

Characters of $\mathrm{Sp}(2m)$ at the powers of Kostant's element



Representation theory · characters of Lie groups

05E10

20G05

17B22

05A15

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At a glance

What the paper contributes

Kostant's element a_p is a point where every irreducible character of a compact simple group takes a value in $\{0, \pm 1\}$. The paper determines the irreducible characters of $Sp(2m)$ at **every power** a_p^k .

- **When a character vanishes, and its value** (Theorem A): it vanishes exactly when $\lambda + \rho$ lies on more walls of the modular root arrangement than ρ does, equivalently when some folded residue class holds more entries than its capacity; otherwise the value is an explicit signed ratio of root products.
- **The absolute value is a product of dimensions** of representations of GL, Sp and odd SO, times $\kappa \in \{1, 2\}$ (Theorem B).
- **Every rational point of the same line:** a non-vanishing character is, up to sign and that factor, such a product times a character of $Sp(2s)$ or $SO(2s)$, $s \leq m$, at a regular element (Theorem 10.1); the points where the closed form holds for all weights are listed (Theorem C), and at the points where all characters are integers the remaining factor is 1, 2, 3, 4 or 6 (Theorem D).

Known and new, as the paper states it. For k prime to $2m+2$, Theorem A is due to Cellini, Möseneder Frajria and Papi; for $\gcd(k, 2m+2)$ odd, Theorems A and B follow from Albion's factorisations of universal characters, with the sign added. New: for $\gcd(k, 2m+2)$ even and $q \geq 3$, the values on the classes not conjugate to that of ρ and the factor κ ; the list of points in Theorem C and the exclusion of the others; in Theorem D, the constants for $q \geq 4$ even with $q \mid 2m$, $q \nmid m$, and for $q = 6$ with $m \equiv 4 \pmod{6}$.

Key tool: an identity of residue multisets (Proposition 5.1), which takes the place of the conjugacy hypothesis in the work of Nadimpalli, Pattanayak and Prasad.

 **Start here: Theorem A:** when the character vanishes, and its value.

Main results:

- **Theorem A:** when the character vanishes, and its value
- **Theorem B:** $|\text{value}| = \kappa \times \text{product of dimensions}$
- **Theorem C:** where the closed form holds for all weights
- **Theorem D:** the constant $c \in \{1, 2, 3, 4, 6\}$

