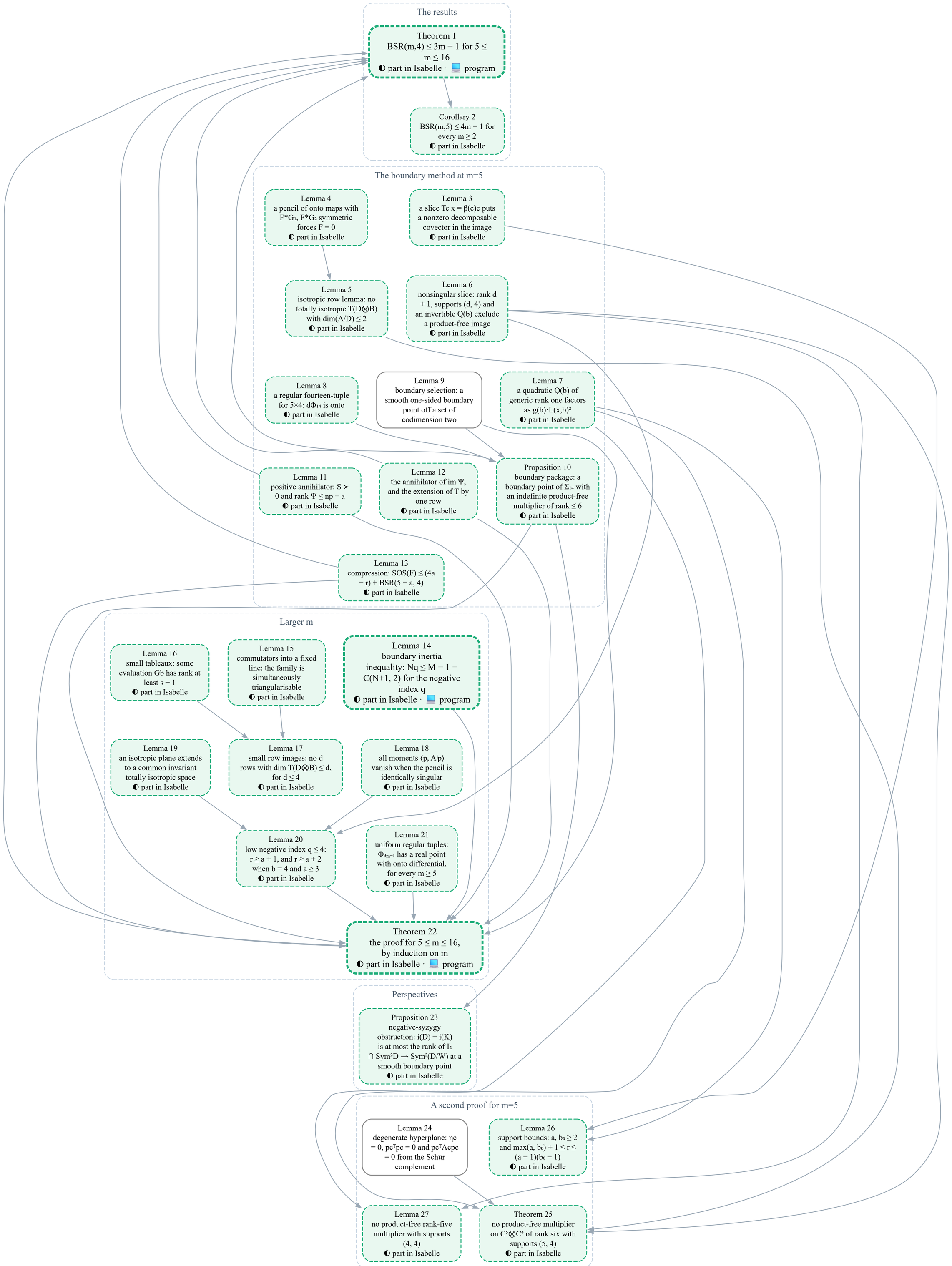
Map of *Every m×4 biquadratic sum of squares is a sum of 3m−1 squares (5 ≤ m ≤ 16)*

