Every m×4 biquadratic sum of squares is a sum of 3m−1 squares (5 ≤ m ≤ 16)

Real algebraic geometry · sums of squares on projective varieties

Carles Marín, Liqun Qi, Chen Ouyang

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At a glance

What the paper contributes

BSR(m,4) ≤ 3m − 1 for 5 ≤ m ≤ 16 (Theorem 1): every real m×4 biquadratic sum of squares is a sum of at most 3m − 1 squares of bilinear forms, one square below the previous bound 3m. In particular BSR(5,4) ≤ 14, the upper half of the conjecture BSR(5,4) = 14 of Qi and Ouyang; with the known lower bounds, 13 ≤ BSR(5,4) ≤ 14 and 16 ≤ BSR(6,4) ≤ 17. The paper does not decide whether BSR(5,4) is 13 or 14.

- **A boundary inertia inequality** (Lemma 17): at a boundary point of the sums of N squares inside the cone, the negative index q of the Hankel multiplier satisfies $N(q-1) \le \dim R_2 - C(N+2, 2)$. The paper says it is, in substance, Theorem 3.4 of Boumal, Voroninski and Bandeira, transported to this setting; its case q = 1 recovers the Barvinok–Pataki bound. For m×4 forms it gives $q \le \lfloor (m+13)/6 \rfloor$, which is at most 4 exactly for m ≤ 16.
- **A case analysis over the support triples (a, b, r)** (Theorem 25): compression to fewer rows for a ≤ m − 2, with the base values BSR(k,4) ≤ 3k − 1 for k = 2, 3, 4; a four-row isotropic lemma (Lemma 20) and a low-negative-index lemma for q ≤ 4 (Lemma 23) for the rest.
- **Explicit regular tuples for every m** (Lemma 24) and a boundary selection lemma whose codimension-two hypothesis is sharp (Lemma 12).
- **Where the method stops:** at m = 17 the inertia bound allows q = 5.
- **Machine checks:** the arithmetic, the enumeration of cases and several linear-algebra steps are checked in Isabelle/HOL (six sessions) or by exact computation; these checks cover individual steps, not the whole theorem.

🚀 **Start here: Theorem 1**: BSR(m,4) ≤ 3m − 1 for 5 ≤ m ≤ 16.

Main results:

- **Theorem 1**: BSR(m,4) ≤ 3m − 1 for 5 ≤ m ≤ 16
- **Lemma 14**: boundary inertia inequality: Nq ≤ M − 1 − C(N+1, 2) for the negative index q
- **Theorem 22**: the proof for 5 ≤ m ≤ 16, by induction on m

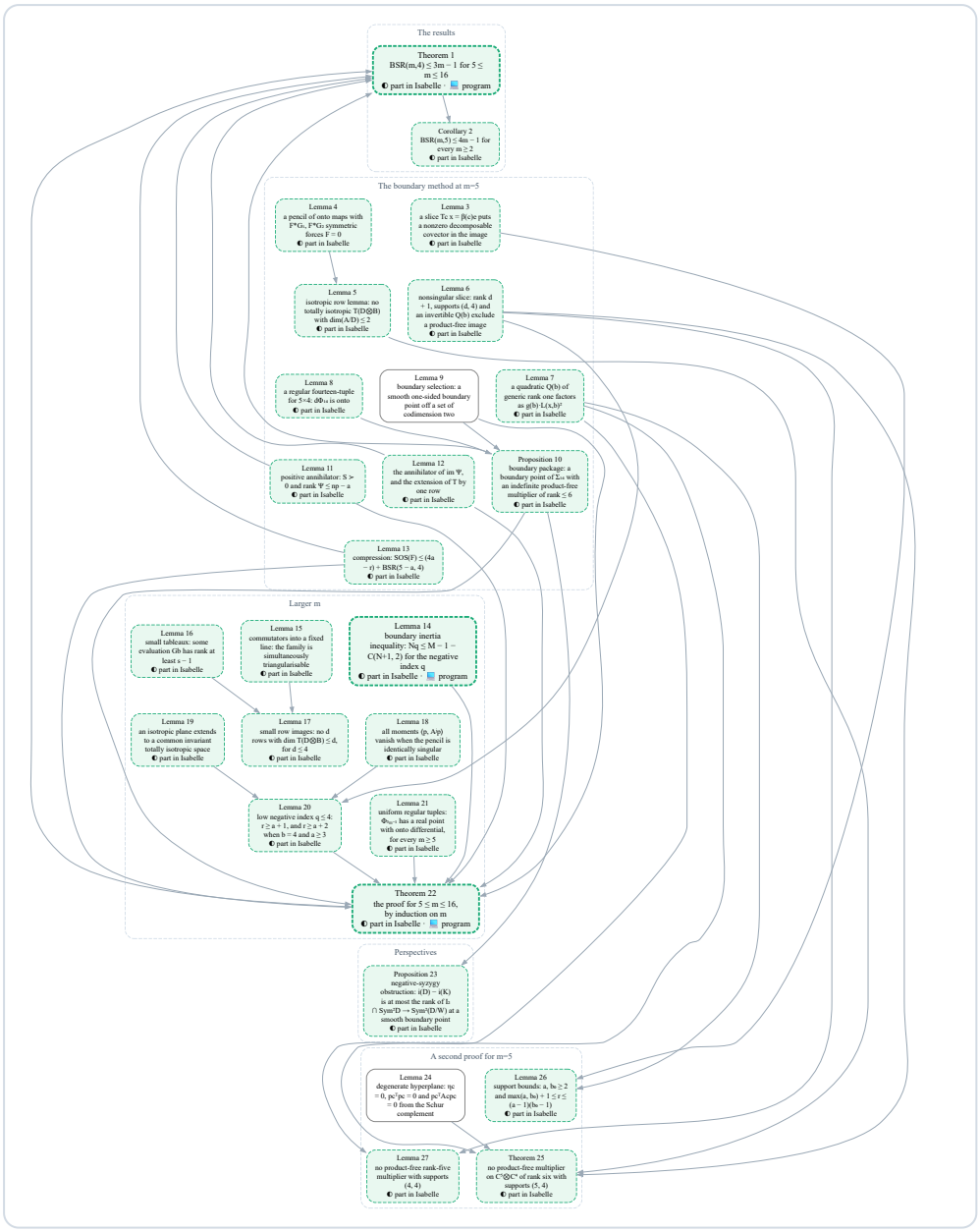
27 results | ✅ **0** entirely in Lean | ◐ **25** partly in Lean | 💻 **3** checked by a program | written proof only

(a result can carry several tags)

🗺️ Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each

statement's “Evidence & scope” block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



📐 Setting (copied from the paper: abstract, §2 and §3)

Let $\mathrm{BSR}(m,n)$ be the least r such that every real biquadratic sum of squares in $m+n$ variables is a sum of r squares of bilinear forms. (Abstract.)

Put $A=\mathbb{R}^5$, $B=\mathbb{R}^4$, $V=A\otimes B$, and

$$R_2=\mathrm{Sym}^2 A\otimes \mathrm{Sym}^2 B,\quad \Phi_k : V^k \longrightarrow R_2,\quad \Phi_k(h)=\sum_{i=1}^{k} h_i^2.$$

Thus $\dim V=20$ and $\dim R_2=150$. Write $\Sigma_k=\operatorname{im}\Phi_k$, $\Sigma=\bigcup_k \Sigma_k$. For $F\in\Sigma$, the least k with $F\in\Sigma_k$ is $\mathrm{SOS}_{\mathbb{R}}(F)$; $\mathrm{BSR}(m,n)$ is the largest real SOS length in size $m\times n$. (§2)

A Hankel multiplier is a symmetric bilinear form H on V with

$$H(x \otimes b, y \otimes c) = H(x \otimes c, y \otimes b). \tag{H}$$

Such forms are precisely the functionals on R_2, evaluated on products of bilinear forms. The first radical consists of the x such that $H(x \otimes B, V) = 0$; the first support is its codimension. Define the second radical and second support in the other factor. The image of $H : V \to V^*$ is called product-free if its complexification contains no nonzero decomposable covector. (§2)

Factor a rank-r multiplier as

$$H = T^* J T, \qquad T : A \otimes B \to E \text{ onto}, \quad \dim E = r,$$

where J is nondegenerate symmetric. Set $T_b x = T(x \otimes b)$; write $\langle\,,\,\rangle$ for J. Condition (H) becomes

$$\langle T_b x, T_c y \rangle = \langle T_c x, T_b y \rangle. \tag{F}$$

(§2)

… extend to $m \times 4$ biquadratic forms after the following modifications, with 14 replaced by $N = 3m - 1$, $\dim V = 20$ by $4m$, $\dim R_2 = 150$ by $M = 5m(m+1)$, the bound $r \le 6$ by $r \le 4m - N = m + 1$, and $\mathrm{BSR}(5 - a, 4)$ by $\mathrm{BSR}(m - a, 4)$ … Throughout, *effective supports* (a, b) means the first and second supports of H (codimensions of the two radicals). (§3)

Throughout the argument, q denotes the negative index of this oriented multiplier. The orientation is fixed by

$$\sum_i H(D_i, D_i) \ge 0 \quad \text{for every syzygy } \sum_i h_i D_i = 0.$$

(§3.1, «Orientation and inertia».)

1 The results

Theorem 1. p. 2

> *Boundary_Inertia_Arith.inertia_m4,*
> *Boundary_Inertia_Arith.q_values,*
> *Boundary_Inertia_Arith.incidence,*
> *Boundary_Inertia_Arith.compression_count,*

> *BSR_m4_Pieces.support_triples_m4, BSR_m4_Pieces.regular_point_m4 For* $5 \le m \le 16$*, every real* $m \times 4$ *biquadratic sum of squares is a sum of at most* $3m - 1$ *squares of real bilinear forms:* $\mathrm{BSR}(m, 4) \le 3m - 1$.

Isabelle · ours · part **show**

Only a part is in Isabelle: pieces only: the m×4 arithmetic of the inertia inequality and the values of ⌊(m+13)/6⌋, the incidence and compression counts, the enumeration of the support triples and the regular tuples; the theorem as a whole is a written proof (the paper: «these checks cover individual steps, not the whole theorem»)

`inertia_m4` · `Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy`, line 36

```
lemma inertia_m4:
  fixes m q :: int
  shows "2*(3*m-1)*(q-1) + (3*m+1)*(3*m) \<le> 2*(5*m*
(m+1)) \<longleftrightarrow> 2*(3*m-1)*(q-1) \<le> m*(m+7)"
```

`q_values` · `Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy`, line 77

```
lemma q_values:
  "\<forall>m::int \<in> {5..10}. (m+13) div 6 = 3" "\
<forall>m::int \<in> {11..16}. (m+13) div 6 = 4" "(17+13)
div 6 = (5::int)"
```

`incidence` · `Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy`, line 82

```
lemma incidence: fixes M m :: int shows "M - 2*(m+3) + 2*
((m-1)+3) = M - 2" by simp
lemma compression_count: fixes a m :: int shows "(4*a -
(a+1)) + (3*(m-a) - 1) = 3*m - 2" by simp
lemma psi_counts: fixes m :: int shows "4*(m+1) - (m-1) =
3*m + 5" "4*m - (m-1) = 3*m + 1" by simp_all
```

`compression_count` · `Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy`, line 83

```
lemma compression_count: fixes a m :: int shows "(4*a -
(a+1)) + (3*(m-a) - 1) = 3*m - 2" by simp
lemma psi_counts: fixes m :: int shows "4*(m+1) - (m-1) =
3*m + 5" "4*m - (m-1) = 3*m + 1" by simp_all
```

`support_triples_m4` · `Isabelle/BSRFourColumnsPieces/BSR_m4_Pieces.thy`, line 50

```
theorem support_triples_m4:
  fixes m a b r :: nat
  assumes m: "5 \<le> m" and "(a,b,r) \<in> triples m" and
"m \<le> a + 1" and "3 \<le> b"
  shows "(a,b,r) \<in> {(m,4,m+1), (m-1,4,m), (m-1,4,m+1),
(m,3,m+1), (m-1,3,m+1), (m-1,3,m)}"
```

regular_point_m4 · Isabelle/BSRFourColumnsPieces/BSR_m4_Pieces.thy, line 381

```
theorem regular_point_m4:
  fixes H :: "nat \<Rightarrow> nat \<Rightarrow> nat \
<Rightarrow> nat \<Rightarrow> real" and n :: nat
  assumes sym: "\<And>a b c d. a < 5 + n \<Longrightarrow>
b < 4 \<Longrightarrow> c < 5 + n \<Longrightarrow> d < 4 \
<Longrightarrow> H a b c d = H c d a b"
    and hankel: "\<And>a b c d. a < 5 + n \<Longrightarrow>
b < 4 \<Longrightarrow> c < 5 + n \<Longrightarrow> d < 4 \
<Longrightarrow> H a b c d = H a d c b"
    and ann: "\<forall>h \<in> S n. \<forall>c < 5 + n. \
<forall>d < 4. av (5 + n) H h c d = 0"
  shows "\<forall>a < 5 + n. \<forall>b < 4. \<forall>c < 5
+ n. \<forall>d < 4. H a b c d = 0"
```

Trust base: ⚙ kernel + code evaluation — **only part of the statement is formal** (see the box). Isabelle: 5 results, no oracles · Isabelle, oracles: q_values (Code_Generator.holds_by_evaluation)

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: pieces only: the m×4 arithmetic of the inertia inequality and the values of ⌊(m+13)/6⌋, the incidence and compression counts, the enumeration of the support triples and the regular tuples; the theorem as a whole is a written proof (the paper: «these checks cover individual steps, not the whole theorem»)

🆕 **Classical and new:** the paper: one square below the previous bound 3m, which follows from BSR(m,3) = 2m [Mar3]; it proves the upper half BSR(5,4) ≤ 14 of the conjecture BSR(5,4) = 14 of Qi and Ouyang [QO, Conjecture 1] and the upper bound 3m − 1 of their progression for m ≤ 16; it does not decide their Question 1 (13 or at least 14).

↳ Used in: Corollary 2 (p. 7)

💡 In plain words (commentary, not part of the paper)

Take a biquadratic form in $x_1, \dots, x_m$ and $y_1, \dots, y_4$ that is a sum of squares of bilinear forms, however many. The theorem says that $3m - 1$ squares always suffice when $5 \le m \le 16$: 14 for $m = 5$, 47 for $m = 16$. The previous bound was $3m$, obtained from $\mathrm{BSR}(m, 3) = 2m$ by adding one column. For $m = 5$ this is the upper half of the conjecture $\mathrm{BSR}(5, 4) = 14$ of Qi and Ouyang; with the lower bound 13 it leaves $13 \le \mathrm{BSR}(5, 4) \le 14$, and the paper does not decide which.

The proof does not search for representations. If the sums of $N = 3m - 1$ squares were not the whole cone, a boundary point inside the cone would carry a Hankel multiplier H that kills every summand (so its rank is at most $4m - N = m + 1$), whose image contains no nonzero decomposable covector $\alpha \otimes \beta$, even over $\mathbb{C}$ (*product-free*), and whose negative index q is at most $\lfloor (m + 13)/6 \rfloor$ by the boundary inertia inequality (Lemma 17). The proof lists the possible triples of row support, column support and rank of H and excludes each one (Theorem 25). It stops at $m = 16$ because at $m = 17$

the inertia bound allows $q = 5$, and the low-negative-index lemma is proved only for $q \leq 4$.

The arithmetic, the enumeration of the cases and several linear-algebra steps are checked in Isabelle/HOL or by exact computation; in the paper's words, these checks cover individual steps, not the whole theorem.

Evidence & scope

- **Program:** the claims of the two language tabs (14 and 15 at m = 5, the lower bound 13, 47/48/43 at m = 16; 210 Gram entries, dim R_2 = 150 and a kernel of dimension 60 at m = 5), with controls that must fail. `lab/comprueba_pestanas.py`
- **Program:** Figures 1 and 2 recomputed against the paper's table and the output of its figure script (r_P, the gaps, the four numerically rank-four points of the 2×3 form), with controls that must fail. `lab/comprueba_lab.py`

Read this theorem in another mathematical language

Read this theorem in:

Commentary, not part of the paper. Take $m = 5$. Every biquadratic form in $x_1, \ldots, x_5, y_1, \ldots, y_4$ that is a sum of squares of bilinear forms is a sum of 14 of them. Before, adding one column to $\mathrm{BSR}(5, 3) = 10$ gave $5 + 10 = 15$; the lower bound is $z_2(5, 4) = 13$, so $\mathrm{BSR}(5, 4)$ is 13 or 14. For $m = 16$: 47 against 48 before, and the lower bound 43.

Commentary, not part of the paper. The same statement for matrices, $m = 5$: a form is $v^\top G v$ with $v = x \otimes y$ and G a symmetric 20×20 matrix (210 entries), and the coefficients are a linear image onto the 150-dimensional space R_2, with a kernel of dimension $210 - 150 = 60$. The positive semidefinite G with $v^\top G v = F$ form the Gram spectrahedron of F, and F is a sum of r squares exactly when it contains a matrix of rank at most r. The theorem says: the Gram spectrahedron of every 5×4 sum of squares contains a matrix of rank at most 14 (for $m \times 4$, $5 \leq m \leq 16$: at most $3m - 1$).

Corollary 2. p. 2

> *BSR_m4_Cor5.clr_bound, BSR_m4_Cor5.clr_bound_choose, BSR_m4_Cor5.add_column_arith, BSR_m4_Cor5.small_cases, BSR_m4_Cor5.cor_five* $\mathrm{BSR}(m, 5) \leq 4m - 1$ *for every* $m \geq 2$. *For* $5 \leq m \leq 10$ *this is one below the bound* $4m$ *of Choi, Lam and Reznick [22] , also in the strict form of Blekherman, Sinn, Smith and Velasco [9 , Theorem 2.1] .*

Isabelle · ours · part **show**

Only a part is in Isabelle: the arithmetic of the proof; the cited inputs (adding one column, the Choi–Lam–Reznick

bound, the values for m = 2, 3, 4) enter as hypotheses

`clr_bound` · Isabelle/BSRFourColumnsCor5/BSR_m4_Cor5.thy, line 21

```
lemma clr_bound:
  fixes m r :: nat
  assumes m: "12 \<le> m" and r: "r * (r + 1) \<le> 15 * m
* (m + 1)"
  shows "r \<le> 4 * m - 1"
```

`clr_bound_choose` · Isabelle/BSRFourColumnsCor5/BSR_m4_Cor5.thy, line 45

```
lemma clr_bound_choose:
  fixes m r :: nat
  assumes m: "12 \<le> m" and r: "(r + 1) choose 2 \<le> 15
* ((m + 1) choose 2)"
  shows "r \<le> 4 * m - 1"
```

`add_column_arith` · Isabelle/BSRFourColumnsCor5/BSR_m4_Cor5.thy, line 85

```
lemma add_column_arith:
  fixes m :: nat
  assumes "5 \<le> m" "m \<le> 16"
  shows "m + (3 * m - 1) = 4 * m - 1"
```

`small_cases` · Isabelle/BSRFourColumnsCor5/BSR_m4_Cor5.thy, line 91

```
lemma small_cases: "(6::nat) \<le> 4 * 2 - 1" "(10::nat) \
<le> 4 * 3 - 1" "(14::nat) \<le> 4 * 4 - 1"
```

`cor_five` · Isabelle/BSRFourColumnsCor5/BSR_m4_Cor5.thy, line 96

```
theorem cor_five:
  fixes bsr :: "nat \<Rightarrow> nat \<Rightarrow> nat"
  assumes add_col: "\<And>m. 5 \<le> m \<Longrightarrow> m
\<le> 16 \<Longrightarrow> bsr m 5 \<le> m + bsr m 4"
    and main: "\<And>m. 5 \<le> m \<Longrightarrow> m \<le>
16 \<Longrightarrow> bsr m 4 \<le> 3 * m - 1"
    and clr: "\<And>m. 12 \<le> m \<Longrightarrow> \
<exists>L. bsr m 5 \<le> L \<and> L * (L + 1) \<le> 15 * m
* (m + 1)"
    and v2: "bsr 2 5 \<le> 6" and v3: "bsr 3 5 \<le> 10"
and v4: "bsr 4 5 \<le> 14"
    and m2: "2 \<le> m"
  shows "bsr m 5 \<le> 4 * m - 1"
```

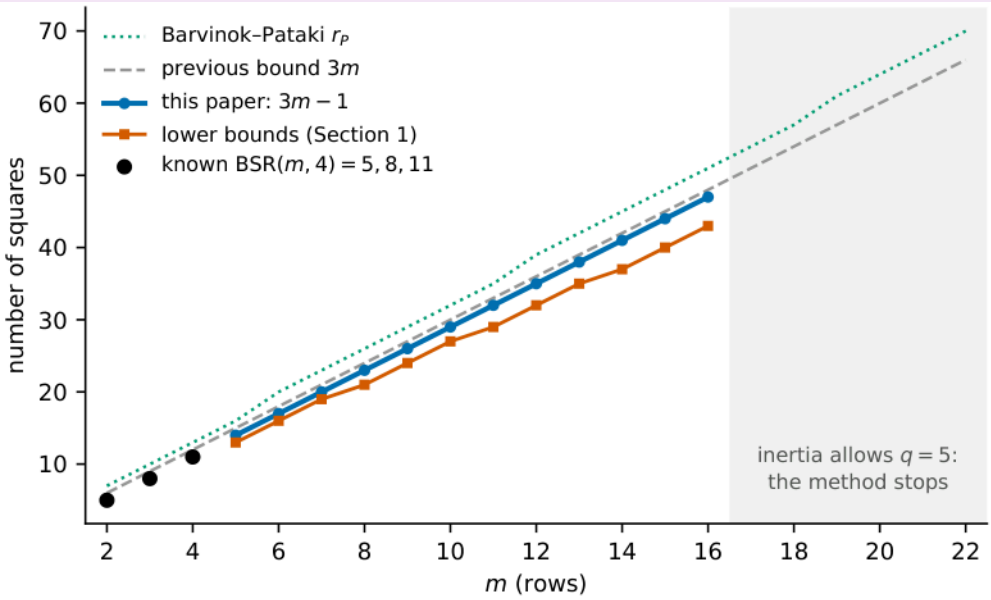
🆕 **Classical and new:** new for $5 \le m \le 10$, one below the Choi–Lam–Reznick bound 4m (also in the strict form of Blekherman–Sinn–Smith–Velasco); confirmed as new by L. Qi.