12 results

✓ 0 formalized in Lean

● 0 with a part in Lean

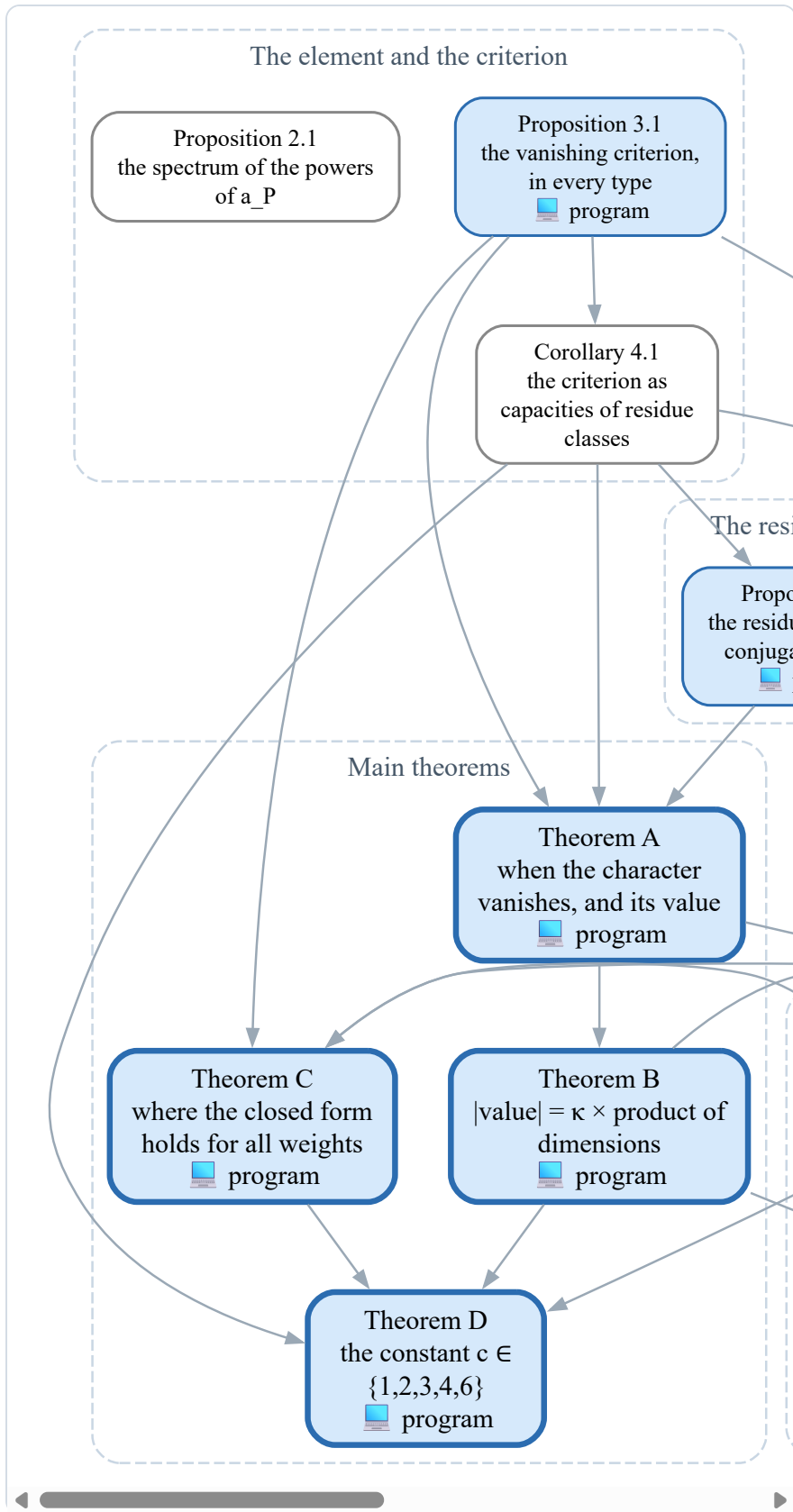
 10 checked by a program

written proof only

(a result can carry several tags)

Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow $A \rightarrow B$ means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



Setting (copied from the paper)

Let $\rho = (m, m-1, \dots, 1)$, with the form normalised so that the short roots $e_i \pm e_j$ have length 2. $g_\theta \in \text{Sp}(2m)$ has eigenvalues $e^{\pm 2\pi i \theta_j}$, $j = 1, \dots, m$; $t = 2m + 2$; Kostant's element is $a_P = g_{1/t}$, and $a_P^k = g_{p/q}$ with $d = \gcd(k, t)$, $q = t/d$, $p = k/d$. For $\ell = \lambda + \rho$: $N_q(x) = \#\{\alpha > 0 : q \mid (x, \alpha)\}$, $Z_q(x) = \prod_{\alpha > 0, q \mid (x, \alpha)} (x, \alpha)$,

$E_{p,q}(\lambda) = \sum_{\alpha>0} (\lfloor p(\ell, \alpha)/q \rfloor - \lfloor p(\rho, \alpha)/q \rfloor)$, over the positive roots $e_i \pm e_j$ ($i < j$) and $2e_i$.
Folded classes $\{c, -c\}$ modulo q ; fixed if $2c \equiv 0$.

1 Main theorems

Theorem A. p. 2

Let λ be a dominant weight of $\mathrm{Sp}(2m)$, $k \geq 1$, and d, q, p as above.