Real algebraic geometry · sums of squares on projective varieties

27 results | 0 entirely in Lean | ◐ 25 partly in Lean | 3 checked by a program | written proof only | (a result can carry several tags)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



What the paper contributes

BSR(m,4) ≤ 3m − 1 for 5 ≤ m ≤ 16 (Theorem 1): every real m×4 biquadratic sum of squares is a sum of at most 3m − 1 squares of bilinear forms, one square below the previous bound 3m. In particular BSR(5,4) ≤ 14, the upper half of the conjecture BSR(5,4) = 14 of Qi and Ouyang; with the known lower bounds, 13 ≤ BSR(5,4) ≤ 14 and 16 ≤ BSR(6,4) ≤ 17. The paper does not decide whether BSR(5,4) is 13 or 14.

- **A boundary inertia inequality** (Lemma 17): at a boundary point of the sums of N squares inside the cone, the negative index q of the Hankel multiplier satisfies $N(q-1) \le \dim R_2 - C(N+2, 2)$. The paper says it is, in substance, Theorem 3.4 of Boumal, Voroninski and Bandeira, transported to this setting; its case q = 1 recovers the Barvinok–Pataki bound. For m×4 forms it gives $q \le \lfloor (m+13)/6 \rfloor$, which is at most 4 exactly for m ≤ 16.
- **A case analysis over the support triples (a, b, r)** (Theorem 25): compression to fewer rows for a ≤ m − 2, with the base values BSR(k,4) ≤ 3k − 1 for k = 2, 3, 4; a four-row isotropic lemma (Lemma 20) and a low-negative-index lemma for q ≤ 4 (Lemma 23) for the rest.
- **Explicit regular tuples for every m** (Lemma 24) and a boundary selection lemma whose codimension-two hypothesis is sharp (Lemma 12).
- **Where the method stops:** at m = 17 the inertia bound allows q = 5.
- **Machine checks:** the arithmetic, the enumeration of cases and several linear-algebra steps are checked in Isabelle/HOL (six sessions) or by exact computation; these checks cover individual steps, not the whole theorem.

🚀 **Start here: Theorem 1**: BSR(m,4) ≤ 3m − 1 for 5 ≤ m ≤ 16.

Main results:

- **Theorem 1**: BSR(m,4) ≤ 3m − 1 for 5 ≤ m ≤ 16
- **Lemma 14**: boundary inertia inequality: Nq ≤ M − 1 − C(N+1, 2) for the negative index q
- **Theorem 22**: the proof for 5 ≤ m ≤ 16, by induction on m

🧭 Fields: where the paper lives and what it connects

Tools taken from other branches

- **Boundary strategy for Pythagoras numbers (Blekherman–Dunbar–Sinn)** → a one-sided boundary point of the sums of 3m − 1 squares inside the cone, off the singular incidence (codimension two), carries an indefinite Hankel multiplier with product-free image and nonnegative second variation on syzygies — Proposition 10, Lemma 9, Lemma 8, Lemma 21
- **Second-order conditions (Burer–Monteiro, Boumal–Voroninski–Bandeira)** → counting syzygies bounds the negative index of the multiplier: $N(q-1) \le \dim R_2 -$

C(N+2, 2), whose case q = 1 is the Barvinok–Pataki bound — Lemma 14
- **Linear algebra over C** → pencils, totally isotropic subspaces, simultaneous triangularisation with commutators into a line, linear spaces of matrices of rank at most one, formal Schur complements and their moments — Lemma 4, Lemma 5, Lemma 15, Lemma 16, Lemma 17, Lemma 18, Lemma 19
- **Completely entangled subspaces (Parthasarathy)** → a product-free image has rank at most (a − 1)(b − 1) — Lemma 26

What the results serve

- **SOS rank of biquadratic forms** ← BSR(m,4) ≤ 3m − 1 for 5 ≤ m ≤ 16; BSR(5,4) ≤ 14 (upper half of the Qi–Ouyang conjecture), 13 ≤ BSR(5,4) ≤ 14 and 16 ≤ BSR(6,4) ≤ 17 — Theorem 1, Theorem 22