↳ Not used by another result of the paper.

Proof

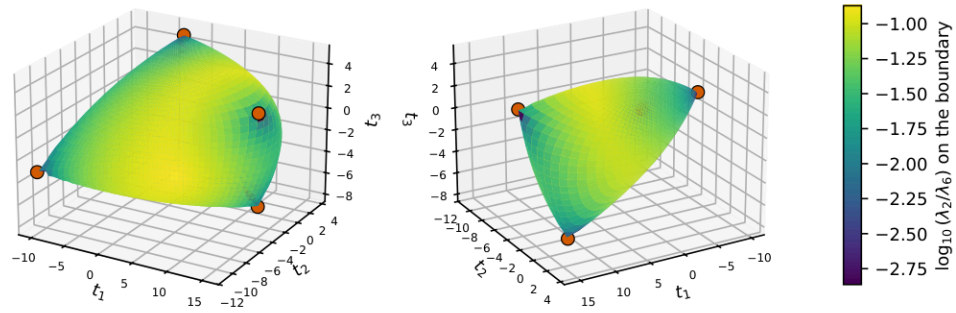
> For $5 \le m \le 16$, adding one column (see below) gives $\mathrm{BSR}(m,5) \le m + \mathrm{BSR}(m,4) \le m + 3m - 1$. For $m \ge 12$, the bound Λ of [22] (the largest r with $\binom{r+1}{2} \le \dim R_2 = 15\binom{m+1}{2}$) is at most $4m-1$, since $r = 4m$ would need $4m(4m+1) \le 15m(m+1)$, that is, $m \le 11$. For $m = 2, 3, 4$, symmetry gives $\mathrm{BSR}(m,5) = \mathrm{BSR}(5,m)$, which is 6, 10 and at most 14 by [24] , [3] and Theorem 1.

Figure 1



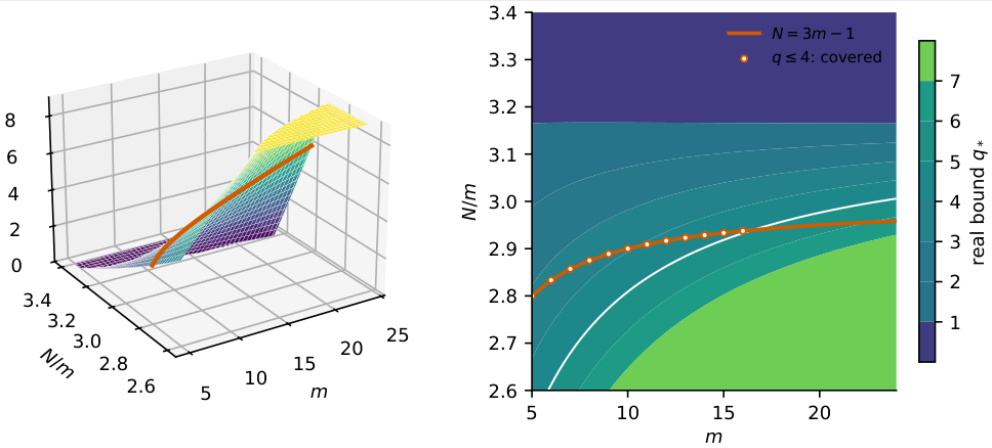
Bounds on $\mathrm{BSR}(m,4)$. Blue: Theorem 1; grey dashed: the previous bound $3m$; vermilion: the lower bounds of the table; black: the known values $5, 8, 11$ for $m = 2, 3, 4$, which the induction uses; green dotted: the Barvinok–Pataki rank r_P. In the shaded range the inertia bound allows $q = 5$ and the method stops. The gap to the lower bound is 1 for $m = 5, 6, 7$ and grows to 4 at $m = 16$.

Figure 2



The Gram spectrahedron [27] of a 2×3 biquadratic sum of squares $F = \sum_{k=1}^{6} \ell_k^2$ (integer coefficients, two views): the positive semidefinite 6×6 matrices G with $F = v^t G v$, an affine three-dimensional family (21 entries, 18 monomials). Representations of F by r squares are the points of rank r. On the boundary the colour is $\log_{10}(\lambda_2/\lambda_6)$. This is a numerical illustration in floating point: on all 9800 rays drawn the boundary point has one numerically zero eigenvalue (rank 5), and a local search finds four numerically rank-four points (vermilion), at the corners; four is $\mathrm{BSR}(2,3)$. The form, the Gram directions and the search are in the supplementary script `laboratorio.py`. In this language Theorem 1 says that the Gram spectrahedron of every $m \times 4$ sum of squares, $5 \le m \le 16$, contains a point of rank at most $3m - 1$.

Figure 3



The boundary inertia inequality for $m \times 4$ forms: the real bound $q_* = 1 + \left(5m(m+1) - \binom{N+2}{2}\right)/N$, so that the negative index (an integer) satisfies $q \le \lfloor q_* \rfloor$, as a function of m and N/m (left, surface; right, its level sets). Along $N = 3m - 1$ (vermilion) $q_* = 22/7 \approx 3.14$ at $m = 5$, $231/47 \approx 4.91$ at $m = 16$ and $127/25 = 5.08$ at $m = 17$, so $\lfloor q_* \rfloor \le 4$ exactly for $5 \le m \le 16$ (white dots), the range of Theorem 1; the white curve is $q_* = 5$.

Table 1: The formal components in Isabelle/HOL (six sessions, none with unproved steps) and the statements of this paper that use them.

Session	Main statements
BoundaryInertia	inertia_pataki_form, inertia_m4, inertia_m4_floor, q_values; incidence, compression_count, psi_counts
BSRFourColumnsPieces	support_triples_m4, regular_point_m4, cross_product_identity
BSRFourColumnsLinear	rank5_polar_kernel, singular_2x2_space, symmetric_trilinear_polarizatio
BSRFourColumnsLinear2	line_decomposable, rank_one_sum_minor, common_line_step_kernel, common_line_step_image, pencil_core
BSRFourColumnsLinear3	symmetric_4linear_polarization, symmetric_5linear_polarization; positive_averaging_nonneg, diagonal_terms_vanish; rank_le_one_space_full, support_count_inequality
BSRFourColumnsSmallRows	row_rank_one_simple, evaluation_step, row_in_line, small_rows_contradiction

2 The boundary method at $m = 5$

Lemma 3. p. 9

> *BSR_m4_Linear2.line_decomposable, BSR_m4_Linear2.line_decomposable_nonzero If $x \neq 0$ and $T_c x = \beta(c) e$ for all c, with $\beta \neq 0$, $e \neq 0$, then $\mathrm{im}(T^* J)$ contains a nonzero decomposable covector.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the factorization (the paper lists «the factorization in Lemma 2» among the formal components)

`line_decomposable` · Isabelle/BSRFourColumnsLinear2/BSR_m4_Linear2.thy, line 27

```
theorem line_decomposable:
  fixes J :: "'e::ab_group_add \<Rightarrow> 'e \
<Rightarrow> 'k::field"
    and sm :: "'k \<Rightarrow> 'e \<Rightarrow> 'e"
    and T :: "'c \<Rightarrow> 'a \<Rightarrow> 'e" and
beta :: "'c \<Rightarrow> 'k"
  assumes Jsm: "\<And>a u w. J (sm a u) w = a * J u w"
    and sm1: "\<And>u. sm 1 u = u"
    and flat: "\<And>b c x y. J (T b x) (T c y) = J (T c x)
(T b y)"
    and Tx: "\<And>c. T c x = sm (beta c) e"
    and bd: "beta d = 1"
  shows "J e (T c y) = beta c * J e (T d y)"
```

`line_decomposable_nonzero` · Isabelle/BSRFourColumnsLinear2/BSR_m4_Linear2.thy, line 49

```
corollary line_decomposable_nonzero:
  fixes J :: "'e::ab_group_add \<Rightarrow> 'e \
<Rightarrow> 'k::field"
    and sm :: "'k \<Rightarrow> 'e \<Rightarrow> 'e"
    and T :: "'c \<Rightarrow> 'a \<Rightarrow> 'e" and
beta :: "'c \<Rightarrow> 'k"
  assumes "\<And>a u w. J (sm a u) w = a * J u w" "\<And>u.
sm 1 u = u"
    and "\<And>b c x y. J (T b x) (T c y) = J (T c x) (T b
y)"
    and "\<And>c. T c x = sm (beta c) e" "beta d = 1" and
"J e (T d y0) \<noteq> 0"
  shows "\<exists>phi. (\<forall>c y. J e (T c y) = beta c
* phi y) \<and> phi y0 \<noteq> 0"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box).
Isabelle: 2 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the factorization (the paper lists «the factorization in Lemma 2» among the formal components)

↳ Used in: Lemma 6, Lemma 26

Proof

> Choose d with $\beta(d) = 1$. Equation (F) gives $\langle e, T_c y\rangle = \beta(c)\langle e, T_d y\rangle$. This is decomposable and nonzero, because T is onto.

Lemma 4. p. 10

> *BSR_m4_Linear2.binform_dehomog, BSR_m4_Linear2.pencil_core Let $G_1, G_2 : B \to S^*$ have the*

property that every nonzero linear combination G_λ is onto. If $F : B \to S$ and F^*G_1, F^*G_2 are symmetric, then $F = 0$.

Isabelle · ours · part **show**

Only a part is in Isabelle: the last step: a binary form of positive degree has a zero (the paper, «What is checked by machine»)

`binform_dehomog` · `Isabelle/BSRFourColumnsLinear2/BSR_m4_Linear2.thy`, line 193

```
lemma binform_dehomog:
  "binform a q s 1 = poly (\<Sum>i\<le>q. monom (a i) i) s"
```

`pencil_core` · `Isabelle/BSRFourColumnsLinear2/BSR_m4_Linear2.thy`, line 197

```
theorem pencil_core:
  assumes q: "q > 0"
  shows "\<exists>s t. (s \<noteq> 0 \<or> t \<noteq> 0) \
<and> binform a q s t = 0"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box).

Isabelle: 2 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the last step: a binary form of positive degree has a zero (the paper, «What is checked by machine»)

↳ Used in: Lemma 5

Proof

If $k = \operatorname{rank} F > 0$, then F^*G_λ has rank k. Symmetry gives $F^*G_\lambda = G_\lambda^* F$, so its kernel equals $\ker F$. It induces a nondegenerate symmetric form on $B/\ker F$ for every $[\lambda] \in \mathbb{P}^1$. Its determinant is a homogeneous polynomial of positive degree k having no zero on $\mathbb{P}^1$, impossible over $\mathbb{C}$.

Lemma 5. (Isotropic row lemma) p. 10

BSR_m4_Pieces.cross_product_identity_real, BSR_m4_Pieces.isotropic_C4, BSR_m4_Pieces.isotropic_C4_independent Assume T is onto, the first radical is zero, and its dual image is product-free. There is no nonzero $D \subset A$ for which $S = T(D \otimes B)$ is totally isotropic and $\dim(A/D) \le 2$.

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces; the remaining arguments of the lemma are proved in the text only (the paper, «What is checked by machine»)

`cross_product_identity_real` · `Isabelle/BSRFourColumnsPieces/BSR_m4_Pieces.thy`, line 496

```
corollary cross_product_identity_real:
  fixes G :: "nat \<Rightarrow> nat \<Rightarrow> real" and
d :: "nat \<Rightarrow> real"
  assumes "i < 3" "j < 3"
  shows "(\<Sum>k<3. crossm d i k * G k j) + (\<Sum>k<3. G
k i * crossm d k j) = crossm (trmul G d) i j"
```

`isotropic_C4` · `Isabelle/BSRFourColumnsPieces/BSR_m4_Pieces.thy`, line 517

```
lemma isotropic_C4:
  "bf4 (\<lambda>k. s * v1 k + t * v2 k) (\<lambda>k. u *
v1 k + w * v2 k) = 0"
```

`isotropic_C4_independent` · `Isabelle/BSRFourColumnsPieces/BSR_m4_Pieces.thy`, line 521

```
lemma isotropic_C4_independent:
  assumes "\<forall>k. s * v1 k + t * v2 k = 0"
  shows "s = 0 \<and> t = 0"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box).
Isabelle: 3 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces; the remaining arguments of the lemma are proved in the text only (the paper, «What is checked by machine»)

↳ Used in: Theorem 25

Proof

The injective map $z \mapsto T^*Jz$, for $z \in S$, takes values in $(A/D)^* \otimes B^*$. If the quotient has dimension zero, then $S = 0$; if it has dimension one, every nonzero image is decomposable. Both possibilities are excluded. For quotient dimension two choose lifts a_1, a_2, and define

$$G_i(b)(z) = \langle z, T(a_i \otimes b)\rangle.$$

If G_λ were not onto, a nonzero $z \in S$ annihilating its image would make the two rows of T^*Jz dependent. This would give a nonzero decomposable covector. Hence every G_λ is onto. For $d \in D$, put $F_d(b) = T(d \otimes b) \in S$. Equation (F) says $F_d^* G_i$ is

symmetric. Lemma 4 gives $F_d = 0$ for every d, contradicting the zero first radical.

Lemma 6. (Nonsingular slice) p. 10

BSR_m4_Nonsingular.jacobi_gives_5, BSR_m4_Nonsingular.five_gives_K, BSR_m4_Nonsingular.bordered_commute, BSR_m4_Nonsingular.nonsingular_slice_core Let $\dim A = d \geq 3$, $\dim B = 4$. *A rank-*$(d+1)$ *multiplier with supports* $(d, 4)$ *and an invertible* $Q(b)$ *cannot have product-free image.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the algebraic core (Jacobi turns (4) into (5), the form of K, the bordered maps commute); the reduction to that core is in the text

jacobi_gives_5 · Isabelle/BSRFourColumnsNonsingular/BSR_m4_Nonsingular.thy, line 167

```
theorem jacobi_gives_5:
  fixes A1 A2 A3 :: "'a::comm_ring_1^'n^'n" and u1 u2 u3 ::
"'a^'n"
  assumes s1: "transpose A1 = A1" and s2: "transpose A2 =
A2" and s3: "transpose A3 = A3"
    and c12: "comm A1 A2 = outer u2 u1 - outer u1 u2"
    and c23: "comm A2 A3 = outer u3 u2 - outer u2 u3"
    and c31: "comm A3 A1 = outer u1 u3 - outer u3 u1"
  shows "symp (A1 *v u2 - A2 *v u1) u3 + symp (A2 *v u3 -
A3 *v u2) u1 + symp (A3 *v u1 - A1 *v u3) u2 = 0"
```

five_gives_K · Isabelle/BSRFourColumnsNonsingular/BSR_m4_Nonsingular.thy, line 272

```
theorem five_gives_K:
  fixes K12 K23 K31 u1 u2 u3 :: "'a::field^'n"
  assumes two: "(2::'a) \<noteq> 0"
    and ind: "\<forall>a b c. a *s u1 + b *s u2 + c *s u3 =
0 \<longrightarrow> a = 0 \<and> b = 0 \<and> c = 0"
    and five: "symp K12 u3 + symp K23 u1 + symp K31 u2 = 0"
  shows "\<exists>t1 t2 t3. K12 = t1 *s u2 - t2 *s u1 \
<and> K23 = t2 *s u3 - t3 *s u2 \<and> K31 = t3 *s u1 - t1
*s u3"
```

bordered_commute · Isabelle/BSRFourColumnsNonsingular/BSR_m4_Nonsingular.thy, line 396

```
theorem bordered_commute:
  fixes Ai Aj :: "'a::comm_ring_1^'n^'n" and ui uj ::
"'a^'n" and ti tj :: 'a
  assumes si: "transpose Ai = Ai" and sj: "transpose Aj =
Aj"
    and cij: "comm Ai Aj = outer uj ui - outer ui uj"
    and kij: "Ai *v uj - Aj *v ui = ti *s uj - tj *s ui"
  shows "bord Ai ui ti \<circ> bord Aj uj tj = bord Aj uj
tj \<circ> bord Ai ui ti"
```

nonsingular_slice_core · Isabelle/BSRFourColumnsNonsingular/BSR_m4_Nonsingular.thy, line 429

```
theorem nonsingular_slice_core:
  fixes A1 A2 A3 :: "'a::field^'n^'n" and u1 u2 u3 ::
"'a^'n"
  assumes two: "(2::'a) \<noteq> 0"
    and s1: "transpose A1 = A1" and s2: "transpose A2 = A2"
and s3: "transpose A3 = A3"
    and c12: "comm A1 A2 = outer u2 u1 - outer u1 u2"
    and c23: "comm A2 A3 = outer u3 u2 - outer u2 u3"
    and c31: "comm A3 A1 = outer u1 u3 - outer u3 u1"
    and ind: "\<forall>a b c. a *s u1 + b *s u2 + c *s u3 =
0 \<longrightarrow> a = 0 \<and> b = 0 \<and> c = 0"
  shows "\<exists>t1 t2 t3. bord A1 u1 t1 \<circ> bord A2
u2 t2 = bord A2 u2 t2 \<circ> bord A1 u1 t1
                  \<and> bord A2 u2 t2 \<circ> bord A3 u3
t3 = bord A3 u3 t3 \<circ> bord A2 u2 t2
                  \<and> bord A3 u3 t3 \<circ> bord A1 u1
t1 = bord A1 u1 t1 \<circ> bord A3 u3 t3"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box).
Isabelle: 4 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the algebraic core (Jacobi turns (4) into (5), the form of K, the bordered maps commute); the reduction to that core is in the text

↳ Used in: Lemma 20 (p. 42), Theorem 25, Lemma 27

Proof

Take $E = \mathbb{C}^d \perp \mathbb{C}$, $J = I$, and $T_b x = (x, 0)$. For a complementary basis c_1, c_2, c_3 write

$$T_i x = (A_i x, u_i^t x).$$

Flatness gives the following exact identities, denoted computation C1:

$$A_i^t = A_i, \quad [A_i, A_j] = u_j u_i^t - u_i u_j^t. \tag{4}$$

Put $K_{ij} = A_i u_j - A_j u_i$, $v \odot w = vw^t + wv^t$. Substituting (4) into Jacobi's identity gives

$$K_{12} \odot u_3 + K_{23} \odot u_1 + K_{31} \odot u_2 = 0. \tag{5}$$

If the u_i span zero dimensions, T is not onto. If they span a line, the last coordinate of T is a nonzero decomposable covector.