1. $N_q(\ell) \geq N_q(\rho)$, and $\chi_\lambda(a_P^k) \neq 0$ if and only if $N_q(\ell) = N_q(\rho)$. Equivalently, each folded class modulo q contains at most d of the entries $\ell_1, \dots, \ell_m$, and each fixed class at most $\lfloor d/2 \rfloor$.
2. If $\chi_\lambda(a_P^k) \neq 0$, then

$$\chi_\lambda(a_P^k) = (-1)^{E_{p,q}(\lambda)} \frac{Z_q(\lambda + \rho)}{Z_q(\rho)}.$$

NEW **Classical and new:** for k prime to $2m+2$ this is due to Cellini, Möseneder Frajria and Papi; for $\gcd(k, 2m+2)$ odd it follows from Albion's factorisations, with the sign $(-1)^E$ added.
New: for $\gcd(k, 2m+2)$ even and $q \geq 3$, the values on the classes not conjugate to that of ρ .

↳ Used in: [Theorem B](#) (p. 7), [Theorem 10.1](#) (p. 21)

Proof

Part (1) is Proposition 3.1 together with Corollary 4.1, since $a_P^k = g_{p/q}$.

For part (2), let $\theta = p/q + \varepsilon$ with $\varepsilon \rightarrow 0$ in (3); the character is continuous in θ . For a positive integer A with $q \nmid A$ one has $\mathrm{sgn} \sin(\pi p A / q) = (-1)^{\lfloor p A / q \rfloor}$, and for $q \mid A$, $\sin(\pi A \theta) = (-1)^{p A / q} \pi A \varepsilon + O(\varepsilon^3)$. For λ dominant every (ℓ, α) and every (ρ, α) with $\alpha > 0$ is a positive integer. By part (1) the numerator and the denominator of (3) carry the same number r_q of vanishing factors, so the powers of $\pi \varepsilon$ cancel, each vanishing factor leaves its exponent, and every factor leaves its sign. Hence

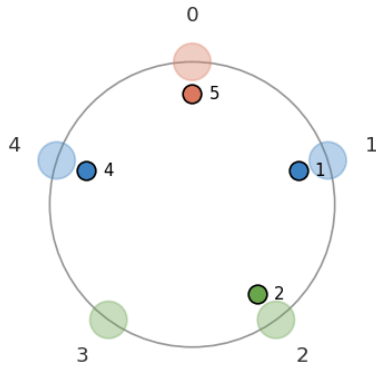
$$\chi_\lambda(a_P^k) = (-1)^{E_{p,q}(\lambda)} \frac{Z_q(\ell)}{Z_q(\rho)} \cdot \frac{\prod_{\alpha>0, q \nmid (\ell, \alpha)} |\sin(\pi p(\ell, \alpha)/q)|}{\prod_{\alpha>0, q \nmid (\rho, \alpha)} |\sin(\pi p(\rho, \alpha)/q)|}.$$

The quantity $|\sin(\pi p j / q)|$ depends only on j modulo q and up to sign, so by Proposition 5.1 and (7) the last quotient is 1.

Figure

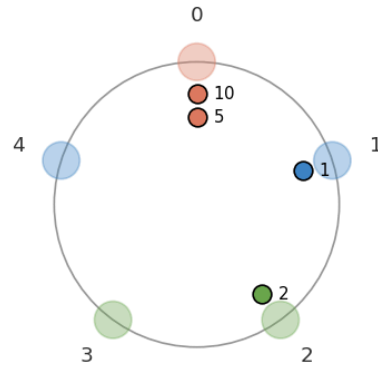
Theorem A in $\text{Sp}(8)$ at a_{ρ}^2 : residues mod $q = 5$, capacity $d = 2$ ($\lfloor d/2 \rfloor = 1$ on fixed classes)

$\lambda = (1, 1, 0, 0)$, $\ell = (5, 4, 2, 1)$
 $\chi_{\lambda} = 2$ (survives)



