzygies, a space $A' \otimes \Lambda^2 B$ of dimension 18. With $H = T^* J T$ as in §2 (now over $\mathbb{R}$, $\dim E = 5$), the second variation is $\sum_i H(D_i, D_i) = J^{\oplus 4}(\Psi z)$ for

$$\Psi : A' \otimes \Lambda^2 B \to E^4, \qquad \Psi(z) = \Big(\sum_j T_{y_j} z_{ij} \Big)_{i=1}^{4}.$$

By Proposition 5, $J^{\oplus 4}$ is non-negative on $\operatorname{im} \Psi$. Since H is indefinite, the positive index p of J is at most 4, and a subspace on which $J^{\oplus 4}$ is non-negative has dimension at most $4p \le 16$. Hence $\operatorname{rk} \Psi \le 16$.

On the other hand $\operatorname{rk} \Psi = 17$. Complexify and compute the annihilator of $\operatorname{im} \Psi$ under $J^{\oplus 4}$: a tuple $(c_i) \in E^4$, i.e. a map $C : B \to E$ with $C y_i = c_i$, annihilates $\operatorname{im} \Psi$ if and only if the matrix $\big(\langle C y_i, T(a \otimes y_j) \rangle_J \big)_{ij}$ is symmetric for every $a \in A'$ (vary each z_{ij}, $i{<}j$, separately). The three maps $C = T(a \otimes \cdot)$, a in a basis of A', are independent and lie in the annihilator by (2). If there were a fourth independent C, it would extend T to $\widetilde{T} : \mathbb{C}^4 \otimes B \to E$ satisfying (2): the identities involving the new row are exactly the symmetry just written, and the one of the new row with itself is automatic. Then $\widetilde{T}^* J \widetilde{T}$ is a multiplier

of rank 5 (as T is onto) with supports 4 and 4 (by independence of the new row, and since the second support was already full), and its image contains no simple tensor: a simple covector in it restricts to $A' \otimes B$ as a simple covector of $\operatorname{im} H$, excluded by Proposition 5, or as zero, and then its representing vector z has $\langle z, T(A' \otimes B)\rangle_J = 0$, so $z = 0$. This contradicts Lemma 2. So the annihilator has dimension 3 and $\operatorname{rk} \Psi = 20 - 3 = 17 > 16$, a contradiction.

Evidence & scope

- **Program:** on the rank-five multiplier with support 3×4 of the supplement of [Mar]: rank $\Psi = 17 > 16$ and a negative value of the second variation; in the 14 numerical candidates rank $\Psi = 17$, and 15 in a control built from five product terms. `supplement.zip: check_proof_steps.py`

Remark 7. p. 5

One triple of quaternionic structures does not suffice: for $z_{ij} = \sum_a \eta^a_{ij} v_a$ with the 't Hooft symbols η^a, the second variation is $\sum_a v_a^\top (\sum_j H_{jj}) v_a$, because $(\eta^a)^\top \eta^b$ has symmetric part $\delta_{ab} I$ and $H_{jk} = H_{kj}$; it only says that $\sum_j H_{jj} \succeq 0$. The argument above uses the whole of $\Lambda^2 B$.

↳ Not used by another result of the paper.

5 Support 3×3, rank four

Proposition 8. p. 5

> *No triple (F, h, H) satisfying the conclusions of Proposition 5 can have $(a, b_0, r) = (3, 3, 4)$.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the final trace count of the case u ∉ W (trace_count); the two second variations are on paper