If their span has dimension two, change the complementary basis so that $u_3 = 0$ and u_1, u_2 are independent. Equation (4)

makes A_3 commute with A_1, A_2. Comparing coefficients of (5) in a basis beginning with u_1, u_2 gives $A_3u_1 = tu_1,\ A_3u_2 = tu_2$ for some t. Put $P = \mathrm{im}(A_3 - tI)$. This space is invariant under every A_i and is orthogonal to u_1, u_2. It is nonzero: otherwise $T_{c_3 - tb} = 0$, contradicting full second support. Equation (4) shows that the restricted operators commute on P. A common eigenvector $x \in P \setminus \{0\}$ then satisfies $T_c x \in \mathbb{C}(x, 0)$ for every c, contrary to Lemma 3.

If u_1, u_2, u_3 are independent, equation (5) implies

$$K_{ij} = t_i u_j - t_j u_i$$

for scalars t_i. To see this, complete the three u_i to a basis. Coefficients involving the additional basis vectors first remove those components of K_{ij}. Coefficients of u_i^2 and $u_i u_j$ give the displayed expression. Consequently

$$\widehat{A}_i = \begin{pmatrix} A_i & u_i \\ u_i^t & t_i \end{pmatrix}$$

are commuting symmetric matrices, by (4). Choose a common eigenvector $z \neq 0$, with eigenvalues λ_i. Since $T_i = \widehat{A}_i T_b$, $\langle z, T_i x \rangle = \lambda_i \langle z, T_b x \rangle$. Thus $T^* J z$ is decomposable; it is nonzero because T is onto. This is the final contradiction.

Lemma 7. (Rank-one factorization) p. 11

BSR_m4_Linear3.rank_le_one_space_full, BSR_m4_Linear3.two_rank_one_contra, BSR_m4_Linear2.rank_one_pair_not_in_rank_le_one_space If the symmetric polynomial matrix $Q(b)$ has homogeneous quadratic entries and generic rank one, then

$$x^t Q(b) x = g(b) L(x, b)^2.$$

Here either L is independent of b, and the first support is one, or g is constant and L is bilinear.

Isabelle · ours · part **show**

Only a part is in Isabelle: the classification of linear spaces of matrices of rank at most one; the remaining arguments are proved in the text only

`rank_le_one_space_full` · `Isabelle/BSRFourColumnsLinear3/BSR_m4_Linear3.thy`, line 428

```
theorem rank_le_one_space_full:
  fixes S :: "(nat \<Rightarrow> nat \<Rightarrow>
'a::field) set"
  assumes n: "0 < n" and k: "0 < k"
    and add: "\<And>A B. A \<in> S \<Longrightarrow> B \
<in> S \<Longrightarrow> (\<lambda>i l. A i l + B i l) \
<in> S"
    and mz: "\<And>M. M \<in> S \<Longrightarrow>
minors_zero M n k"
  shows "(\<exists>u. (\<exists>i<n. u i \<noteq> 0) \<and>
(\<forall>M\<in>S. \<exists>\<phi>. \<forall>i<n. \
<forall>l<k. M i l = u i * \<phi> l))
       \<or> (\<exists>a. (\<exists>l<k. a l \<noteq> 0) \
<and> (\<forall>M\<in>S. \<exists>\<psi>. \<forall>i<n. \
<forall>l<k. M i l = \<psi> i * a l))"
```

two_rank_one_contra · Isabelle/BSRFourColumnsLinear3/BSR_m4_Linear3.thy, line 389

```
lemma two_rank_one_contra:
  fixes M N :: "nat \<Rightarrow> nat \<Rightarrow>
'a::field" and u v :: "nat \<Rightarrow> 'a" and a b ::
"nat \<Rightarrow> 'a"
  assumes add: "\<And>A B. A \<in> S \<Longrightarrow> B \
<in> S \<Longrightarrow> (\<lambda>i l. A i l + B i l) \
<in> S"
    and mz: "\<And>A. A \<in> S \<Longrightarrow>
minors_zero A n k"
    and MS: "M \<in> S" and NS: "N \<in> S"
    and rM: "\<And>i l. i < n \<Longrightarrow> l < k \
<Longrightarrow> M i l = u i * a l"
    and rN: "\<And>i l. i < n \<Longrightarrow> l < k \
<Longrightarrow> N i l = v i * b l"
    and indu: "\<not> dep n u v" and indb: "\<not> dep k a
b"
  shows False
```

rank_one_pair_not_in_rank_le_one_space · Isabelle/BSRFourColumnsLinear2/BSR_m4_Linear2.thy, line 104

```
corollary rank_one_pair_not_in_rank_le_one_space:
  fixes u v :: "nat \<Rightarrow> 'a::idom" and a b :: "nat
\<Rightarrow> 'a"
  assumes "\<exists>i<n. \<exists>j<n. u i * v j \<noteq> u
j * v i" "\<exists>l<k. \<exists>m<k. a l * b m \<noteq> a
m * b l"
    and add: "\<And>M N. M \<in> S \<Longrightarrow> N \
<in> S \<Longrightarrow> (\<lambda>i l. M i l + N i l) \
<in> S"
    and "(\<lambda>i l. u i * a l) \<in> S" "(\<lambda>i l.
v i * b l) \<in> S"
    and "\<And>M. M \<in> S \<Longrightarrow> minors_zero M
n k"
  shows False
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box).
Isabelle: 3 results, no oracles

↳ Used in: Theorem 25, Lemma 26, Lemma 27

Proof

Over the fraction field $\mathbb{C}(b)$, a symmetric rank-one matrix gives a scalar times the square of a linear form in x. Clear denominators and make that linear form primitive over the UFD $\mathbb{C}[b]$. Gauss's lemma makes the scalar a polynomial. Bihomogeneity permits L to be chosen homogeneous in b; if its degree is e, then $\deg g + 2e = 2$, so $e = 0$ or 1. If $e = 0$, write $L(x) = \lambda^t x$. Separate polarization of $q(x, b) = x^t Q(b) x = g(b)(\lambda^t x)^2$ in both factors gives

$$H(x \otimes b,\, y \otimes c) = G(b, c)\,(\lambda^t x)(\lambda^t y),$$

with G the polar form of g; hence every x with $\lambda^t x = 0$ lies in the first radical, and the first support is at most one.

Lemma 8. (Regular fourteen-tuple) p. 11

BSR_m4_Pieces.regular_point_m4 The differential $d\Phi_{14}$ is onto at

$$S = \{x_i y_i : 1 \le i \le 4\} \cup \{x_i y_j + x_j y_i : 1 \le i < j \le 4\} \cup \{x_5 y_j :$$

Isabelle · ours · part **show**

Only a part is in Isabelle: the regular tuple; the paper also confirms it by an exact rank computation (C6)

`regular_point_m4` · `Isabelle/BSRFourColumnsPieces/BSR_m4_Pieces.thy`, line 381

```
theorem regular_point_m4:
  fixes H :: "nat \<Rightarrow> nat \<Rightarrow> nat \
<Rightarrow> nat \<Rightarrow> real" and n :: nat
  assumes sym: "\<And>a b c d. a < 5 + n \<Longrightarrow>
b < 4 \<Longrightarrow> c < 5 + n \<Longrightarrow> d < 4 \
<Longrightarrow> H a b c d = H c d a b"
    and hankel: "\<And>a b c d. a < 5 + n \<Longrightarrow>
b < 4 \<Longrightarrow> c < 5 + n \<Longrightarrow> d < 4 \
<Longrightarrow> H a b c d = H a d c b"
    and ann: "\<forall>h \<in> S n. \<forall>c < 5 + n. \
<forall>d < 4. av (5 + n) H h c d = 0"
  shows "\<forall>a < 5 + n. \<forall>b < 4. \<forall>c < 5
+ n. \<forall>d < 4. H a b c d = 0"
```

↳ Used in: Proposition 10

Proof

Suppose a Hankel multiplier kills every element of S. The last four elements make x_5 a first-radical vector. The first ten span

the symmetric tensors in the remaining identified 4×4 factors. For any x, y in these factors,

$$H(x \otimes y, x \otimes y) = -H(x \otimes y, y \otimes x) = -H(x \otimes x, y \otimes y) =$$

The first equality uses a symmetric tensor in the kernel; the second uses (H); the last uses $x \otimes x$ in the kernel. Separate polarization gives $H = 0$. Thus the annihilator of $S \cdot V$ is zero and $S \cdot V = R_2$, proving surjectivity. This explicit identity, not a numerical rank test, is computation C6.

Lemma 9. (Boundary selection) p. 11

Let $\Omega \subset \mathbb{R}^d$ be a connected open semialgebraic set, let $K \subset \mathbb{R}^d$ be a closed semialgebraic set with $\Omega \cap \operatorname{int} K \neq \varnothing$ and $\Omega \not\subset K$, and let $Z \subset \mathbb{R}^d$ be semialgebraic with $\dim Z \leq d-2$. Then there are a point $F \in \Omega \cap \partial K$ with $F \notin Z$, an open ball $U \ni F$ and a C^∞ function g on U with $g(F) = 0$, $dg_F \neq 0$ and $K \cap U = \{g \geq 0\}$.

↳ Used in: Proposition 10, Theorem 22 (p. 49)

Proof

Put $K' = \overline{\operatorname{int} K} \subset K$. Then $\Omega \cap \operatorname{int} K' \neq \varnothing$, and $\Omega \not\subset K'$ because $K' \subset K$. A semialgebraic set $E \subset \Omega$ with $\dim E \leq d-2$ does not disconnect Ω: two points of $\Omega \setminus E$ are joined by a polygonal path in Ω, and the pairs of points whose segment meets E form a set of dimension at most $(d-2)+1+d<2d$, so the vertices can be moved off it. Since $\Omega \cap \operatorname{int} K'$ and $\Omega \setminus K'$ are disjoint, open and nonempty, the frontier $\Omega \cap \partial K'$ therefore has dimension $d-1$. Take a C^∞ semialgebraic Whitney stratification of $\mathbb{R}^d$ compatible with K, K' and Z; it satisfies the frontier condition. Let S be a stratum of dimension $d-1$ in $\Omega \cap \partial K'$. The closure of another stratum meets S only if it has dimension d, so the union Y of the closures of the strata of dimension at most $d-1$ other than S, together with the closure $\overline{Z}$ (semialgebraic, with $\dim \overline{Z} = \dim Z \leq d-2$), is closed and meets S in a set of dimension at most $d-2$. Choose $F \in S \setminus Y$ and a ball $U \ni F$ disjoint from Y in which S is a connected hypersurface $\{g = 0\}$ with $dg \neq 0$. Each of the two components of $U \setminus S$ is covered by open strata of dimension d, hence lies in one of them, and so lies either in $\operatorname{int} K$ or outside K. If both lay in K, F would be interior to K'; if both lay outside K, then

$\operatorname{int} K' \cap U \subset S$ would be empty, contradicting $F \in \overline{\operatorname{int} K'}$. So one side lies in $\operatorname{int} K$ and the other outside K, and after changing the sign of g, $K \cap U = \{g \geq 0\}$ ($S \subset K$ since K is closed). Finally $F \in \partial K$, and $F \notin Z$.

Proposition 10. (Boundary package) p. 12

Boundary_Inertia_Arith.incidence If $\Sigma_{14} \neq \Sigma$, there exist $F \in \operatorname{int} \Sigma \cap \partial \Sigma_{14}$ and a nonzero indefinite real Hankel multiplier H with the following properties. The real SOS length of F is fourteen. For every representation $F = \sum_{i=1}^{14} h_i^2$, the forms h_i are independent, their span W lies in $\ker H$, and the complex annihilator of W contains no nonzero simple tensor. In particular $r = \operatorname{rank} H \leq 6$ and its image is product-free. For every syzygy $\sum_i h_i D_i = 0$,

$$\sum_i H(D_i, D_i) \geq 0. \tag{11}$$

Isabelle · ours · part **show**

Only a part is in Isabelle: the incidence count of the m×4 form (M − 2(m+3) + (2m+4) = M − 2); the rest is a written proof

`incidence` · `Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy`, line 82

```
lemma incidence: fixes M m :: int shows "M - 2*(m+3) + 2*
((m-1)+3) = M - 2" by simp
lemma compression_count: fixes a m :: int shows "(4*a -
(a+1)) + (3*(m-a) - 1) = 3*m - 2" by simp
lemma psi_counts: fixes m :: int shows "4*(m+1) - (m-1) =
3*m + 5" "4*m - (m-1) = 3*m + 1" by simp_all
```

🆕 **Classical and new:** the paper: the existence of a suitable boundary point follows the stratification argument of [BDS, §2] and [Mar3, Proposition 3]; here it is proved in full through Lemma 12.

↳ Used in: Theorem 1 (p. 4), Theorem 22 (p. 49), Proposition 23 (p. 55)

Proof

Each Σ_k is semialgebraic, as a polynomial image, and is closed. To prove closedness, let $\tau(F)$ be the sum of the coefficients of all $x_i^2 y_j^2$. If $h_\nu = \sum_{ij} c_{\nu ij} x_i y_j$, then

$$\tau\left(\sum_{\nu=1}^{k} h_\nu^2\right) = \sum_{\nu,i,j} c_{\nu ij}^2 .$$

For a convergent sequence of outputs the representing tuples are bounded, and a convergent subsequence proves closedness. A PSD Gram on V has rank at most twenty, so $\Sigma = \Sigma_{20}$. Its interior is nonempty, since the positive definite Grams form an open set and the linear multiplication map onto R_2 is surjective. Lemma 8 gives Σ_{14} nonempty interior.

Put $\Omega = \operatorname{int} \Sigma$. It is open and convex. If Σ_{14} is proper in Σ, its complement in Ω is nonempty; otherwise closedness and $\overline{\Omega} = \Sigma$ would give equality. Lemma 9, applied to Ω, $K = \Sigma_{14}$ and the singular incidence below as Z, gives a point of $\partial \Sigma_{14} \cap \Omega$ off Z with a local defining function g, $\Sigma_{14} = \{ g \geq 0 \}$ near it.

Consider a complex point of $\mathbb{P}^4 \times \mathbb{P}^3$ with neither projective factor real. It and its conjugate are distinct in both factors. Complex changes of coordinates put them at (e_1, e_1), (e_2, e_2). Each first jet of a $(2,2)$-form has $1 + 4 + 3 = 8$ coefficients. At the first point they are coefficients of

$$x_1^2 y_1^2, \quad x_1 x_i y_1^2 \ (i \neq 1), \quad x_1^2 y_1 y_j \ (j \neq 1).$$

The eight coefficients at the second point are disjoint from these. Therefore the joint complex jet map has rank sixteen. Conjugation exchanges the two jets, so imposing them on real forms has real codimension sixteen. The point moves in real dimension $2(4+3) = 14$. Written in real and imaginary coordinates, the singular incidence, including the conditions that neither factor is real, is semialgebraic, so its projection to forms is semialgebraic by Tarski–Seidenberg. The singular incidence, and hence its projection to forms, has real dimension at most

$$150 - 16 + 14 = 148. \tag{12}$$

These jet coordinates and the count are computation C7. A singular point with a real factor is impossible for an interior SOS form: fixing that nonzero real factor gives a positive definite real quadratic form in the other factor, whose complex gradient has no nonzero kernel. This incidence is the set Z used above, and F is the point given by Lemma 9.

Let g be the function of Lemma 9, nonnegative on Σ_{14} near F, and $\ell = dg_F \neq 0$. For every tuple h representing F, the function $g \circ \Phi_{14}$ has a local minimum at h. Its first derivative gives $H(h_i, v) = 0$ for every i, v, where $H(u, v) = \ell(uv)$. For a syzygy D, $d\Phi_{14}(D) = 0$; hence its second derivative gives (11). This uses the same g, ℓ, H at every representing tuple.

We have $\ell(F) = 0$. Since $F \in \operatorname{int} \Sigma$, the nonzero functional ℓ can be neither nonnegative nor nonpositive on Σ. Equivalently, H is indefinite. If F had at most thirteen squares, pad that representation with a zero slot. Applying (11) to a variation only in that slot would give $H(v, v) \geq 0$ for all v, a contradiction. Thus its length is fourteen. A linear dependence among fourteen summands could be removed by an orthogonal change of summand coordinates, producing a zero slot. The summands are therefore independent, and $W \subset \ker H$ gives $r \leq 20 - 14 = 6$. Finally, a simple covector in $W^{\perp}$ gives a Segre point where every h_i vanishes; $F = \sum h_i^2$ and all its first derivatives vanish there. This contradicts the choice of F. Since $\operatorname{im} H \subset W^{\perp}$, its image is product-free.

Lemma 11. (Positive annihilator; $n \geq 3$) p. 14

BSR_m4_Linear3.positive_averaging_nonneg, BSR_m4_Linear3.diagonal_terms_vanish, Boundary_Inertia_Arith.psi_counts, BSR_m4_C8.wedge_pair_nonneg, BSR_m4_C8.S_nonneg, BSR_m4_C8.S_zero_radical, BSR_m4_C8.inertia_subtraction *Suppose a radical slice $x_0 \otimes L$, with x_0 in the first radical, $L \subseteq B$ and $\dim L = n \geq 3$, lies in a boundary span W. Let A' be a complement of the first radical, of dimension a, and restrict H to $A' \otimes L$. Assume that $T : A' \otimes L \to E$ is onto, that the restriction has zero first radical in A', and that it has full second support on L. Choose orthogonal summand coordinates so that $h_i = x_0 \otimes y_i$, $1 \leq i \leq n$, where y_i is a basis of L. Put $T_i a = T(a \otimes y_i)$, $H_{ij} = T_i^* J T_j$, and $S = \sum_i H_{ii}$. For the alternating map*

$$\Psi : A' \otimes \Lambda^2 L \longrightarrow E^n, \qquad \Psi(z)_i = \sum_j T_j z_{ij}, \quad z_{ij} = -z_{ji},$$

if p is the positive index of J, then

$$S \succ 0, \qquad \operatorname{rank} \Psi \leq np - a. \tag{13}$$

isabelle · ours · part **show**

Only a part is in Isabelle: the averaging step, the counts of Ψ and the two steps of computation C8 (single-wedge pairings give S ≥ 0 and the radical statement; inertia subtraction for a diagonal metric); the reduction of the metric to diagonal form and the rest are proved in the text

positive_averaging_nonneg · Isabelle/BSRFourColumnsLinear3/BSR_m4_Linear3.thy, line 301

```
theorem positive_averaging_nonneg:
  fixes h :: "nat \<Rightarrow> real"
  assumes n2: "2 \<le> n" and pairs: "\<And>i j. i < j \
<Longrightarrow> j < n \<Longrightarrow> 0 \<le> h i + h j"
  shows "0 \<le> (\<Sum>i<n. h i)"
```

diagonal_terms_vanish · Isabelle/BSRFourColumnsLinear3/BSR_m4_Linear3.thy, line 316

```
theorem diagonal_terms_vanish:
  fixes h :: "nat \<Rightarrow> 'k::field_char_0"
  assumes n3: "3 \<le> n" and pairs: "\<And>i j. i < n \
<Longrightarrow> j < n \<Longrightarrow> i \<noteq> j \
<Longrightarrow> h i + h j = 0" and i: "i < n"
  shows "h i = 0"
```

psi_counts · Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy, line 84

```
lemma psi_counts: fixes m :: int shows "4*(m+1) - (m-1) =
3*m + 5" "4*m - (m-1) = 3*m + 1" by simp_all
```

wedge_pair_nonneg · Isabelle/BSRFourColumnsC8/BSR_m4_C8.thy, line 189

```
theorem wedge_pair_nonneg:
  assumes "i \<noteq> j"
  shows "H i i a a + H j j a a \<ge> 0"
```

S_nonneg · Isabelle/BSRFourColumnsC8/BSR_m4_C8.thy, line 199

```
theorem S_nonneg:
  assumes "2 \<le> CARD('i)"
  shows "S a a \<ge> 0"
```

S_zero_radical · Isabelle/BSRFourColumnsC8/BSR_m4_C8.thy, line 218