$\{0\}$: 1 of 1 ok
 $\{1,4\}$: 2 of 2 ok
 $\{2,3\}$: 1 of 2 ok

$\lambda = (6, 2, 0, 0)$, $\ell = (10, 5, 2, 1)$
 $\chi_{\lambda} = 0$ (vanishes)



$\{0\}$: 2 of 1 OVERFLOW
 $\{1,4\}$: 1 of 2 ok
 $\{2,3\}$: 1 of 2 ok

Theorem A at a glance (commentary, not part of the paper): the numbers ℓ_i placed on the residues modulo q ; folded classes $\{c, -c\}$ share a colour. The character vanishes exactly when some class holds more than its capacity. Values computed exactly in $\mathbb{Z}[\zeta_q]$ by `figuras/fig_reloj_A.py` (a check in the blueprint's source, published with the blueprint).

In plain words (commentary, not part of the paper)

On the line of ρ the Weyl character formula is, up to the phase $z^{-(\lambda, \rho)}$, a product over the positive roots of $(1 - z^{(\ell, \alpha)}) / (1 - z^{(\rho, \alpha)})$ (equation (2) of the paper), and at a primitive q -th root of unity a factor vanishes exactly when q divides its root value. Part (1): the character survives exactly when ℓ has no more vanishing factors than ρ ; in type C this is a capacity rule on the residues of the ℓ_i (Corollary 4.1). Part (2): approaching the point along the line, each vanishing factor leaves its root value, which gives $Z_q(\ell) / Z_q(\rho)$; every factor leaves a sign, which gives $E_{p,q}$; and the remaining sines cancel, because ℓ and ρ have the same root values modulo q up to sign (Proposition 5.1). For $\gcd(k, 2m + 2) = 1$ this is the type- C case of Cellini–Möseneder Fradjia–Papi; for d odd the paper says that Theorems A and B are not new in substance (Albion, Ayer–Kumari), and for q odd Reeder gives part (2) up to a constant. For d even and $q \geq 3$ the values on the classes not Weyl-conjugate to that of ρ , together with κ , are new (Example 6.1).

Evidence & scope

- **Not formalized in Lean.** Proof on paper.
- **Value and sign, checked by program `signo_Fq.py`** (supplement): 18 904 surviving weights, $m = 2, \dots, 8$, every k with $1 \leq k < 2m + 2$, $\lambda_1 \leq 9$ ($m \leq 4$), ≤ 7 ($m = 5, 6$), ≤ 5 ($m = 7, 8$); this box is in the program, not in the paper. Reference value: the character as an exact integer determinant built from the spectrum of Proposition 2.1. Exact arithmetic; value, modulus and sign each 18 904 of 18 904

(`signo_Fq_OUT.txt`). Its only negative control shows that the formula needs the survival hypothesis; there is no mutation of $E_{p,q}$.

- **Additional check, not part of the supplement:** `comprueba_traduccion_aP.py` (a check in the blueprint's source, published with the blueprint), run when this blueprint is built: 2336 cases, $m = 2, \dots, 5$, every power $k < 2m + 2$, $\lambda_1 \leq 4$, character computed exactly in $\mathbb{Z}[\zeta_q]$ by the principal specialisation (equation (2), the starting point of the proof). Part (1) as the occupancy rule and part (2) as the closed form: 0 failures. Mutation (bound of the fixed classes changed to d): detected in 567 cases (`comprueba_traduccion_aP_OUT.txt`).
- **Sign form of Remark 6.2:** `valor_signo_recuento.py` V3, 11 972 surviving weights, exact, 0 failures.
- **Example 6.1:** `cifras_alco.py` E1 reproduces -2 and $+2$, $Z_3(\ell) = 6$, $Z_3(\rho) = 3$; the character is an exact polynomial in $\mathbb{Z}[z]$, evaluated at the root of unity in double precision (tolerance 10^{-9}). The individual root values and the absence of w with $w\ell \equiv \rho \pmod{3}$ are not printed by any program.