`trace_count` · `Isabelle/BSRFourFour/BSR44_Pieces.thy`, line 92 — eigenvalues at most 1, at most four non-zero, one negative: their sum is less than 3

```
lemma trace_count:
  fixes lam :: "'i \<Rightarrow> real" and I :: "'i set"
  assumes fin: "finite I" and le1: "\<And>i. i \<in> I \
```

```
<Longrightarrow> lam i \<le> 1"
    and rk: "card {i \<in> I. lam i \<noteq> 0} \<le> 4"
and neg: "j \<in> I" "lam j < 0"
  shows "(\<Sum>i\<in>I. lam i) < 3"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box).
Isabelle: 1 result, no oracles

Proof: in this note · **Covers:** not formalized · **Cite:** Proposition 5, p. 4, doi:10.5281/zenodo.24055900

↳ Used in: Theorem 1 (p. 3)

Proof

Let x_0, y_0 span the real radicals of the two factors, A', B' complements and $u = x_0 \otimes y_0$. The space $U = x_0 \otimes B + A \otimes y_0$ has dimension 7 and lies in $K = \ker H$, $\dim K = 12$; W is a hyperplane of K, and $K = U \oplus K_0$ with $K_0 = K \cap (A' \otimes B')$.

Case $u \notin W$. Then $W \cap (x_0 \otimes B) = x_0 \otimes L$ with $\dim L = 3$ and L projecting isomorphically onto $B/\langle y_0 \rangle$. Orthogonalizing on the preimage of this intersection, $h_i = x_0 \otimes y_i$, $i \le 3$, for a basis of L. The restriction of H to $A' \otimes L$ keeps the indices, the rank four and the support three in the first factor. For $v_1, v_2, v_3 \in A'$ put $D_i = \sum_{j,p} \varepsilon_{ijp}\, v_p \otimes y_j$ for $i \le 3$ and $D_i = 0$ otherwise; they are syzygies by antisymmetry, and with H_{jl} the blocks of the restriction and $S = \sum_j H_{jj}$, Proposition 5 and the contraction of the alternating symbols give, as in [3 , § 5] ,

$$I_3 \otimes S - H|_{A' \otimes L} \succeq 0.$$

The diagonal blocks sum to $2S$, so $S \succeq 0$; if $Sx = 0$, the three diagonal values of this positive semidefinite form at $x \otimes y_j$ sum to zero, so each $x \otimes y_j$ is in its kernel, $H(x \otimes y_j, \cdot) = 0$, and $x = 0$ by full support in A'. So $S \succ 0$, and after the congruence by $S^{-1/2}$ in the three blocks the matrix of H has trace 3, all eigenvalues at most 1, rank four and at least one negative eigenvalue: its trace is less than 3, a contradiction.

Case $u \in W$. An orthogonal change makes h_1 a non-zero multiple of u. Let ℓ be a real linear functional on K with $W = \ker \ell$. For $v = \sum v_{ij} a_i \otimes b_j \in A' \otimes B'$ (bases a_i, b_j) the lift $P(v) = \sum_{ij} v_{ij} (a_i \otimes y_0)(x_0 \otimes b_j) \in \operatorname{Sym}^2 U$ has product uv in R_2. If $\ell|_U = 0$ all these products lie in WK; otherwise $\operatorname{Sym}^2 U / (U \cap W) U$ is one-dimensional and the class is measured by $(\ell|_U)^2$. So $uv \in WK$ whenever

$$\phi(v) = \sum_{ij} v_{ij}\, \ell(a_i \otimes y_0)\, \ell(x_0 \otimes b_j) = 0.$$

The functional ϕ on the active block is zero or simple. If it is non-zero it does not vanish on K_0, since otherwise it would lie in the image of the active part of H, which contains no simple tensor. Choose v in the active block with $H(v,v)<0$; adding a real element of K_0 keeps this value and achieves $\phi(v) = 0$. Then $h_1 v \in WK$: write $-h_1 v = \sum_i h_i k_i$ with $k_i \in K$. The $D_i = \delta_{i1} v + k_i$ form a syzygy with second variation $H(v,v)<0$, a contradiction.

By Propositions 6 and 8 no case of Lemma 3 remains: $\Sigma_{11} = \Sigma$, and with [4 , Theorem 3.1] Theorem 1 follows. □

6 Checks and status

Computed with exact arithmetic (SymPy): the rank 100 at the regular point (the control with only the four diagonal summands gives 58); on the rank-five multiplier with support 3×4 of the supplement of [3] (factors exchanged), $\operatorname{rk} \Psi = 17 > 16$ and a negative value of the second variation; the identity (5), the Schur expansion of case (ii) and the rank 16 of the polar of $(x^{\top} b)^2$. Numerically, a Newton search targeting rank-five Hankel multipliers with support 3×4 and product-free image produced 14 numerical candidates; the estimated rank of Ψ was 17 in all 14 cases and 15 in one control built from five product terms. For all 312 tested perturbations $P + \varepsilon b^2$ of the Qi–Cui witness P, $\varepsilon \in \{10^{-4}, 10^{-2}, 1\}$, the solver found approximate representations using eleven bilinear squares. The Isabelle/HOL session `BSRFourFour` of our public repository of formalizations verifies the algebraic vanishing assertion in Lemma 4, the integer enumeration of support and rank triples at the end of Lemma 3, the injectivity of the polar of $(x^{\top} b)^2$, and the final trace contradiction in Proposition 8. Lemmas 2 and 3 themselves, Proposition 5 and the second variations are not formalized.

For real positive semidefinite Hankel multipliers, partial transposition fixes the associated state, and the full-support exclusion at rank five in this case follows from Chen and Đoković's results on PPT states of rank $N + 1$ [2 , Theorems 43 and 45] . Lemma 2 treats complex symmetric bilinear multipliers, and our boundary argument requires an indefinite real one. We have not located the full complex statement of Lemmas 2 and 3 in the literature.

The companion site

This note has a companion website, its *blueprint*, at https://karlesmarin.github.io/blueprints/bsr-four-four/. There one finds:

- *The statements.* Theorem 1, Lemmas 2, 3 and 4 and Propositions 5, 6 and 8, copied verbatim with their numbers and proofs.
- *The dependencies.* What each statement uses and where it is used, as a list and as a graph.
- *The map.* The result, the tools it rests on and the question it answers.
- *A lab in the browser.* The rank of the products at the regular point of Lemma 4, for $2 \leq n \leq 5$, with its controls.
- *Formal proofs.* For each statement, which part is checked in Isabelle/HOL and which is not.
- *The note itself.* Its PDF, opened at the right page from every statement.

Acknowledgement

We worked with an AI assistant.

References

1

G. Blekherman, A. Dunbar, R. Sinn, Pythagoras numbers for ternary forms, arXiv:2410.17123v2.

2

L. Chen, D. Ž. Đoković, Properties and construction of extreme bipartite states having positive partial transpose, Comm. Math. Phys. 323 (2013), 241–284; arXiv:1203.1364.

3

C. Marín, For $m \geq 2$, every $m \times 3$ biquadratic sum of squares is a sum of $2m$ squares, Zenodo, doi:10.5281/zenodo.24051362, 2026.

4

L. Qi, C. Cui, The second-order augmented Zarankiewicz number, arXiv:2610.04448v1.