```
theorem S_zero_radical:
  assumes n3: "3 \<le> CARD('i)" and S0: "S a a = 0"
  shows "H i j a b = 0"
```

inertia_subtraction · Isabelle/BSRFourColumnsC8/BSR_m4_C8.thy, line 274

```
theorem inertia_subtraction:
  fixes L M :: "((real^'p::finite) \<times>
(real^'q::finite)) set"
  assumes "subspace L" and "subspace M"
```

```
    and Lnn: "\<And>l. l \<in> L \<Longrightarrow> Jd l l \
<ge> 0"
    and Mpd: "\<And>m. m \<in> M \<Longrightarrow> m \
<noteq> 0 \<Longrightarrow> Jd m m > 0"
    and orth: "\<And>l m. l \<in> L \<Longrightarrow> m \
<in> M \<Longrightarrow> Jd l m = 0"
  shows "dim L + dim M \<le> CARD('p)"
```

↳ Used in: Theorem 1 (p. 4), Theorem 22 (p. 49)

Proof

The change of summand coordinates exists because the map $\mathbb{R}^{14} \to W$, sending coordinate vectors to the independent summands, is an isomorphism. Choose an orthonormal basis of the preimage of the fiber, then extend it orthogonally. The variations

$$D_i = \sum_j z_{ij} \otimes y_j \ (i \le n), \qquad D_i = 0 \ (i{>}n)$$

are syzygies by antisymmetry. Thus $J^{\oplus n}$ is nonnegative on $L_\Psi = \mathrm{im}\Psi$.

A single (i,j)-wedge with coefficient $a \in A'$ gives $H_{ii}(a,a) + H_{jj}(a,a) \ge 0$. Summing all pairs gives $(n-1)S(a,a) \ge 0$. If $S(a,a) = 0$, every one of these wedge images is null in L_Ψ, hence orthogonal to all of L_Ψ. Pairing its (i,j)-wedge with the (i,k)-wedge of any b, for three distinct indices, gives $H_{jk}(a,b) = 0$. Pairing equal wedges gives $H_{ii}(a,b) + H_{jj}(a,b) = 0$. As $n \ge 3$, all diagonal terms also vanish. Therefore a belongs to the first radical of the restriction. Effective support gives $a = 0$, proving $S \succ 0$.

For $a \in A'$, set $C_a = (T_i a)_i$. Flatness makes C_a orthogonal to L_Ψ, and

$$J^{\oplus n}(C_a, C_b) = S(a,b).$$

These natural annihilator vectors span an a-dimensional positive definite space M. The nondegenerate space $M^\perp$ has positive index $np - a$. It contains the nonnegative space L_Ψ, whose dimension cannot exceed that positive index. This proves (13). The single-wedge pairings and inertia subtraction are computation C8.

Lemma 12. (Annihilator and extension) p. 14

Boundary_Inertia_Arith.psi_counts The annihilator of $\mathrm{im}\Psi$*, with respect to* $J^{\oplus n}$*, consists exactly of the maps* $C : L \to E$ *satisfying*

$$\langle Cy_i, T(a \otimes y_j)\rangle = \langle Cy_j, T(a \otimes y_i)\rangle \quad \text{for every } a, i, j. \quad ($$

It contains the a*-dimensional space* $C_a = T(a \otimes \cdot)$*. Assume that* $L = B$*, or that* $B = L \oplus L'$ *with* L' *contained in the second radical of* H *(one of the two holds in every application below). Any additional independent map extends* T *by one first-factor row, preserving rank, flatness, product-freeness, and full second support, and increasing first support by one.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the counts of Ψ used with it; the lemma is proved in the text

`psi_counts` · `Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy`, line 84

```
lemma psi_counts: fixes m :: int shows "4*(m+1) - (m-1) =
3*m + 5" "4*m - (m-1) = 3*m + 1" by simp_all
```

↳ Used in: Theorem 1 (p. 4), Theorem 22 (p. 49)

Proof

Pair an arbitrary tuple $(Cy_i)_i$ with each single-wedge image. Its vanishing is exactly (14). The natural maps satisfy (14) by flatness, and they are independent by zero first radical. Define the new row of T' to be C. Equation (14) is the cross-flatness condition with each old row; the self-pairing condition holds because J is symmetric. The map remains onto the same E. A first-radical relation involving the new row would express C as a natural map, and an old second-radical relation remains excluded by the old full second support.

If $T'^{*}Jz$ were a nonzero simple covector, its restriction to the old tensor space $A' \otimes L$ would be either a simple covector or zero. A simple covector $\alpha \otimes \lambda$ on $A' \otimes L$ pulls back along the projection $A \otimes B \to A' \otimes L$ (kernel $x_0 \otimes B + A \otimes L'$, for a complement $B = L \oplus L'$ inside the second radical when $L \neq B$) to a simple covector of V^* lying in $\mathrm{im}H$; this is excluded because $\mathrm{im}H$ is product-free. In the latter case z is orthogonal to the old surjective image E, hence $z = 0$, also impossible. This argument uses one additional map; it requires no mutual compatibility between different additional maps.

Lemma 13. (Compression) p. 15

> *Boundary_Inertia_Arith.compression_count For the boundary form F, if H has first support a and rank r, then*
>
> $$\mathrm{SOS}_{\mathbb{R}}(F) \le (4a - r) + \mathrm{BSR}(5 - a, 4). \tag{15}$$

Isabelle · ours · part **show**

Only a part is in Isabelle: the m×4 count (4a − (a+1)) + (3(m − a) − 1) = 3m − 2; the lemma is proved in the text

`compression_count` · `Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy`, line 83

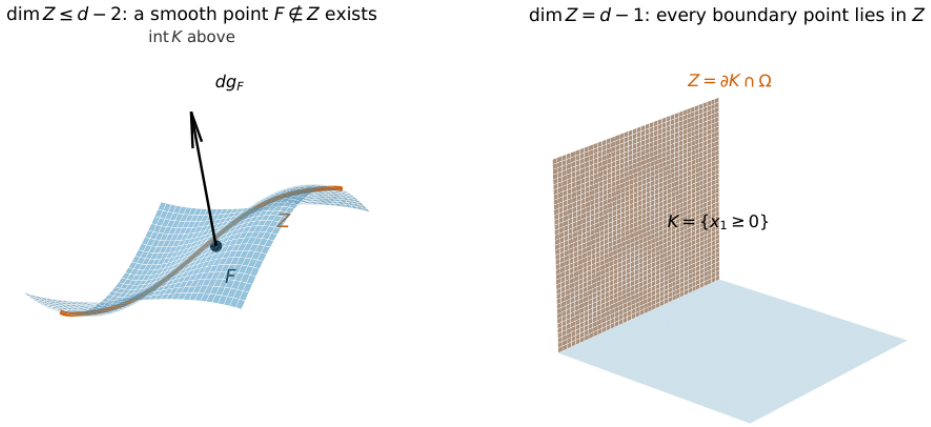
```
lemma compression_count: fixes a m :: int shows "(4*a -
(a+1)) + (3*(m-a) - 1) = 3*m - 2" by simp
lemma psi_counts: fixes m :: int shows "4*(m+1) - (m-1) =
3*m + 5" "4*m - (m-1) = 3*m + 1" by simp_all
```

↳ Used in: Theorem 1 (p. 4), Theorem 22 (p. 49)

Proof

> Choose $A = A' \oplus A_0$, where A_0 is the first radical. The projections to $A' \otimes B$ of the fourteen summands lie in the kernel of the restricted multiplier, of dimension $4a - r$. An orthogonal change of summand coordinates therefore makes at most $4a - r$ summands have nonzero A'-projection. All remaining summands lie in $A_0 \otimes B$. Replace their sum of squares by a representation using at most $\mathrm{BSR}(5 - a, 4)$ squares.

Figure 4



Lemma 9 in $d = 3$ (schematic). Left: $\dim Z \le d - 2$, a curve on the boundary; off it there is a smooth boundary point F with $dg_F \neq 0$ and K on one side. Right: $\dim Z = d - 1$ is not enough: with $K = \{x_1 \ge 0\}$ and $Z = \partial K \cap \Omega$ every boundary point lies in Z.

3 Larger m

Lemma 14. (Boundary inertia inequality) p. 15

Boundary_Inertia_Arith.inertia_pataki_form, Boundary_Inertia_Arith.inertia_q1_is_pataki, Boundary_Inertia_Arith.inertia_m4, Boundary_Inertia_Arith.inertia_m4_floor Let V be a finite-dimensional real vector space of forms, and let R_2 be the span of their products. Put

$$v = \dim V, \qquad M = \dim R_2, \qquad \Phi_N(h) = \sum_{i=1}^{N} h_i^2, \qquad \Sigma_N =$$

Let Σ be the full sum-of-squares cone.

Suppose $F \in \Sigma_N \cap \operatorname{int} \Sigma$ admits a C^2 local defining function g with

$$g(F) = 0, \qquad dg_F \neq 0, \qquad g \ge 0 \quad \text{on } \Sigma_N \text{ near } F.$$

Define

$$H(u,w) = dg_F(uw), \qquad H = T^{*}JT, \qquad T : V \to E \text{ onto},$$

Then

$$Nq \le M - 1 - \binom{N+1}{2}. \tag{I}$$

Isabelle · ours · part **show**

Only a part is in Isabelle: the arithmetic: the two forms of the inequality, its case q = 1, the m×4 form 2(3m−1)(q−1) ≤ m(m+7) and its equivalence with 6q ≤ m + 13 for m ≥ 5; the inequality itself is a written proof

`inertia_pataki_form` · `Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy`, line 19

```
lemma inertia_pataki_form:
  fixes N q M :: int
  shows "2*N*q \<le> 2*M - 2 - N*(N+1) \
<longleftrightarrow> 2*N*(q-1) + (N+2)*(N+1) \<le> 2*M"
```

`inertia_q1_is_pataki` · `Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy`, line 25

```
lemma inertia_q1_is_pataki:
  fixes N M :: int
  shows "2*N*(1-1) + (N+2)*(N+1) \<le> 2*M \
<longleftrightarrow> (N+2)*(N+1) \<le> 2*M"
```

`inertia_m4` · `Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy`, line 36

```
lemma inertia_m4:
  fixes m q :: int
  shows "2*(3*m-1)*(q-1) + (3*m+1)*(3*m) \<le> 2*(5*m*
(m+1)) \<longleftrightarrow> 2*(3*m-1)*(q-1) \<le> m*(m+7)"
```

`inertia_m4_floor` · `Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy`, line 70

```
lemma inertia_m4_floor:
  fixes m q :: int
  assumes "m \<ge> 5"
  shows "2*(3*m-1)*(q-1) \<le> m*(m+7) \
<longleftrightarrow> 6*q \<le> m + 13"
```

🆕 **Classical and new:** the paper, remark (i): «in substance, Theorem 3.4 of Boumal, Voroninski and Bandeira [BVB]», applied at a boundary point of Σ_N where the second-order conditions of [BSVZ] hold; the lemma gives

the self-contained proof needed, without a constant-rank assumption; its case q = 1 recovers the Barvinok–Pataki bound.

↳ Used in: Theorem 22 (p. 49)

Proof

For every representing tuple $F = \Phi_N(h)$, the function $g \circ \Phi_N$ has a local minimum at h. Its first derivative gives

$$H(h_i, u) = 0 \qquad (i = 1, \dots, N,\ u \in V).$$

Its second derivative along a syzygy $\sum_i h_i D_i = 0$ gives

$$\sum_i H(D_i, D_i) \geq 0. \tag{I1}$$

The Hessian term of g vanishes because $d\Phi_N(D) = 0$. The same g and multiplier apply at every representing tuple.

The multiplier is nonzero because products span R_2. Also $dg_F(F) = \sum_i H(h_i, h_i) = 0$. Since F is interior to Σ, a nonzero linear functional vanishing at F cannot be nonnegative or nonpositive on all of Σ. Thus H is indefinite.

If F had length below N, pad a shorter representation with a zero slot. In (I1), vary only that slot. This would force H to be positive semidefinite, a contradiction. Therefore F has minimum length N. Every representing N-tuple is linearly independent, since an orthogonal change of summands would otherwise produce a zero slot.

Put

$$W = \operatorname{span}\{h_1, \dots, h_N\}, \qquad K = \ker H.$$

Then $\dim W = N$ and $W \subset K$. Multiplication $\operatorname{Sym}^2 W \to R_2$ is injective: a nonzero symmetric relation matrix R would make $I + tR$ a Gram matrix for the same form. Moving to a singular positive semidefinite endpoint would express F with at most $N - 1$ squares.

Consider the entire syzygy space

$$\mathcal{Z} = \ker(W \otimes V \longrightarrow R_2).$$

The nonzero functional dg_F annihilates the multiplication image, so

$$\dim \mathcal{Z} \geq Nv - M + 1.$$

Project $\mathcal{Z}$ to $W \otimes E$ using $\mathrm{id}_W \otimes T$. The kernel is precisely the kernel of multiplication on $W \otimes K$. The multiplication image of $W \otimes K$ contains the independent products $\operatorname{Sym}^2 W$. Hence this projection kernel has dimension at most

$$N \dim K - \binom{N+1}{2}.$$

Writing $r = p + q$ and $\dim K = v - r$, the projected syzygy space consequently has dimension at least

$$Nv - M + 1 - N(v-r) + \binom{N+1}{2} = Nr + \binom{N+1}{2} - M$$

By (I1), this projected space is nonnegative for $J^{\oplus N}$. Since that form is nondegenerate and has positive index Np, the space has dimension at most Np. Therefore

$$Nr + \binom{N+1}{2} - M + 1 \le Np.$$

Substituting $r = p + q$ gives (I).

💡 **In plain words (commentary, not part of the paper)**

Take a point F of the sums of N squares that lies inside the cone and where they have a one-sided C^2 defining function g ($g(F) = 0$, $dg_F \neq 0$, $g \geq 0$ on Σ_N near F): a boundary point. The differential gives a symmetric form $H(u, w) = dg_F(uw)$ on the forms. The lemma bounds its negative index q (with the orientation fixed by the second-order condition): $Nq \leq M - 1 - \binom{N+1}{2}$, equivalently $N(q-1) \leq M - \binom{N+2}{2}$, with $M = \dim R_2$.

Why it is true: every representation $F = \sum h_i^2$ is a local minimum of $g \circ \Phi_N$, so H kills the summands and is nonnegative, summed over the N slots, along every syzygy $\sum_i h_i D_i = 0$. There are at least $Nv - M + 1$ independent syzygies; after projecting through T at least $Nr + \binom{N+1}{2} - M + 1$ of them survive, and a space on which $J^{\oplus N}$ is nonnegative has dimension at most Np. Writing $r = p + q$ gives the inequality. Each negative direction beyond the first costs N dimensions of syzygies.

For $q = 1$ it says that the sums of N squares have a boundary inside the cone only if $\binom{N+2}{2} \leq M$: the Barvinok–Pataki bound. For $m \times 4$ forms and $N = 3m - 1$ the room $M - \binom{N+2}{2}$ is exactly $m(m+7)/2$, and the bound becomes $q \leq \lfloor (m+13)/6 \rfloor$ for $m \geq 5$: 3 for $m = 5, \ldots, 10$, 4 for $m = 11, \ldots, 16$, 5 at $m = 17$ (the Isabelle lemma `q_values`).

Origin, as the paper states it: the inequality is, in substance, Theorem 3.4 of Boumal, Voroninski and Bandeira (a bound on the negative eigenvalues of the dual multiplier at second-order critical points of the Burer–Monteiro factorization), applied at a boundary point; the lemma gives the self-contained proof the paper needs, without a constant-rank assumption.

Evidence & scope

- **Program:** Figure 3 recomputed: q_* in exact fractions (22/7, 114/29, 231/47, 127/25, 384/65, 413/68), the room m(m+7)/2, ⌊q_*⌋ ≤ 4 exactly for 5 ≤ m ≤ 16 and ⌊q_*⌋ = ⌊(m+13)/6⌋ up to m = 200, with controls that must fail. `lab/comprueba_lab.py`

Lemma 15. (Common-line commutators) p. 17

> *BSR_m4_Linear2.common_line_step_kernel, BSR_m4_Linear2.common_line_step_image, BSR_m4_Linear3.symmetric_4linear_polarization, BSR_m4_Linear3.symmetric_5linear_polarization Let $\mathcal{C}$ be a family of endomorphisms of a finite-dimensional complex vector space S. Suppose that a fixed subspace $K \subset S$, of dimension at most one, satisfies*
>
> $$[C, D](S) \subset K \qquad (C, D \in \mathcal{C}).$$
>
> *Then $\mathcal{C}$ is simultaneously triangularizable.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the key step (the kernel and image cases); the induction is in the text

`common_line_step_kernel` · `Isabelle/BSRFourColumnsLinear2/BSR_m4_Linear2.thy`, line 122

```
theorem common_line_step_kernel:
  fixes X Y :: "'v::ab_group_add \<Rightarrow> 'v" and sm
:: "'k::field \<Rightarrow> 'v \<Rightarrow> 'v" and k ::
'v
  assumes addX: "\<And>u w. X (u + w) = X u + X w"
    and smX: "\<And>c u. X (sm c u) = sm c (X u)"
    and sm0: "\<And>u. sm 0 u = 0"
    and sm1: "\<And>u. sm 1 u = u"
    and smmul: "\<And>c d u. sm c (sm d u) = sm (c * d) u"
    and Y0: "Y 0 = 0"
    and comm: "\<And>w. \<exists>c. X (Y w) - Y (X w) = sm
c k"
    and v: "X v = 0" and nz: "X (Y v) \<noteq> 0"
  shows "\<exists>m. X m = k"
```

`common_line_step_image` · `Isabelle/BSRFourColumnsLinear2/BSR_m4_Linear2.thy`, line 148

```
theorem common_line_step_image:
  fixes X Z :: "'v::ab_group_add \<Rightarrow> 'v" and sm
```

```
:: "'k::field \<Rightarrow> 'v \<Rightarrow> 'v" and k ::
'v
  assumes addX: "\<And>u w. X (u + w) = X u + X w"
    and subX: "\<And>u w. X (u - w) = X u - X w"
    and smX: "\<And>c u. X (sm c u) = sm c (X u)"
    and comm: "\<And>w. \<exists>c. X (Z w) - Z (X w) = sm
c k"
    and kim: "X m = k"
  shows "Z (X w) \<in> range X"
```

symmetric_4linear_polarization · Isabelle/BSRFourColumnsLinear3/BSR_m4_Linear3.thy, line 143

```
theorem symmetric_4linear_polarization:
  fixes B :: "'v::ab_group_add \<Rightarrow> 'v \
<Rightarrow> 'v \<Rightarrow> 'v \<Rightarrow>
'k::field_char_0"
  assumes add1: "\<And>x x' y z w. B (x + x') y z w = B x y
z w + B x' y z w"
    and s12: "\<And>x y z w. B x y z w = B y x z w"
    and s23: "\<And>x y z w. B x y z w = B x z y w"
    and s34: "\<And>x y z w. B x y z w = B x y w z"
    and diag: "\<And>x. B x x x x = 0"
  shows "B x y z w = 0"
```

symmetric_5linear_polarization · Isabelle/BSRFourColumnsLinear3/BSR_m4_Linear3.thy, line 205

```
theorem symmetric_5linear_polarization:
  fixes B :: "'v::ab_group_add \<Rightarrow> 'v \
<Rightarrow> 'v \<Rightarrow> 'v \<Rightarrow> 'v \
<Rightarrow> 'k::field_char_0"
  assumes add1: "\<And>x x' y z w v. B (x + x') y z w v = B
x y z w v + B x' y z w v"
    and s12: "\<And>x y z w v. B x y z w v = B y x z w v"
    and s23: "\<And>x y z w v. B x y z w v = B x z y w v"
    and s34: "\<And>x y z w v. B x y z w v = B x y w z v"
    and s45: "\<And>x y z w v. B x y z w v = B x y z v w"
    and diag: "\<And>x. B x x x x x = 0"
  shows "B x y z w v = 0"
```

↳ Used in: Lemma 17 (p. 35)

Proof

Induct on dim S. If every member is scalar, there is nothing to prove. Otherwise subtract an eigenvalue from a nonscalar member to obtain a nonzero singular operator X.

If $\ker X$ is invariant under the entire family, it is a nonzero proper common invariant subspace. Otherwise choose $Y \in \mathcal{C}$ and $v \in \ker X$ with $XYv \neq 0$. Then

$$0 \neq XYv = [X, Y]v \in K \cap \operatorname{im} X.$$

Thus $K \subset \operatorname{im} X$, and for every $Z \in \mathcal{C}$,

$$ZXw = XZw - [X, Z]w \in \operatorname{im} X.$$

> Hence $\operatorname{im} X$ is a common proper invariant subspace.
>
> On a resulting common invariant subspace U, restricted commutators have image in $K \cap U$; on the quotient their images lie in $(K + U)/U$. Both have dimension at most one. Apply induction to U and S/U.

Lemma 16. (Small tableaux) p. 17

> *BSR_m4_Linear.singular_2x2_space, BSR_m4_Linear.singular_2x2_space_complex Let*
>
> $$\varphi : S \hookrightarrow W \otimes B^*$$
>
> *be injective, with no nonzero rank-one tensor in its image. Write* $G_b(s) = \varphi(s)(b)$ *and* $s = \dim S$. *If* $2 \le s \le 3$, *some* G_b *has rank at least* $s - 1$, *without a restriction on* $\dim B$. *The same conclusion holds for* $s = 4$ *when* $\dim B \le 4$.

Isabelle · ours · part **show**

Only a part is in Isabelle: the 2×2 step (a linear space of singular 2×2 matrices has a common kernel or image); the rest is in the text

`singular_2x2_space` · `Isabelle/BSRFourColumnsLinear/BSR_m4_Linear.thy`, line 165

```
theorem singular_2x2_space:
  fixes S :: "('a::field \<times> 'a \<times> 'a \<times>
'a) set"
  assumes sing: "\<And>a b c d. (a,b,c,d) \<in> S \
<Longrightarrow> a * d = b * c"
    and add: "\<And>a b c d a' b' c' d'. (a,b,c,d) \<in> S
\<Longrightarrow> (a',b',c',d') \<in> S \<Longrightarrow>
(a+a',b+b',c+c',d+d') \<in> S"
  shows "common_kernel S \<or> common_image S"
```

`singular_2x2_space_complex` · `Isabelle/BSRFourColumnsLinear/BSR_m4_Linear.thy`, line 347

```
corollary singular_2x2_space_complex:
  fixes S :: "(complex \<times> complex \<times> complex \
<times> complex) set"
  assumes "\<And>a b c d. (a,b,c,d) \<in> S \
<Longrightarrow> a * d = b * c"
    and "\<And>a b c d a' b' c' d'. (a,b,c,d) \<in> S \
<Longrightarrow> (a',b',c',d') \<in> S \<Longrightarrow>
(a+a',b+b',c+c',d+d') \<in> S"
  shows "common_kernel S \<or> common_image S"
```

🆕 **Classical and new:** the paper: the step on spaces of rank-at-most-one maps is the classical description [Fla, AL]; its last step is [Drn, Lemma 1.8].

↳ Used in: Lemma 17 (p. 35)

Proof

For $s = 2$, the assertion follows from injectivity. For $s = 3$, suppose all G_b have rank at most one. A linear space of rank-at-most-one maps has a common image line or a common kernel hyperplane: two rank-one maps whose image lines and covector lines are both independent have a rank-two sum (this is the classical description of spaces of matrices of rank at most one [18 , 19] ; the last step is [20 , Lemma 1.8]). A common kernel contradicts injectivity; a common image line makes each nonzero $\varphi(s)$ rank one.

Now let $s = 4$. Padding B with zero coordinates permits $\dim B = 4$. Maximum rank at most one is excluded as above. Suppose the maximum is two. Choose bases with

$$G_{b_0} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad G_b = \begin{pmatrix} A_b & B_b \\ C_b & D_b \end{pmatrix}.$$

The Schur complement of $G_{b_0} + tG_b$ vanishes. Its first three coefficients give

$$D_b = 0, \qquad C_b B_b = 0, \qquad C_b A_b B_b = 0.$$

If some B_b is invertible, polarization gives $C_c = 0$ for every c. All evaluation images then lie in a fixed two-plane. A four-dimensional subspace of $\mathbb{C}^2 \otimes (\mathbb{C}^4)^*$ contains a rank-one tensor: write its two rows as maps $R_1, R_2 : S \to (\mathbb{C}^4)^*$. If R_1 is singular, use a nonzero vector in its kernel; otherwise use an eigenvector of $R_1^{-1} R_2$.

Thus every B_b is singular. A linear space of singular 2×2 matrices has a common kernel or a common image. To see this, normalize one rank-one member to E_{11}. Every other member has the form $\left(\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right)$, and $bc = 0$ identically on the linear space.

A common kernel of the B_b gives a common kernel of all G_b, impossible. Hence $B_b = u\beta_b$ for a fixed $u \neq 0$. The identity $C_b B_b = 0$ gives $C_b u = 0$ for all b. If all C_b vanish, use the fixed-two-plane argument. Otherwise $\ker C_b = \mathbb{C}u$ generically, and $C_b A_b B_b = 0$ gives $A_b u \in \mathbb{C}u$, first generically and then identically. Thus $G_b(u, 0)$ always lies in a fixed line. The nonzero tensor $\varphi(u, 0)$ has rank one, a final contradiction.

Lemma 17. (Small row images) p. 18

BSR_m4_Small_Rows.row_rank_one_simple, BSR_m4_Small_Rows.evaluation_step, BSR_m4_Small_Rows.row_in_line, BSR_m4_Small_Rows.small_rows_contradiction Let $T : A \otimes B \to E$ be a surjective complex linear map to a nondegenerate symmetric space. Assume flatness, zero first radical, and that $T^*J(E)$ contains no nonzero simple tensor.

There is no d-dimensional subspace $D \subset A$ with

$$\dim T(D \otimes B) \leq d$$

when $d = 2$ or 3. There is also no such subspace for $d = 4$ if $\dim B \leq 4$.

In particular, the three-row and four-row isotropic lemmas hold.

Isabelle · ours · part **show**

Only a part is in Isabelle: the three algebraic steps (the rank-one row observation, the evaluation step and the final contradiction), with the dimension counts and the common eigenvector of Lemma 18 as hypotheses

`row_rank_one_simple` · `Isabelle/BSRFourColumnsSmallRows/BSR_m4_Small_Rows.thy`, line 30

```
theorem row_rank_one_simple:
  fixes J :: "'e \<Rightarrow> 'e \<Rightarrow> 'k::field"
    and sm :: "'k \<Rightarrow> 'e \<Rightarrow> 'e"
    and T :: "'a \<Rightarrow> 'b \<Rightarrow> 'e" and lam
:: "'b \<Rightarrow> 'k"
  assumes Jsm: "\<And>t u w. J (sm t u) w = t * J u w"
    and flat: "\<And>x y b c. J (T x b) (T y c) = J (T x c)
(T y b)"
    and row: "\<And>b. T a b = sm (lam b) e"
    and b0: "lam b0 \<noteq> 0"
  shows "\<exists>alpha mu. \<forall>c d. J e (T c d) =
alpha c * mu d"
```

`evaluation_step` · `Isabelle/BSRFourColumnsSmallRows/BSR_m4_Small_Rows.thy`, line 85

```
theorem evaluation_step:
  fixes J :: "'e::zero \<Rightarrow> 'e \<Rightarrow>
'k::zero" and T :: "'a \<Rightarrow> 'b \<Rightarrow> 'e"
  assumes flat: "\<And>x y b c. J (T x b) (T y c) = J (T x
c) (T y b)"
    and J0: "\<And>w. J 0 w = 0"
    and zero: "T a b0 = 0"
  shows "J (T a b) (T c b0) = 0"
```

`row_in_line` · `Isabelle/BSRFourColumnsSmallRows/BSR_m4_Small_Rows.thy`, line 100

```
corollary row_in_line:
  fixes J :: "'e::zero \<Rightarrow> 'e \<Rightarrow>
```

```
'k::zero" and T :: "'a \<Rightarrow> 'b \<Rightarrow> 'e"
    and sm :: "'k \<Rightarrow> 'e \<Rightarrow> 'e"
  assumes "\<And>x y b c. J (T x b) (T y c) = J (T x c) (T
y b)" and "\<And>w. J 0 w = 0" and "T a b0 = 0"
    and line: "\<And>z. (\<forall>c. J z (T c b0) = 0) \
<Longrightarrow> \<exists>t. z = sm t w"
  shows "\<exists>lam. \<forall>b. T a b = sm (lam b) w"
```

small_rows_contradiction · Isabelle/BSRFourColumnsSmallRows/BSR_m4_Small_Rows.thy, line 125

```
theorem small_rows_contradiction:
  fixes G :: "'b \<Rightarrow> 'e::zero \<Rightarrow> 'c \
<Rightarrow> 'k::field"
    and C :: "'b \<Rightarrow> 'e \<Rightarrow> 'e" and sm
:: "'k \<Rightarrow> 'e \<Rightarrow> 'e"
    and lam :: "'b \<Rightarrow> 'k" and z :: 'e
  assumes factor: "\<And>b u. G b u = G b0 (C b u)"
    and homog: "\<And>t u c. G b0 (sm t u) c = t * G b0 u
c"
    and eigen: "\<And>b. C b z = sm (lam b) z" and z_ne: "z
\<noteq> 0"
    and inj: "\<And>u. (\<forall>b c. G b u c = 0) \
<Longrightarrow> u = 0"
    and norank1: "\<And>u mu w. (\<forall>b c. G b u c = mu
b * w c) \<Longrightarrow> (\<forall>b c. G b u c = 0)"
  shows False
```

↳ Used in: Lemma 20 (p. 42)

Proof

Every nonzero row map $T_a : B \to E$ has rank at least two. Indeed, rank zero contradicts zero first radical. If $T_a(b) = \lambda(b)e$ has rank one, flatness gives

$$\lambda(b)\langle e, T(c,d)\rangle = \lambda(d)\langle e, T(c,b)\rangle.$$

Consequently T^*Je is a simple tensor. It is nonzero by surjectivity and nondegeneracy, a contradiction.

Suppose $S = T(D \otimes B)$ has dimension $s \le d$. The preceding observation excludes $s \le 1$. Define

$$F_a(b) = T(a,b) \in S, \qquad G_b(z)(c) = \langle z, T(c,b)\rangle.$$

The tableau $z \mapsto G_\bullet(z)$ is injective and contains no nonzero rank-one tensor. Lemma 16 supplies b_0 with $\dim\ker G_{b_0} \le 1$.

Mixed flatness says

$$G_b F_a(c) = G_c F_a(b).$$

If $F_a(b_0) = 0$, then $F_a(B) \subset \ker G_{b_0}$. The row-rank observation forces $a = 0$. Thus evaluation at b_0 is injective on

D, so $d \le s$. Hence $s = d$, and evaluation at b_0 identifies D with S.

Write, under this identification,

$$F_a(b) = C_b a, \qquad C_{b_0} = I.$$

Then

$$G_b = G_{b_0} C_b, \qquad G_{b_0}[C_b, C_c] = 0.$$

All commutators have image in the fixed space $\ker G_{b_0}$, of dimension at most one. Lemma 15 supplies a common eigenvector $z \neq 0$, with $C_b z = \lambda(b) z$. Therefore

$$G_b z = \lambda(b) G_{b_0} z.$$

If $G_{b_0} z = 0$, tableau injectivity fails. Otherwise this is a nonzero rank-one tableau member. Both alternatives are impossible.

Lemma 18. (All moments) p. 19

BSR_m4_Moments.unperturbed_coeff_nonzero, BSR_m4_Moments.moments_vanish, BSR_m4_Moments.shift_moments Let W be a real nondegenerate symmetric space. In Witt coordinates (e, W, f) suppose

$$T_0(z,w) = (z,w,0), \qquad T_c(z,w) = (\alpha z + \langle \ell, w\rangle,\ pz + Aw,\ \langle p,$$

where A is self-adjoint. If $(T_0 + tT_c)^* J (T_0 + tT_c)$ is singular identically in t, then

$$\langle p, A^j p\rangle = 0 \qquad (j \ge 0).$$

Isabelle · ours · part **show**

Only a part is in Isabelle: the unperturbed coefficients and the induction over the moments; the word analysis of the Neumann expansion enters as a hypothesis

`unperturbed_coeff_nonzero` · `Isabelle/BSRFourColumnsMoments/BSR_m4_Moments.thy`, line 116

```
corollary unperturbed_coeff_nonzero:
  "fps_nth (1 - (1 + inverse (1 + fps_X)) ^ 2 :: real fps)
j \<noteq> 0"
```

moments_vanish · Isabelle/BSRFourColumnsMoments/BSR_m4_Moments.thy, line 122

```
theorem moments_vanish:
  fixes mu a :: "nat \<Rightarrow> real"
    and Wd :: "nat \<Rightarrow> 'w set" and idx :: "nat \
<Rightarrow> 'w \<Rightarrow> nat" and cf :: "nat \
<Rightarrow> 'w \<Rightarrow> real"
  assumes fin: "\<And>j. finite (Wd j)"
    and low: "\<And>j w. w \<in> Wd j \<Longrightarrow> idx
j w + 2 \<le> j"
    and dec: "\<And>j. a j = (-1) ^ (j + 1) * real (j + 3)
* mu j + (\<Sum>w\<in>Wd j. cf j w * mu (idx j w))"
    and zero: "\<And>j. a j = 0"
  shows "mu j = 0"
```

shift_moments · Isabelle/BSRFourColumnsMoments/BSR_m4_Moments.thy, line 155

```
theorem shift_moments:
  fixes B :: "'v::real_vector \<Rightarrow> 'v" and phi ::
"'v \<Rightarrow> real" and p :: 'v and alpha :: real
  assumes B: "linear B" and phi: "linear phi" and mom: "\
<And>k. phi ((B ^^ k) p) = 0"
  shows "phi ((((\<lambda>x. B x + alpha *\<^sub>R x) ^^ j)
p) = 0"
```

↳ Used in: Lemma 20 (p. 42)

Proof

Set $B = A - \alpha I$, $\tau = t/(1+t\alpha)$ and $\mu_j = \langle p, B^j p\rangle$. After division of $T_0 + tT_c$ by $1+t\alpha$, the scalar Schur complement, divided by τ^2, is

$$\mu_0 - \left\langle (2I+\tau B)p,\, \left((I+\tau B)^2 + \tau^2(\ell p^* + p\ell^*)\right)^{-1}(2I\right.$$

The W-block is invertible as a formal power series, so (M1) vanishes identically.

Without the rank-two perturbation, (M1) equals

$$\mu_0 - \left\langle p,\, \left(I + 2(I+\tau B)^{-1} + (I+\tau B)^{-2}\right)p\right\rangle.$$

Its coefficient of τ^j is $(-1)^{j+1}(j+3)\mu_j$, including $-3\mu_0$ for $j=0$. In one variable these are the coefficients of $1 - \left(1 + \frac{1}{1+x}\right)^2 = -\frac{2x+3}{(1+x)^2}$, and none of them vanishes.

Expand the perturbed inverse by its formal Neumann series. Put $X = I + \tau B$, $u = (2I+\tau B)p$ and $E = \tau^2(\ell p^* + p\ell^*)$. Then

$$\left(X^2 + E\right)^{-1} = \sum_{n\ge 0} (-1)^n X^{-2}\left(EX^{-2}\right)^n,$$

and the term with $n \geq 1$ contributes τ^{2n} times a sum of words

$$(u^* X^{-2} a_1)\,(b_1^* X^{-2} a_2) \cdots (b_{n-1}^* X^{-2} a_n)\,(b_n^* X^{-2} u), \qquad (a_k, b$$

Each word contains a factor $\langle p, g(\tau B)p\rangle$ for a power series g. Indeed, if $a_1 = p$ the first factor is such, because $u = (2I + \tau B)p$. Otherwise $(a_1, b_1) = (\ell, p)$ and the second factor begins with p^*; it is of this form if $a_2 = p$, and otherwise $b_2 = p$. Continuing, either some $a_k = p$ closes a factor $p^* X^{-2} p$, or all $a_k = \ell$, all $b_k = p$, and the last factor is $p^* X^{-2}(2I + \tau B)p$. All operators in these factors are functions of τB, so the factor expands as $\sum_i c_i \tau^i \mu_i$. Since the word carries the prefactor τ^{2n} with $n \geq 1$, its coefficient of τ^j is a sum of products, each containing some μ_i with $i \leq j - 2$. Now argue by induction. If $\mu_0 = \cdots = \mu_{j-1} = 0$, the coefficient of τ^j in (M1) reduces to $(-1)^{j+1}(j+3)\mu_j$, which must vanish; hence $\mu_j = 0$. Finally $A = B + \alpha I$, so the moments with A also vanish.

Lemma 19. (Invariant extension of an isotropic plane) p. 20

BSR_m4_Linear.symmetric_trilinear_polarization, BSR_m4_Polar.pol_form.polar, BSR_m4_Invariant.iso_setup.T_zero, BSR_m4_Invariant.iso_setup.R_zero, BSR_m4_Invariant.iso_setup.U_zero, BSR_m4_Invariant.iso_setup.invariant_dim3, BSR_m4_Invariant.iso_setup.invariant_dim2 Let W be a real symmetric space of negative index at most three. Let p_c and self-adjoint B_c depend linearly on c. Suppose $P = \mathrm{span}\{p_c\}$ is an isotropic two-plane,

$$B_c p_d = B_d p_c, \qquad [B_c, B_d]P \subset P, \qquad [B_c, B_d]P^{\perp} \subset P,$$

and $\langle p_c, B_c^j p_c\rangle = 0$ for $j = 1, 2, 3, 4$. Then P is contained in a common invariant totally isotropic space P' of dimension two or three.

Isabelle · ours · part **show**

Only a part is in Isabelle: every polarization and positivity step (multilinear polarization, B_c P inside its orthogonal, the invariant space in dimensions two and three); that negative index at most 3 gives dimension at most 3, and the positivity of the quotient in dimension 3 (Witt), are in the text

symmetric_trilinear_polarization · Isabelle/BSRFourColumnsLinear/BSR_m4_Linear.thy, line 361

```
theorem symmetric_trilinear_polarization:
  fixes B :: "'v::ab_group_add \<Rightarrow> 'v \
<Rightarrow> 'v \<Rightarrow> 'k::field_char_0"
  assumes add1: "\<And>x x' y z. B (x + x') y z = B x y z +
B x' y z"
    and s12: "\<And>x y z. B x y z = B y x z"
    and s23: "\<And>x y z. B x y z = B x z y"
    and diag: "\<And>x. B x x x = 0"
  shows "B x y z = 0"
```

polar · Isabelle/BSRFourColumnsInvariant/BSR_m4_Polar.thy, line 155

```
theorem polar:
  assumes diag: "\<And>x. f (\<lambda>_. x) = 0"
  shows "f v = 0"
```

T_zero · Isabelle/BSRFourColumnsInvariant/BSR_m4_Invariant.thy, line 133

```
lemma T_zero: "ip (p c) (B d (p e)) = 0"
```

R_zero · Isabelle/BSRFourColumnsInvariant/BSR_m4_Invariant.thy, line 199

```
lemma R_zero: "ip (B c (p d)) (B e (p f)) = 0"
```

U_zero · Isabelle/BSRFourColumnsInvariant/BSR_m4_Invariant.thy, line 328

```
lemma U_zero: "ip (B i (B j (p k))) (B l (p m)) = 0"
```

invariant_dim3 · Isabelle/BSRFourColumnsInvariant/BSR_m4_Invariant.thy, line 373

```
theorem invariant_dim3:
  assumes pos: "\<And>x. (\<And>y. y \<in> Pp \
<Longrightarrow> ip x y = 0) \<Longrightarrow> ip x x = 0 \
<Longrightarrow> x \<in> Pp"
  shows "x \<in> Pp \<Longrightarrow> B i x \<in> Pp"
```

invariant_dim2 · Isabelle/BSRFourColumnsInvariant/BSR_m4_Invariant.thy, line 459

```
theorem invariant_dim2:
  assumes inP: "\<And>c d. B c (p d) \<in> range p"
  shows "Pp = range p" and "x \<in> range p \
<Longrightarrow> B i x \<in> range p"
```

↳ Used in: Lemma 20 (p. 42)

Proof

The trilinear form $\langle p_c, B_d p_e\rangle$ is fully symmetric. Its diagonal vanishes, so $B_c P \subset P^\perp$.

The four-linear form

$$R(c,d,e,f) = \langle B_c p_d, B_e p_f\rangle$$

is fully symmetric. Interchange within either pair uses $B_c p_d = B_d p_c$; interchange of the operator indices changes its value by $\langle p_d, [B_c, B_e] p_f\rangle = 0$. Its diagonal is the second moment, hence $R = 0$. Thus

$$P' = P + \sum_c B_c P$$

is totally isotropic and has dimension at most three. If $P' = P$, it is already invariant.

Suppose $\dim P' = 3$. The quotient $P'^\perp/P'$ is positive definite. The five-linear form

$$U(i,j,k,l,m) = \langle B_i B_j p_k, B_l p_m\rangle$$

is fully symmetric: swapping i, j introduces $\langle [B_i, B_j] p_k, B_l p_m\rangle = 0$; swapping the outer operator indices introduces $\langle B_j p_k, [B_i, B_l] p_m\rangle = 0$; the other swaps follow from $B_j p_k = B_k p_j$. Its diagonal is the third moment, so $U = 0$. Together with $R = 0$, this gives $B_i P' \subset P'^\perp$.

The squared norm of $B_c^2 p_c$ is the fourth moment. Hence $B_c^2 p_c \in P'$ by positivity of $P'^\perp/P'$. Modulo P', the trilinear map $(i,j,k) \mapsto B_i B_j p_k$ is symmetric, using $B_j p_k = B_k p_j$ and $[B_i, B_j] p_k \in P$. Its diagonal is zero; polarization gives $B_i B_j p_k \in P'$ for all indices. Therefore $B_i P' \subset P'$.

Lemma 20. (Low negative index) p. 20

BSR_m4_LowIndex.low_setup.low_invariant, BSR_m4_LowIndex.rank2_setup.rank2_invariant, BSR_m4_LowIndex.generic_to_all, BSR_m4_LowIndex.count_r_ge_a1, BSR_m4_LowIndex.count_kh Let H be a nonzero real Hankel multiplier with effective supports (a,b), where $b \le 4$. Suppose its complex image contains no nonzero simple tensor and its negative index satisfies $q \le 4$. Put $r = \operatorname{rank} H$. Then

$$r \geq a + 1.$$

If $b = 4$ *and* $a \geq 3$*, then*

$$r \geq a + 2.$$

Isabelle · ours · part **show**

Only a part is in Isabelle: the polarization steps, the rank-two case and the integer counts; the Schur coefficients (L2) are an exact computation, and the steps quoted from other lemmas enter as hypotheses

`low_invariant` · Isabelle/BSRFourColumnsInvariant/BSR_m4_LowIndex.thy, line 132

```
theorem low_invariant:
  assumes pos: "\<And>x. (\<And>e. ip x (p e) = 0) \
<Longrightarrow> ip x x = 0 \<Longrightarrow> x \<in> range
p"
  shows "B c (p d) \<in> range p"
```

`rank2_invariant` · Isabelle/BSRFourColumnsInvariant/BSR_m4_LowIndex.thy, line 209

```
theorem rank2_invariant:
  assumes pos: "\<And>x. (\<And>y. y \<in> range Pc \
<Longrightarrow> ip x y = 0) \<Longrightarrow> ip x x = 0 \
<Longrightarrow> x \<in> range Pc"
  assumes x: "x \<in> range Pc"
  shows "A x \<in> range Pc"
```

`generic_to_all` · Isabelle/BSRFourColumnsInvariant/BSR_m4_LowIndex.thy, line 263

```
theorem generic_to_all:
  fixes A :: "'c::real_normed_vector \<Rightarrow>
'v::real_vector \<Rightarrow> 'v" and Pset :: "'v set"
  assumes A_add: "\<And>c c' x. A (c + c') x = A c x + A c'
x"
    and A_sc: "\<And>s c x. A (s *\<^sub>R c) x = s *\
<^sub>R A c x"
    and Psub: "subspace Pset"
    and open_set: "open S" "S \<noteq> {}" and gen: "\
<And>c x. c \<in> S \<Longrightarrow> x \<in> Pset \
<Longrightarrow> A c x \<in> Pset"
  shows "x \<in> Pset \<Longrightarrow> A c x \<in> Pset"
```

`count_r_ge_a1` · Isabelle/BSRFourColumnsInvariant/BSR_m4_LowIndex.thy, line 307

```
theorem count_r_ge_a1:
  fixes a s k h q r :: nat
  assumes a_def: "a = s + k"
    and L1h: "0 < k \<Longrightarrow> 2 \<le> h \<and> h \
<le> q" and hq: "q \<le> 4"
    and hrs: "0 < k \<Longrightarrow> h \<le> r - s"
    and SR: "0 < k \<Longrightarrow> h \<le> k \
<Longrightarrow> False"
```

```
    and K0: "k = 0 \<Longrightarrow> r \<le> a \
<Longrightarrow> False"
  shows "a + 1 \<le> r"
```

count_kh · Isabelle/BSRFourColumnsInvariant/BSR_m4_LowIndex.thy, line 329

```
theorem count_kh:
  fixes a s k h q r :: nat
  assumes a_def: "a = s + k"
    and L1h: "0 < k \<Longrightarrow> 2 \<le> h \<and> h \
<le> q" and hq: "q \<le> 4"
    and hrs: "0 < k \<Longrightarrow> h \<le> r - s"
    and SR: "0 < k \<Longrightarrow> h \<le> k \
<Longrightarrow> False"
    and ra: "r = a + 1" and kpos: "0 < k"
  shows "(k = 1 \<and> h = 2) \<or> (k = 2 \<and> h = 3) \
<or> (k = 3 \<and> h = 4)"
```

↳ Used in: Theorem 22 (p. 49)

Proof

Remove the factor radicals and write $H = T^{*}JT$, with $T : A \otimes B \to E$ onto and J of signature (p, q). Choose a generic real b_0 maximizing both $s = \operatorname{rank} T_{b_0}$ and $\operatorname{rank}(T_{b_0}^{*} J T_{b_0})$. Set

$$U = \operatorname{im} T_{b_0}, \quad k = a - s, \quad h = \dim(U \cap U^{\perp}).$$

The linear terms of bordered $(s+1)$-minors give $T_c(\ker T_{b_0}) \subset U$. Flatness also gives orthogonality to U, hence

$$T_c(\ker T_{b_0}) \subset U \cap U^{\perp}. \tag{L1}$$

Every nonzero complex row map has rank at least two, by the row-rank argument of Lemma 17. Thus $k{>}0$ implies

$$2 \le h \le q \le 4.$$

Suppose $r \le a$. If $k = 0$, then $r = a$ and T_{b_0} is an isomorphism. Under this identification all other slices are self-adjoint and pairwise commuting. A common complex eigenvector gives a rank-one row, impossible. If $k{>}0$, then

$$k = a - s \ge r - s \ge h.$$

An h-plane in $\ker T_{b_0}$ maps into the isotropic h-plane $U \cap U^{\perp}$, contradicting Lemma 17. Hence $r \ge a + 1$.

Now let $b = 4$ and $r = a + 1$. Lemma 6 excludes an invertible $Q(b)$, so every $Q(b)$ is singular.

First suppose $k>0$. By (L1) and the small-row lemma, $k<h$. Since $h \le r-s = k+1$, necessarily

$$(k,h) = (1,2), (2,3), (3,4).$$

Choose real Witt coordinates

$$E = W_0 \perp \mathbb{H}^h, \qquad A = W_0 \oplus U_0^h \oplus Z^k, \qquad T_{b_0}(w,u,z) =$$

Flatness and (L1) give

$$T_c = \begin{pmatrix} A_c & P_c & 0 \\ L_c & B_c & R_c \\ P_c^* & Q_c & 0 \end{pmatrix}, \qquad A_c^* = A_c, \quad Q_c^t = Q_c.$$

The Schur complement along W_0 vanishes identically: b_0 maximizes $\mathrm{rank} Q(b)$, so $\mathrm{rank} Q(b_0+\tau c) \le \mathrm{rank} Q(b_0) = \dim W_0$ for every c and τ, while the W_0-block of $Q(b_0+\tau c)$ is invertible near $\tau = 0$; rank additivity under block elimination forces the complement to vanish. Its successive U_0-block coefficients give

$$Q_c = 0, \qquad P_c^* P_c = 0, \qquad P_c^* A_c P_c = 0, \qquad P_c^* A_c^2 P$$

The coefficients after preceding identities are imposed are, respectively,

$$2Q_c, \quad -3P_c^* P_c, \quad 4P_c^* A_c P_c, \quad -5P_c^* A_c^2 P_c.$$

(In the raw expansion the blocks L_c and B_c do appear; with $G_j = P_c^* A_c^j P_c$ the τ^3 coefficient is $4G_1 - 2(G_0 B_c + B_c^t G_0)$ and the τ^4 coefficient is $-5G_2 + 3(G_1 B_c + B_c^t G_1) - B_c^t G_0 B_c + 4(G_0 L_c P_c + (L_c P_c)^t G_0)$, so they drop out exactly once the preceding identities $G_0 = 0$, $G_1 = 0$ are imposed; R_c never enters.) These coefficients do not depend on the dimensions. Write J_0 for the metric on W_0, so that $J_0^2 = 1$ in a Witt basis, $A_c^* = J_0 A_c^t J_0$ and $P_c^* = P_c^t J_0$. The W_0-block of $Q(b_0+\tau c)$ is $J_0 + N(\tau)$ with $N(\tau) = O(\tau)$, and its inverse is the finite recurrence $J_0 \sum_{j \le 4} (-N(\tau) J_0)^j$ modulo τ^5. Expanding the Schur complement with this inverse is a computation in the free algebra generated by the blocks $A_c, P_c, P_c^t, L_c, L_c^t, B_c, B_c^t, Q_c$ and J_0, with the relation $J_0^2 = 1$. Since the preceding identities are monomials (Q_c, $P_c^* P_c$, $P_c^* A_c P_c$), imposing them deletes the words that contain them.

The coefficients above are what remains, for every $\dim W_0$ and every h. Polarization and cross-flatness imply $P_c^* P_d = 0$ for all c, d. Therefore

$$P = \sum_c \operatorname{im} P_c$$

is totally isotropic in W_0, and $\dim P \le q - h$.

For $(k, h) = (3, 4)$, $P = 0$ and the seven rows $U_0 \oplus Z$ map into an isotropic four-space. For $(k, h) = (2, 3)$, $\dim P \le 1$ and the five rows $U_0 \oplus Z$ map into an isotropic space of dimension at most four. Both contradict Lemma 17.

For $(k, h) = (1, 2)$, the same argument works if $\dim P \le 1$. Suppose $\dim P = 2$. Then $P^\perp / P$ is positive definite. If some P_c has rank two, it is generically onto P; (L2) gives $A_c P \subset P^\perp$ and makes every vector of $A_c P$ null. Thus $A_c P \subset P$, first generically and then for every c.

If all P_c have rank at most one, the rank-one matrix-space classification gives either a common image line or $P_c = p_c \phi$ with a fixed nonzero covector ϕ. The first option contradicts $\dim P = 2$. Mixed flatness in the second option gives

$$A_c p_d - A_d p_c \in P.$$

The first moment in (L2) polarizes to $A_c P \subset P^\perp$. The second moment and the positive quotient give $A_c p_c \in P$. Polarizing this relation and combining it with the displayed difference yields $A_c p_d \in P$ for all c, d.

Thus P is invariant in either case. The five rows $P \oplus U_0^2 \oplus Z$ map into the isotropic four-space $P \oplus \mathbb{R}^2 e$, again impossible.

It remains that $k = 0$. The generic slice image is a degenerate hyperplane, with one-dimensional radical. In Witt coordinates it has the form

$$T_{b_0}(z, w) = (z, w, 0),$$

and flatness gives

$$T_c(z, w) = (\alpha_c z + \langle \ell_c, w \rangle, p_c z + A_c w, \eta_c z + \langle p_c, w \rangle), \qquad A_c^*$$

The linear Schur coefficient is $2\eta_c$, so $\eta_c = 0$. Lemma 18 applies.

Let $P = \operatorname{span}_{\mathbb{R}}\{p_c\}$. The zeroth moment makes P totally isotropic. The covector $T^{*}Je$ is

$$(z, w; c) \longmapsto \langle p_c, w \rangle.$$

If $P = 0$, surjectivity fails; if $\dim P = 1$, this is a nonzero simple covector. Since W_0 has negative index $q - 1 \le 3$, we obtain

$$2 \le \dim P \le 3.$$

Put $B_c = A_c - \alpha_c I$. Flatness gives

$$B_c p_d = B_d p_c$$

and

$$[B_c, B_d]x = \ell_d\langle p_c, x\rangle - \ell_c\langle p_d, x\rangle + p_d\langle \ell_c, x\rangle - p_c\langle \ell_d, x$$

In particular, these commutators send both P and $P^{\perp}$ into P.

If $\dim P = 2$, Lemma 19 supplies a common invariant isotropic space $P' \supset P$ of dimension two or three.

If $\dim P = 3$, it is maximal isotropic. Polarizing $\langle p_c, B_c p_c\rangle = 0$ gives $B_c P \subset P^{\perp}$. Since $\langle B_c p_c, B_c p_c\rangle = 0$ and $P^{\perp}/P$ is positive definite, $B_c p_c \in P$. Polarization and $B_c p_d = B_d p_c$ give $B_c P \subset P$. Set $P' = P$.

In either case P' is invariant under every A_c, and

$$T\big((\mathbb{R}z \oplus P') \otimes B\big) \subset \mathbb{R}e \oplus P'.$$

The source row space and the totally isotropic target both have dimension three or four. After complexification, Lemma 17 gives a contradiction. Thus $r = a + 1$ is impossible, proving $r \ge a + 2$.

Lemma 21. (Uniform regular tuples) p. 23

BSR_m4_Pieces.regular_step, BSR_m4_Pieces.regular_point_m4
For every $m \ge 5$, Φ_{3m-1} on $m \times 4$ bilinear forms has a real point with surjective differential.

Isabelle · ours · part **show**

Only a part is in Isabelle: the regular tuples for every m (the paper, «What is checked by machine»)

regular_step · Isabelle/BSRFourColumnsPieces/BSR_m4_Pieces.thy, line 244

```
lemma regular_step:
  fixes H :: "nat \<Rightarrow> nat \<Rightarrow> nat \
<Rightarrow> nat \<Rightarrow> real" and m :: nat and T ::
"(nat \<Rightarrow> nat \<Rightarrow> real) set" and hs
  assumes sym: "\<And>a b c d. a < Suc m \<Longrightarrow>
b < 4 \<Longrightarrow> c < Suc m \<Longrightarrow> d < 4 \
<Longrightarrow> H a b c d = H c d a b"
    and hankel: "\<And>a b c d. a < Suc m \<Longrightarrow>
b < 4 \<Longrightarrow> c < Suc m \<Longrightarrow> d < 4 \
<Longrightarrow> H a b c d = H a d c b"
    and supp: "\<forall>h \<in> T. \<forall>a \<ge> m. \
<forall>b. h a b = 0" and hsT: "hs \<in> T" and hs3: "\
<forall>a. hs a 3 = 0"
    and span: "\<forall>r < m. \<exists>h \<in> T. \
<forall>a. h a 3 = (if a = r then 1 else 0)"
    and ann: "\<forall>h \<in> (T - {hs}) \<union> {(\
<lambda>a b. hs a b + ev m 3 a b), ev m 0, ev m 1, ev m 2}.
                \<forall>c < Suc m. \<forall>d < 4. av (Suc
m) H h c d = 0"
    and IH: "(\<forall>h \<in> T. \<forall>c < m. \
<forall>d < 4. av m H h c d = 0) \<Longrightarrow> (\
<forall>a < m. \<forall>b < 4. \<forall>c < m. \<forall>d <
4. H a b c d = 0)"
  shows "\<forall>a < Suc m. \<forall>b < 4. \<forall>c <
Suc m. \<forall>d < 4. H a b c d = 0"
```

regular_point_m4 · Isabelle/BSRFourColumnsPieces/BSR_m4_Pieces.thy, line 381

```
theorem regular_point_m4:
  fixes H :: "nat \<Rightarrow> nat \<Rightarrow> nat \
<Rightarrow> nat \<Rightarrow> real" and n :: nat
  assumes sym: "\<And>a b c d. a < 5 + n \<Longrightarrow>
b < 4 \<Longrightarrow> c < 5 + n \<Longrightarrow> d < 4 \
<Longrightarrow> H a b c d = H c d a b"
    and hankel: "\<And>a b c d. a < 5 + n \<Longrightarrow>
b < 4 \<Longrightarrow> c < 5 + n \<Longrightarrow> d < 4 \
<Longrightarrow> H a b c d = H a d c b"
    and ann: "\<forall>h \<in> S n. \<forall>c < 5 + n. \
<forall>d < 4. av (5 + n) H h c d = 0"
  shows "\<forall>a < 5 + n. \<forall>b < 4. \<forall>c < 5
+ n. \<forall>d < 4. H a b c d = 0"
```

↳ Used in: Theorem 22 (p. 49)

Proof

For $m = 5$, take

$$\mathcal{S}_5 = \{x_i y_i : 0 \le i \le 3\} \cup \{x_i y_j + x_j y_i : 0 \le i < j \le 3\} \cup \{x_4$$

A Hankel multiplier annihilating this tuple has x_4 as a first radical vector. On the remaining block its kernel contains every symmetric tensor. Therefore

$$H(x \otimes y, x \otimes y) = -H(x \otimes y, y \otimes x) = -H(x \otimes x, y \otimes y) =$$

Separate polarization gives $H = 0$, proving regularity.

Maintain two additional properties: h_m^* has zero y_3 coefficient, and the y_3 coefficients of $\mathcal{S}_m \setminus \{h_m^*\}$ span all old rows. They hold for the base tuple.

Introduce $z = x_m$ and define

$$\mathcal{S}_{m+1} = (\mathcal{S}_m \setminus \{h_m^*\}) \cup \{h_m^* + zy_3,\ zy_0,\ zy_1,\ zy_2\}, \qquad h_{m+}^*$$

The cardinality increases by three, and the two additional properties persist.

Suppose H annihilates the new tuple. Since zy_0, zy_1, zy_2 lie in its kernel, Hankel symmetry leaves only the following possible entries involving z:

$$k(a) = H(zy_3, ay_3), \qquad \gamma = H(zy_3, zy_3).$$

Pairing zy_3 with the unchanged old forms forces $k = 0$ by their column-three spanning property. The old restriction of H now annihilates the whole old tuple, including h_m^*, so it is zero by induction. Finally $H(zy_3, h_m^* + zy_3) = 0$ gives $\gamma = 0$. Thus $H = 0$, proving regularity.

Theorem 22. p. 23

Boundary_Inertia_Arith.inertia_m4_floor, Boundary_Inertia_Arith.q_values, Boundary_Inertia_Arith.incidence, Boundary_Inertia_Arith.psi_counts, BSR_m4_Pieces.triples_card
For $5 \le m \le 16$,

$$\mathrm{BSR}(m, 4) \le 3m - 1.$$

Isabelle · ours · part **show**

Only a part is in Isabelle: pieces: the inertia arithmetic, the values q ≤ 4, the incidence count, the counts of Ψ and the number of support triples; the theorem is a written proof

`inertia_m4_floor` · `Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy`, line 70

```
lemma inertia_m4_floor:
  fixes m q :: int
  assumes "m \<ge> 5"
  shows "2*(3*m-1)*(q-1) \<le> m*(m+7) \
<longleftrightarrow> 6*q \<le> m + 13"
```

q_values · Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy, line 77

```
lemma q_values:
  "\<forall>m::int \<in> {5..10}. (m+13) div 6 = 3" "\
<forall>m::int \<in> {11..16}. (m+13) div 6 = 4" "(17+13)
div 6 = (5::int)"
```

incidence · Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy, line 82

```
lemma incidence: fixes M m :: int shows "M - 2*(m+3) + 2*
((m-1)+3) = M - 2" by simp
lemma compression_count: fixes a m :: int shows "(4*a -
(a+1)) + (3*(m-a) - 1) = 3*m - 2" by simp
lemma psi_counts: fixes m :: int shows "4*(m+1) - (m-1) =
3*m + 5" "4*m - (m-1) = 3*m + 1" by simp_all
```

psi_counts · Isabelle/BoundaryInertia/Boundary_Inertia_Arith.thy, line 84

```
lemma psi_counts: fixes m :: int shows "4*(m+1) - (m-1) =
3*m + 5" "4*m - (m-1) = 3*m + 1" by simp_all
```

triples_card · Isabelle/BSRFourColumnsPieces/BSR_m4_Pieces.thy, line 41

```
lemma triples_card:
  "card (triples 5) = 9" "card (triples 6) = 14" "card
(triples 7) = 21" "card (triples 8) = 29"
  "card (triples 9) = 38" "card (triples 10) = 48" "card
(triples 11) = 60" "card (triples 12) = 72"
  "card (triples 13) = 86" "card (triples 14) = 101" "card
(triples 15) = 117" "card (triples 16) = 134"
```

↳ Used in: Theorem 1 (p. 4)

Proof

Induct on m, using

$$\mathrm{BSR}(2,4) \le 5, \quad \mathrm{BSR}(3,4) \le 8, \quad \mathrm{BSR}(4,4) \le 11.$$

Put $N = 3m - 1$ and $M = 5m(m+1)$.

Suppose $\Sigma_N \neq \Sigma$. The regular tuple of Lemma 21 permits Proposition 10 (in its $m \times 4$ form) to be applied. Its singular-incidence exception has codimension at least two: a first jet has $m + 3$ coefficients, while a conjugate pair with neither factor

real moves in dimension $2m+4$. The incidence dimension is at most

$$M - 2(m+3) + (2m+4) = M - 2.$$

Thus the boundary proposition supplies a form F of minimum length N, a representing span W of dimension N, and an indefinite real multiplier H with product-free complex image, such that

$$W \subset K = \ker H, \qquad r = \operatorname{rank} H \le m+1,$$

with nonnegative normal second variation on every syzygy.

The boundary inertia inequality (Lemma 14, with $v = 4m$) gives

$$q \le \left\lfloor \frac{m^2 + 13m - 2}{6m - 2} \right\rfloor \le 4.$$

Let (a, b) be the effective supports. The small-support exclusions leave $a \ge 3$, $b \in \{3, 4\}$.

If $a \le m - 2$, compression and induction give

$$\mathrm{SOS}_{\mathbb{R}}(F) \le 4a - r + \mathrm{BSR}(m-a, 4) \le 4a - (a+1) + 3(m$$

a contradiction. Here $r \ge a + 1$ is the weak inequality of Lemma 20, and $\mathrm{BSR}(k, 4) \le 3k - 1$ holds for every $k = m - a \ge 2$: by induction for $5 \le k < m$, and by the base values $5, 8, 11$ for $k = 2, 3, 4$, which lie exactly on the line $3k - 1$.

Compression stops at $a = m - 2$: for $a = m - 1$ it would need $\mathrm{BSR}(1, 4) = 4 > 3 \cdot 1 - 1$. For $a = m$ or $m - 1$, the constraints $b \in \{3, 4\}$, $a + 1 \le r \le m + 1$, and $r \ge a + 2$ when $b = 4$ (Lemma 20) leave exactly the following triples; in particular the low-index lemma excludes $(m, 4, m+1)$ and $(m-1, 4, m)$. The remaining triples are

$$(m-1, 4, m+1), \quad (m, 3, m+1), \quad (m-1, 3, m+1), \quad (r$$

Figure 1: Every candidate integer triple (a, b, r) for $m = 10$ (row support, column support, rank of $H \le m + 1$; not all are realized by multipliers), coloured by the step of the proof of Theorem 22 that excludes it, with the number of triples in each class. Small supports, the weak inequality $r \ge a + 1$ and

compression remove all but six triples; the low-index lemma removes two more, and the last four (large markers) each need their own argument. The same partition is checked for every $5 \le m \le 16$ by the script that draws the figure.

For $(m-1, 4, m+1)$, $W = K$. The positive-annihilator lemma has fiber dimension four. Its annihilator has exactly $m-1$ dimensions: an additional map would extend H to supports $(m, 4)$ and rank $m+1$, contradicting the strong low-index inequality. The rank of Ψ is the dimension of its target, $4(m+1)$, minus that of its annihilator (the target carries the nondegenerate form $J^{\oplus 4}$), that is $4(m+1) - (m-1) = 3m+5$; and $p = r - q \le m$ since $q \ge 1$. Hence

$$3m + 5 = \operatorname{rank} \Psi \le 4p - (m-1) \le 3m + 1,$$

impossible.

For $(m, 3, m+1)$, exchange the factors. The fiber dimension is m, the coefficient support has dimension three, and the annihilator has exactly three dimensions by the same extension argument. The target has dimension $m(m+1)$, so $\operatorname{rank}\Psi = m(m+1) - 3$, and again $p \le m$. Hence

$$m^2 + m - 3 = \operatorname{rank} \Psi \le mp - 3 \le m^2 - 3,$$

again impossible.

For the remaining triples choose radical generators x_0, y_0, put $u = x_0 \otimes y_0$, and set

$$U = x_0 \otimes B + A \otimes y_0.$$

For active bases a_i, b_j and $v = \sum v_{ij} a_i \otimes b_j$, the rectangle identities give

$$uv = \sum_{ij} v_{ij} (a_i \otimes y_0)(x_0 \otimes b_j). \tag{R}$$

For $(m-1, 3, m+1)$, $W = K$. Choose summand coordinates with $h_1 = cu$, $c \neq 0$. Take an active v with $H(v,v) < 0$. Equation (R) implies $h_1 v \in WK$, so write

$$-h_1 v = \sum_i h_i k_i, \qquad k_i \in K.$$

Then $D_i = \delta_{i1} v + k_i$ is a syzygy with
$\sum_i H(D_i, D_i) = H(v, v) < 0$, a contradiction.

For $(m-1, 3, m)$, $\dim K = N + 1$, so $W = \ker \lambda$ for a nonzero functional λ on K. Write $K = U \oplus K_0$, with K_0 the active kernel.

If $u \notin W$, then

$$W \cap (x_0 \otimes B) = x_0 \otimes L,$$

where $\dim L = 3$ and L projects isomorphically onto $B/\mathbb{R} y_0$. The restricted multiplier retains rank m, first support $m-1$, and signature (p, q). The positive-annihilator lemma has fiber dimension three. An extra annihilator map would extend first support to m without increasing rank, contradicting $r \geq m+1$. The target has dimension $3m$, so
$\operatorname{rank} \Psi = 3m - (m-1) = 2m+1$, while $p \leq m-1$ since $q \geq 1$. Therefore

$$2m + 1 = \operatorname{rank} \Psi \leq 3p - (m-1) \leq 2m - 2,$$

impossible.

If $u \in W$, arrange $h_1 = cu$. When $\lambda|_U = 0$, equation (R) gives $uv \in WK$ for every active v. Otherwise

$$\operatorname{Sym}^2 U / (U \cap W) U$$

is one-dimensional, measured by $(\lambda|_U)^2$. Thus $uv \in WK$ whenever

$$\varphi(v) = \sum_{ij} v_{ij} \lambda(a_i \otimes y_0) \lambda(x_0 \otimes b_j) = 0.$$

The active covector φ is zero or decomposable. If nonzero, it cannot annihilate K_0, since that would put it in the product-free image of the active multiplier.

Start with a negative active v and, when necessary, add an element of K_0 to impose $\varphi(v) = 0$. Its value under H remains negative. The syzygy construction above again contradicts nonnegative second variation.

All support cases are excluded. Hence $\Sigma_N = \Sigma$.

💡 **In plain words (commentary, not part of the paper)**

This is Theorem 1 proved for every $5 \leq m \leq 16$ at once, by induction on m. The paper’s paragraph «The mechanism in three counts» says why it works

and where it stops.

(a) *The rank budget.* The summands of a shortest representation span an N-dimensional space killed by H, so
$\operatorname{rank} H \le \dim V - N = 4m - (3m-1) = m+1$.

(b) *The room for negative directions.* By the boundary inertia inequality (Lemma 17), $q \le q_* = 1 + \frac{m(m+7)}{2(3m-1)}$; the lemmas of Section 3 handle $q \le 4$, and $\lfloor q_* \rfloor \le 4$ exactly for $m \le 16$ ($q_* = 22/7$ at $m=5$, $231/47$ at $m=16$, $127/25$ at $m=17$).

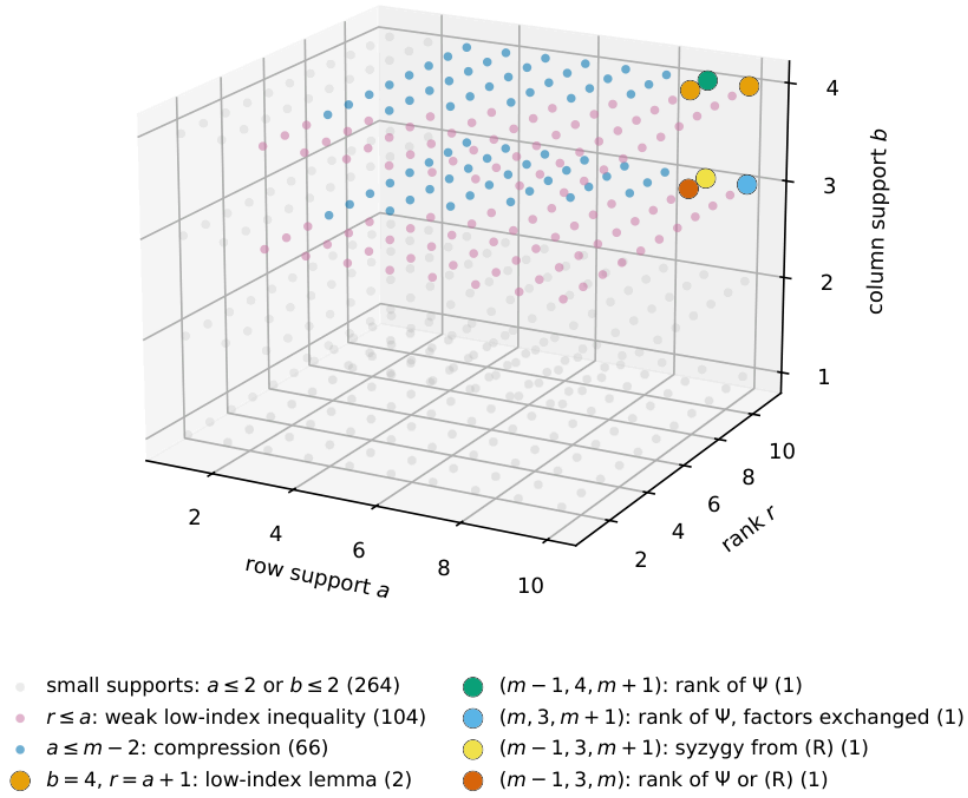
(c) *The induction line.* Compression puts all but at most $4a-r$ summands on the $m-a$ rows outside the row support, so
$\mathrm{SOS}_{\mathbb{R}}(F) \le 4a - r + \mathrm{BSR}(m-a,4)$; when $a \le m-2$, with $r \ge a+1$ and $\mathrm{BSR}(k,4) \le 3k-1$ for $k = m-a \ge 2$, this is at most $3m-2 < N$, a contradiction. The known values $5, 8, 11$ for $k = 2, 3, 4$ lie exactly on the line $3k-1$, so the induction closes; it cannot start from $k=1$, since $\mathrm{BSR}(1,4) = 4 > 2$. That is why the row supports $a = m-1$ and $a = m$ are left to the isotropic and low-index lemmas.

What remains: the low-index lemma ($r \ge a+2$ when $b=4$) removes $(m,4,m+1)$ and $(m-1,4,m)$, and the four triples $(m-1,4,m+1)$, $(m,3,m+1)$, $(m-1,3,m+1)$, $(m-1,3,m)$ are excluded one by one, by the rank of the alternating map Ψ against the positive index, or by a syzygy whose second variation is negative. Figure 5 colours the candidate triples for $m=10$ by the step that excludes them; the lab below counts them for any $5 \le m \le 16$ (for $m=10$: 264, 104, 66, 2 and four single triples, 440 in all).

Evidence & scope

- **Program:** Figure 5 recomputed: every candidate triple in exactly one class for each $5 \le m \le 16$, the last four and the two of the low-index lemma exactly those of the proof, and the class counts 264, 104, 66, 2, 1, 1, 1, 1 at $m = 10$; with controls that must fail.
  `lab/comprueba_lab.py`

Figure 5



Every candidate integer triple (a, b, r) for $m = 10$ (row support, column support, rank of $H \leq m + 1$; not all are realized by multipliers), coloured by the step of the proof of Theorem 22 that excludes it, with the number of triples in each class. Small supports, the weak inequality $r \geq a + 1$ and compression remove all but six triples; the low-index lemma removes two more, and the last four (large markers) each need their own argument. The same partition is checked for every $5 \leq m \leq 16$ by the script that draws the figure.

4 Perspectives

Proposition 23. (Negative-syzygy obstruction) p. 26

> *BSR_m4_Trace.trace_pos, BSR_m4_Trace.syzygy_negative Let F be a smooth boundary point of Σ_s with multiplier H as in Proposition 10, and let $W \subseteq K = \ker H$ be the s-dimensional span of the summands of a representation of F. Let N be a q-dimensional subspace on which H is negative definite, $D = K \oplus N$ and $k = \dim K$. Then*

$$i(D) - i(K) \;\le\; \operatorname{rk}\bigl(I_2 \cap \operatorname{Sym}^2 D \to \operatorname{Sym}^2(D/W)\bigr) \;\le\; \binom{k-s}$$

Isabelle · ours · part **show**

Only a part is in Isabelle: the trace inequality behind the second-variation step; the dimension count is in the text, and the certificates of Appendix B are an exact computation

`trace_pos` · `Isabelle/BSRFourColumnsTrace/BSR_m4_Trace.thy`, line 229

```
theorem trace_pos:
  fixes Y G :: "real^'n::finite^'n"
  assumes Y: "mpd Y" and G: "mpsd G" and G0: "G \<noteq> 0"
  shows "0 < mtr Y G"
```

`syzygy_negative` · `Isabelle/BSRFourColumnsTrace/BSR_m4_Trace.thy`, line 284

```
theorem syzygy_negative:
  fixes H :: "'a::real_vector \<Rightarrow> 'a \
<Rightarrow> real"
    and nv :: "'k::finite \<Rightarrow> 'a"
    and Y :: "real^'k^'k"
  assumes Hb: "bilinear H" and Hs: "\<And>x y. H x y = H y
x"
    and N: "subspace N" and Hneg: "\<And>x. x \<in> N \
<Longrightarrow> x \<noteq> 0 \<Longrightarrow> H x x < 0"
    and nN: "\<And>i. nv i \<in> N" and nz: "\<exists>i. nv
i \<noteq> 0"
    and Y: "mpd Y"
  shows "(\<Sum>i\<in>UNIV. \<Sum>j\<in>UNIV. Y $ i $ j * H
(nv i) (nv j)) < 0"
```

↳ Not used by another result of the paper.

Proof

Choose a complement C of W in D, so that $\operatorname{Sym}^2 D = (W \odot D) \oplus \operatorname{Sym}^2 C$ and $\operatorname{Sym}^2 C \cong \operatorname{Sym}^2(D/W)$. The kernel of the projection of $I_2 \cap \operatorname{Sym}^2 D$ onto $\operatorname{Sym}^2 C$ is $I_2 \cap (W \odot D)$. Its subspace $I_2 \cap (W \odot K)$ has dimension at most $i(K)$. If the displayed inequality failed, some element of $I_2 \cap (W \odot D)$ would not lie in $W \odot K$. Writing it as $\sum_i h_i \odot g_i$, with $h_1, \dots, h_s$ orthonormal summands of the representation, gives a syzygy $\sum_i h_i g_i = 0$ in R_2 with $g_i = k_i + n_i \in K \oplus N$ and not all n_i zero. Since K is the radical of H, its second variation is $\sum_i H(g_i, g_i) = \sum_i H(n_i, n_i) < 0$, against the nonnegative second variation of Proposition 10.

A A second proof for $m = 5$

Lemma 24. (Degenerate hyperplane calculation) p. 27

Suppose $\dim A = d$, $\dim E = d+1$, *a generic* T_b *is injective, and every* $Q(b)$ *is singular. Choose a Witt decomposition*

$$E = W \perp \langle e, f\rangle, \quad \dim W = d-1, \quad e^2 = f^2 = 0, \quad \langle e, f\rangle = 1$$

and coordinates with $T_b(w, z) = (w, z, 0)$. *Then, for every* c,

$$T_c(w,z) = (A_c w + p_c z,\ \ell_c^t w + t_c z,\ p_c^t w + \eta_c z), \quad A_c^t = A_c,$$

and

$$\eta_c = 0, \qquad p_c^t p_c = 0, \qquad p_c^t A_c p_c = 0. \tag{6}$$

↳ Used in: Theorem 25, Lemma 27

Proof

A degenerate hyperplane in a nondegenerate $(d+1)$-space has a one-dimensional radical, so the stated Witt coordinates exist. Flatness with b gives the displayed form of T_c. Suppress the subscript c. The blocks of $Q(b+\tau c)$ are the following exact multiplication, computation C2:

$$\begin{aligned} \mathcal{M} &= I + 2\tau A + \tau^2(A^2 + \ell p^t + p\ell^t),\\ \nu &= 2\tau p + \tau^2(Ap + tp + \eta\ell),\\ \omega &= 2\tau\eta + \tau^2(p^t p + 2t\eta). \end{aligned}$$

Since $\mathcal{M}(0) = I$, the formal Schur complement $\omega - \nu^t\mathcal{M}^{-1}\nu$ is defined. With $\nu = \tau\nu_1 + \tau^2\nu_2$, $\nu_1 = 2p$, $\nu_2 = Ap + tp + \eta\ell$ and $\mathcal{M}^{-1} = I - 2\tau A + O(\tau^2)$,

$$\nu^t\mathcal{M}^{-1}\nu = 4\tau^2 p^t p + \tau^3\left(2\nu_1^t\nu_2 - 2\nu_1^t A\nu_1\right) + O(\tau^4) = 4\tau^2 p^t p$$

and ω has no τ^3 term. Hence the first three coefficients of $\omega - \nu^t\mathcal{M}^{-1}\nu$ are

$$2\eta, \quad 2t\eta - 3p^t p, \quad 4p^t A p - 4tp^t p - 4\eta p^t\ell.$$

> Its vanishing follows from $\det Q(b+\tau c)=0$. Successively setting these coefficients to zero proves (6).

Theorem 25. (Full-support rank-six exclusion) p. 28

> *BSR_m4_Linear.symmetric_trilinear_polarization, BSR_m4_C3.c3_tensor_vanishes, BSR_m4_C3.c3_inclusion*
> *There is no complex Hankel multiplier on $\mathbb{C}^5 \otimes \mathbb{C}^4$ of rank six, supports $(5,4)$, and product-free image.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the polarization of symmetric trilinear forms (case (ii)) and computation C3; the rest is proved in the text only

`symmetric_trilinear_polarization` · `Isabelle/BSRFourColumnsLinear/BSR_m4_Linear.thy`, line 361

```
theorem symmetric_trilinear_polarization:
  fixes B :: "'v::ab_group_add \<Rightarrow> 'v \
<Rightarrow> 'v \<Rightarrow> 'k::field_char_0"
  assumes add1: "\<And>x x' y z. B (x + x') y z = B x y z +
B x' y z"
    and s12: "\<And>x y z. B x y z = B y x z"
    and s23: "\<And>x y z. B x y z = B x z y"
    and diag: "\<And>x. B x x x = 0"
  shows "B x y z = 0"
```

`c3_tensor_vanishes` · `Isabelle/BSRFourColumnsC3/BSR_m4_C3.thy`, line 251

```
theorem c3_tensor_vanishes:
  fixes p :: "'b::finite \<Rightarrow> 'a::field_char_0^'n"
and A :: "'b \<Rightarrow> 'a^'n^'n" and t :: "'b \
<Rightarrow> 'a"
  assumes sym: "\<And>i. transpose (A i) = A i"
    and flat: "\<And>i j. A i *v p j + t j *s p i = A j *v
p i + t i *s p j"
    and iso: "\<And>i j. dotv (p i) (p j) = 0"
    and cubic: "\<And>c. dotv (pc p c) ((Ac A c - msc (tc t
c) (mat 1)) *v pc p c) = 0"
  shows "dotv (p i) ((A j - msc (t j) (mat 1)) *v p k) = 0"
```

`c3_inclusion` · `Isabelle/BSRFourColumnsC3/BSR_m4_C3.thy`, line 293

```
theorem c3_inclusion:
  fixes p :: "'b::finite \<Rightarrow> 'a::field_char_0^'n"
and A :: "'b \<Rightarrow> 'a^'n^'n" and t :: "'b \
<Rightarrow> 'a"
  assumes sym: "\<And>i. transpose (A i) = A i"
    and flat: "\<And>i j. A i *v p j + t j *s p i = A j *v
p i + t i *s p j"
    and iso: "\<And>i j. dotv (p i) (p j) = 0"
    and cubic: "\<And>c. dotv (pc p c) ((Ac A c - msc (tc t
c) (mat 1)) *v pc p c) = 0"
    and maxiso: "\<And>v. (\<forall>i. dotv v (p i) = 0) \
<Longrightarrow> \<exists>l. v = (\<Sum>i\<in>UNIV. l i *s
p i)"
```

```
  shows "\<exists>\<nu>. Ac A c *v (\<Sum>i\<in>UNIV. \<mu>
i *s p i) = (\<Sum>i\<in>UNIV. \<nu> i *s p i)"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box).
Isabelle: 3 results, no oracles

Proof: in this note · **Covers:** not formalized

↳ Not used by another result of the paper.

Proof

Suppose such a multiplier exists. Use s, U, Q from (1)–(3).

Case (i): some $Q(b)$ is invertible. Lemma 6, with $d = 5$, gives a contradiction.

Case (ii): $s = 5$, but every $Q(b)$ is singular. Apply Lemma 24; now $W = \mathbb{C}^4$. The vectors p_c, linear in c, span a totally isotropic space $P \subset W$ by (6) and polarization, so $\dim P \leq 2$. If $P = 0$, then $T^*Je = 0$, contrary to surjectivity. If $\dim P = 1$, then

$$(T^*Je)(w, z; c) = p_c^t w$$

is a nonzero decomposable covector. It remains to exclude $\dim P = 2$.

The mixed w, z coefficients of flatness give, for all i, j,

$$A_i p_j + t_j p_i = A_j p_i + t_i p_j.$$

Set $\widetilde{A}_i = A_i - t_i I$. The tensor $\mathcal{B}(i, j, k) = p_i^t \widetilde{A}_j p_k$ is symmetric in all three indices: symmetry of A_j interchanges the first and third indices, and $\widetilde{A}_j p_k = \widetilde{A}_k p_j$ interchanges the last two. Its diagonal cubic is zero by (6) and total isotropy of P. Polarization therefore gives $\mathcal{B} = 0$. It follows that $\widetilde{A}_c P \subset P^\perp = P$, and hence $A_c P \subset P$. This polarization and inclusion are computation C3. For $D = P \oplus \mathbb{C}z \subset A$, of dimension three, the formula for T_c now gives

$$T(D \otimes B) \subset P \oplus \mathbb{C}e.$$

The right-hand space is totally isotropic. Lemma 5 applies since $\dim(A/D) = 2$, giving a contradiction.

Case (iii): $s \leq 4$. Since $\ker T_b \neq 0$, (2)–(3) force exactly the alternatives

$$s = 3, \quad \operatorname{rank} Q(b) = 1; \qquad s = 4, \quad \operatorname{rank} Q(b) = 2. \quad (7$$

In the first alternative, Lemma 7 either gives first support one or writes $q(x, b) = gL(x, b)^2$ with g constant and L bilinear. A 5×4 bilinear form has first support at most four. Both conclusions contradict first support five.

For the second alternative, let $\mathbb{C}x(b) = \ker T_b$ generically, and let $N_b = U \cap U^{\perp}$, of dimension two. Equation (1) says $T_c x(b) \in N_b$ for every c; (2) makes the map $c \mapsto T_c x(b)$ have rank exactly two. Choose a rational local representative $x(b)$. Differentiating $T_b x(b) = 0$ gives

$$T_b(dx_b(c)) = -T_c x(b).$$

Because T_b identifies $A/\mathbb{C}x(b)$ with U, the projective kernel map $\mathbb{P}(B) \dashrightarrow \mathbb{P}(A)$ has generic differential rank two. In characteristic zero its image closure X is an irreducible surface. Its linear span $\mathbb{P}(D)$ has $\dim D \geq 3$.

For every c, $q(x(b), c) = 0$, since N_b is isotropic. Also $\operatorname{rank} Q(c) \leq 2$ for every c, by closedness of rank conditions. Each nonzero quadric $q(\,\cdot\,, c)$ therefore factors into two linear forms over $\mathbb{C}$. Irreducibility puts X, and hence its span, in one of the two hyperplanes. Thus $q(d, c) = 0$ for all $d \in D, c \in B$. At least one such quadric is nonzero, so $\dim D \leq 4$. Polarizing first in d, then in c, and using (H), gives

$$H(D \otimes B, D \otimes B) = 0.$$

Consequently $T(D \otimes B)$ is totally isotropic, while $\dim(A/D) \leq 2$. Lemma 5 supplies the contradiction.

Lemma 26. (Support bounds) p. 29

BSR_m4_Pieces.support_triples_m4, BSR_m4_Pieces.triples_5, BSR_m4_Pieces.triples_6, BSR_m4_Linear3.support_count_inequality A nonzero product-free multiplier with supports $a \leq 5$, $b_0 \leq 4$ satisfies

$$a, b_0 \geq 2, \qquad \max(a, b_0) + 1 \leq r \leq (a - 1)(b_0 - 1). \quad (8)$$

Isabelle · ours · part **show**

Only a part is in Isabelle: the enumeration of the support triples and the support count; the bounds themselves are

proved in the text

support_triples_m4 · Isabelle/BSRFourColumnsPieces/BSR_m4_Pieces.thy, line 50

```
theorem support_triples_m4:
  fixes m a b r :: nat
  assumes m: "5 \<le> m" and "(a,b,r) \<in> triples m" and
"m \<le> a + 1" and "3 \<le> b"
  shows "(a,b,r) \<in> {(m,4,m+1), (m-1,4,m), (m-1,4,m+1),
(m,3,m+1), (m-1,3,m+1), (m-1,3,m)}"
```

triples_5 · Isabelle/BSRFourColumnsPieces/BSR_m4_Pieces.thy, line 33

```
lemma triples_5: "triples 5 = {(3,3,4),(3,4,5),(3,4,6),
(4,3,5),(4,3,6),(4,4,5),(4,4,6),(5,3,6),(5,4,6)}"
```

triples_6 · Isabelle/BSRFourColumnsPieces/BSR_m4_Pieces.thy, line 36

```
lemma triples_6: "triples 6 =
  {(3,3,4),(3,4,5),(3,4,6),(4,3,5),(4,3,6),(4,4,5),(4,4,6),
(4,4,7),(5,3,6),(5,3,7),(5,4,6),(5,4,7),(6,3,7),(6,4,7)}"
```

support_count_inequality · Isabelle/BSRFourColumnsLinear3/BSR_m4_Linear3.thy, line 564

```
theorem support_count_inequality:
  fixes a b r :: nat
  assumes a: "1 \<le> a" and b: "1 \<le> b" and r: "1 \<le>
r"
    and hyp: "a * b - 1 \<le> (r - 1) + (a + b - 2)"
  shows "(a - 1) * (b - 1) + 1 \<le> r"
```

Trust base: ⚙ kernel + code evaluation — **only part of the statement is formal** (see the box). Isabelle: 2 results, no oracles · Isabelle, oracles: triples_5 (Code_Generator.holds_by_evaluation); triples_6 (Code_Generator.holds_by_evaluation)

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the enumeration of the support triples and the support count; the bounds themselves are proved in the text

🆕 **Classical and new:** the paper, remark (ii): the bound rank $H \le (a-1)(b-1)$ is Parthasarathy's maximal dimension of a completely entangled subspace [Par].

↳ Not used by another result of the paper.

Proof

Restrict to complements of the factor radicals; this preserves rank and product-freeness. If a factor has dimension one, every

nonzero image covector is decomposable. Otherwise the projective image, of dimension $r-1$, is disjoint from the Segre variety of dimension $a+b_0-2$ in $\mathbb{P}^{ab_0-1}$. The projective dimension theorem gives the upper bound in (8).

For the first lower bound, suppose $r \le a \le 4$. If generic T_b is not injective, (2)–(3) contradict $\min(s-1, r-s) \le 1$. If it is injective, then $r = a$ and it is an isomorphism onto E. Identify $A = E$ by this map. Flatness makes the operators $T_b^{-1}T_c$ self-adjoint and pairwise commuting. A common eigenvector contradicts Lemma 3. Exchanging the two factors proves $r \ge b_0 + 1$, since $b_0 \le 4$.

For $a = 5$, suppose $r \le 5$. The injective case has just been settled. Otherwise (2)–(3) force $r = 5$, $s = 3$ and generic $\operatorname{rank} Q = 1$. Lemma 7 gives first support one or at most $b_0 \le 4$, a contradiction. This proves the repaired first-support-five lower bound.

Lemma 27. (Rank five in 4×4) p. 29

BSR_m4_Linear.rank5_polar_kernel, BSR_m4_Pieces.triples_5 A rank-five multiplier with supports $(4,4)$ cannot have product-free image.

Isabelle · ours · part **show**

Only a part is in Isabelle: the kernel of the polar map (9); the rest is proved in the text

`rank5_polar_kernel` · `Isabelle/BSRFourColumnsLinear/BSR_m4_Linear.thy`, line 27

```
theorem rank5_polar_kernel:
  fixes X :: "nat \<Rightarrow> nat \<Rightarrow>
'a::field_char_0"
  assumes zero: "\<And>i j. i < 4 \<Longrightarrow> j < 4 \
<Longrightarrow> (\<Sum>k<4. X k k) * (if i = j then 1 else
0) + X j i = 0"
    and ij: "i < 4" "j < 4"
  shows "X i j = 0"
```

`triples_5` · `Isabelle/BSRFourColumnsPieces/BSR_m4_Pieces.thy`, line 33

```
lemma triples_5: "triples 5 = {(3,3,4),(3,4,5),(3,4,6),
(4,3,5),(4,3,6),(4,4,5),(4,4,6),(5,3,6),(5,4,6)}"
```

Trust base: kernel + code evaluation — **only part of the statement is formal** (see the box). Isabelle: 1 result, no oracles · Isabelle, oracles: triples_5

(Code_Generator.holds_by_evaluation)

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the kernel of the polar map (9); the rest is proved in the text

↳ Not used by another result of the paper.

Proof

Restrict H to $A' \otimes B$, with A' a complement of the first radical ($\dim A' = 4$); rank, flatness and product-freeness are kept, as in the support lemma. An invertible slice is excluded by Lemma 6 with $d = 4$. If generic T_b has rank four but every $Q(b)$ is singular, apply Lemma 24. The p_c span a totally isotropic subspace of the three-dimensional nondegenerate space W, hence have span of dimension at most one. Span zero gives $T^{*}Je = 0$, contradicting surjectivity; span one makes $T^{*}Je$ decomposable and nonzero.

Otherwise (2)–(3), with $r = 5$, force $s = 3$ and generic $\operatorname{rank} Q = 1$. Lemma 7 either gives first support one or $q(x, b) = g(x^{t}Pb)^{2}$, with $g \neq 0$ constant. Full first support makes the 4×4 matrix P invertible. Change bases and rescale to make $q = (x^{t}b)^{2}$. Its Hankel polar map on 4×4 matrices is

$$X \longmapsto \tfrac{1}{2}\big(\operatorname{tr}(X)I + X^{t}\big). \tag{9}$$

This identity is computation C4. If its value is zero, $X^{t} = -\operatorname{tr}(X)I$; taking traces gives $5\operatorname{tr}(X) = 0$, and then $X = 0$. Thus its rank is sixteen, contradicting rank five.

B Certificates for the negative-syzygy obstruction

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper, p. 31.

C Formula sheet

No numbered statement in this section.

Its figures and tables are below; its text is in the paper, p. 32.

quantity	formula
main bound (Theorem 1)	$\mathrm{BSR}(m,4) \le 3m-1$ for $5 \le m \le 16$; previous $\mathrm{BSR}(m,n) \le m + \mathrm{BSR}(m,n-1)$, so $3m$
dimensions ($m \times n$)	$\dim V = mn$, $\dim R_2 = M = \binom{m+1}{2}\binom{n+1}{2}$; for $m \times 4$, $M = 5m(m+1)$, $N = 3m-1$
boundary inertia (Lemma 14)	$N(q-1) \le M - \binom{N+2}{2}$; for $m \times 4$: $2(3m-1)(q-1) \le m(m+7)$, i.e. $q \le \lfloor (m+13)/6 \rfloor$ ($m \ge 5$); ≤ 4 iff $m \le 16$, ≤ 5 iff $m \le 22$
Barvinok–Pataki (case $q=1$)	$\Sigma_N = \Sigma$ for $N \ge r_P = \max\{r : \binom{r+1}{2} \le M\}$
rank of the multiplier	$\mathrm{rank} H \le \dim V - N = m+1$; product-free image: $\mathrm{rank} H \le (a-1)(b-1)$ (Parthasarathy)
compression (Lemma 13)	$\mathrm{SOS}_{\mathbb{R}}(F) \le (4a-r) + \mathrm{BSR}(m-a,4) \le 3m-2$ for $a \le m-2$, $r \ge a+1$
low-index lemma (Lemma 20)	$q \le 4$: $r \ge a+1$, and $r \ge a+2$ when $b=4$
remaining triples	$(m-1,4,m+1)$, $(m,3,m+1)$, $(m-1,3,m+1)$, $(m-1,3,m)$ (Figure 5)
singular incidence	dimension $\le M - 2(m+3) + (2m+4) = M-2$, codimension two (Proposition 10, proof of Theorem 22); used through Lemma 9
slopes	$\mathrm{BSR}(m,2) = m+1$, $\mathrm{BSR}(m,3) = 2m$, here $3m-1$; Pataki slope $\sqrt{n(n+1)/2}$; $(n-1)^2 > n(n+1)/2$ iff $n \ge 5$

The companion site

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper, p. 33.

Acknowledgement

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper (PDF).

Tables

Every table of the paper, in its order: click one to unfold it, with where the paper cites it, how its numbers were checked, and its data as CSV. The typeset table is one click away, on its page of the PDF.

Table without a number, in the text of §1 The results

m	5	6	7	8	9	10	11	12	13	14	15	16
$\mathrm{BSR}(m,4) \le$	**14**	**17**	**20**	**23**	**26**	**29**	**32**	**35**	**38**	**41**	**44**	**47**
previous, $3m$	15	18	21	24	27	30	33	36	39	42	45	48
lower bound	13	16	19	21	24	27	29	32	35	37	40	43

Colour: blue shade = the number's size within its column.

⬇ CSV

Table without a number, in the text of §1 The results

m	$N = 3m-1$	**rank budget** $m+1$	**room** $m(m+7)/2$	q_*	$\lfloor q_* \rfloor$
5	14	6	30	$22/7 \approx 3.14$	3
10	29	11	85	$114/29 \approx 3.93$	3
16	47	17	184	$231/47 \approx 4.91$	4
17	50	18	204	$127/25 = 5.08$	5
22	65	23	319	$384/65 \approx 5.91$	5
23	68	24	345	$413/68 \approx 6.07$	6

Colour: blue shade = the number's size within its column.

⬇ CSV

Table 1. The formal components in Isabelle/HOL (six sessions, none with unproved steps) and the statements of this paper that use them.

Session	Main statements
BoundaryInertia	inertia_pataki_form, inertia_m4, inertia_m4_floor, q_values; incidence, compression_count, psi_counts
BSRFourColumns\-Pieces	support_triples_m4, regular_point_m4, cross_product_identity
BSRFourColumns\-Linear	rank5_polar_kernel, singular_2x2_space, symmetric_trilinear_polarization
BSRFourColumns\-Linear2	line_decomposable, rank_one_sum_minor, common_line_step_kernel, common_line_step_image, pencil_core
BSRFourColumns\-Linear3	symmetric_4linear_polarization, symmetric_5linear_polarization; positive_averaging_nonneg, diagonal_terms_vanish; rank_le_one_space_full, support_count_inequality
BSRFourColumns\-SmallRows	row_rank_one_simple, evaluation_step, row_in_line, small_rows_contradiction

p. 7 in the PDF · ⬇ CSV · cited in §1 The results

Table without a number, in the text of §B Certificates for the negative-syzygy obstruction

(a,b,r)	(p,q)	k	q	$i(D)$	$i(K)\le$	loss $\le$	$i(D)-i(K)-\text{loss}\ge$
$(3,3,4)$	$(1,3)$	16	3	40	16	21	3
$(4,3,5)$	$(2,3)$	15	3	26	9	15	2
$(4,3,5)$	$(1,4)$	15	4	40	9	21	10

$(4,4,6)$	$(2,4)$	14	4	22	4	15	3
$(5,3,6)$	$(4,2)$	14	2	10	3	6	1
$(5,3,6)$	$(3,3)$	14	3	15	3	10	2
$(5,3,6)$	$(2,4)$	14	4	21	3	15	3
$(4,4,6)$	$(4,2)$	14	2	8	4	3*	1
$(4,4,6)$	$(3,3)$	14	3	12	4	4*	4
$(4,3,5)$	$(3,2)$	15	2	18	9	7*	2
$(3,3,4)$	$(2,2)$	16	2	29	16	12*	1

Colour: blue shade = the number's size within its column.

⬇ CSV

Table without a number, in the text of §C Formula sheet

quantity	formula
main bound (Theorem 1)	$\boxed{\mathrm{BSR}(m,4) \le 3m-1}$ for $5 \le m \le 16$; previous $\mathrm{BSR}(m,n) \le m + \mathrm{BSR}(m,n-1)$, so $3m$
dimensions ($m \times n$)	$\dim V = mn$, $\dim R_2 = M = \binom{m+1}{2}\binom{n+1}{2}$; for $m \times 4$, $M = 5m(m+1)$, $N = 3m-1$
boundary inertia (Lemma 14)	$N(q-1) \le M - \binom{N+2}{2}$; for $m \times 4$: $2(3m-1)(q-1) \le m(m+7)$, i.e. $q \le \lfloor (m+13)/6 \rfloor$ $(m \ge 5)$; ≤ 4 iff $m \le 16$, ≤ 5 iff $m \le 22$
Barvinok–Pataki (case $q=1$)	$\Sigma_N = \Sigma$ for $N \ge r_P = \max\{r : \binom{r+1}{2} \le M\}$
rank of the multiplier	$\mathrm{rank} H \le \dim V - N = m+1$; product-free image: $\mathrm{rank} H \le (a-1)(b-1)$ (Parthasarathy)
compression (Lemma 13)	$\mathrm{SOS}_{\mathbb{R}}(F) \le (4a-r) + \mathrm{BSR}(m-a,4) \le 3m-2$ for $a \le m-2$, $r \ge a+1$
low-index lemma (Lemma 20)	$q \le 4$: $r \ge a+1$, and $r \ge a+2$ when $b=4$
remaining triples	$(m-1,4,m+1)$, $(m,3,m+1)$, $(m-1,3,m+1)$, $(m-1,3,m)$ (Figure 5)
singular incidence	dimension $\le M - 2(m+3) + (2m+4) = M-2$, codimension two (Proposition 10, proof of Theorem 22); used through Lemma 9
slopes	$\mathrm{BSR}(m,2) = m+1$, $\mathrm{BSR}(m,3) = 2m$, here $3m-1$; Pataki slope $\sqrt{n(n+1)/2}$; $(n-1)^2 > n(n+1)/2$ iff $n \ge 5$

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