Read this theorem in another mathematical language

Read this theorem in:

Commentary, not part of the paper. χ_λ is the character of the irreducible representation of $\mathrm{Sp}(2m)$ with highest weight λ , evaluated at the power $a_P^k = g_{p/q}$ of Kostant's principal element of the maximal torus. The criterion counts the positive roots α on which $\ell = \lambda + \rho$ is divisible by q : the character is nonzero exactly when ℓ lies on as few of these root hyperplanes modulo q as ρ does, and then its value is a signed ratio of the products of those root values.

Commentary, not part of the paper. Let $\zeta = e^{2\pi i/q}$. Then $\chi_\lambda(a_P^k) = \mathrm{sp}_\lambda(z_1^{\pm 1}, \dots, z_m^{\pm 1})$, the symplectic Schur function at $z_j = \zeta^{pj}$. Put $\ell_i = \lambda_i + m - i + 1$, so $\ell_1 > \dots > \ell_m \geq 1$, and $\rho = (m, \dots, 1)$.

- **Vanishing.** Reduce $\ell_1, \dots, \ell_m$ modulo q and identify each residue c with $-c$. The value is nonzero exactly when every class holds at most d of the ℓ_i , and the classes of 0 and (for q even) $q/2$ hold at most $\lfloor d/2 \rfloor$.
- **Value.** For $x = (x_1, \dots, x_m)$ let $Z(x)$ be the product of those numbers among $x_i - x_j$, $x_i + x_j$ ($i < j$) and $2x_i$ that are divisible by q . If the value is nonzero, then $\mathrm{sp}_\lambda(\dots) = (-1)^E Z(\ell)/Z(\rho)$, where E is the sum, over the same m^2 numbers $y(x)$, of $\lfloor py(\ell)/q \rfloor - \lfloor py(\rho)/q \rfloor$.

Commentary, not part of the paper. The $2m$ eigenvalues of a_P^k are the roots of unity $\zeta^{\pm pj}$, $\zeta = e^{2\pi i/q}$; as a multiset they are d copies of the q -th roots of unity with two letters removed (Proposition 2.1). A character value is a priori an element of $\mathbb{Z}[\zeta]$; at these points every character value is a rational integer (Proposition 8.1). Theorem A identifies that integer. With $\ell_i = \lambda_i + m - i + 1$, count the coincidences $\zeta^{p\ell_i} = \zeta^{\pm p\ell_j}$ ($i < j$) and $\zeta^{p\ell_i} = \zeta^{-p\ell_i}$ (that is, $\zeta^{p\ell_i} = \pm 1$). The value is 0 exactly when there are more of them than for ρ ; otherwise it is $\pm Z_q(\ell)/Z_q(\rho)$, a ratio of products of the integers $\ell_i \pm \ell_j$ and $2\ell_i$ that are divisible by q . Those coincidences are exactly the divisibilities: $\zeta^{p\ell_i} = \zeta^{\mp p\ell_j}$ if and only if $q \mid \ell_i \pm \ell_j$, and $\zeta^{p\ell_i} = \zeta^{-p\ell_i}$ if and only if $q \mid 2\ell_i$, because p is prime to q .

Let λ be as in Theorem A with $\chi_\lambda(a_P^k) \neq 0$. Then

$$|\chi_\lambda(a_P^k)| = \kappa(\lambda) \dim V_{\nu^{(0)}}^{\mathrm{Sp}_{2n_0}} \dim V_{\nu^{(q/2)}}^{\mathrm{SO}_{2n_{q/2}+1}} \prod_{0 < c < q/2} \dim V_{\nu^{(c)}}^{\mathrm{GL}_{n_c}},$$

where n_c is the number of ℓ_i in the class c , the factor for $q/2$ is omitted when q is odd, and $\kappa(\lambda) = 1$ if d is odd or if the class 0 holds fewer than $d/2$ of the ℓ_i , and $\kappa(\lambda) = 2$ otherwise.

NEW **Classical and new:** for $\gcd(k, 2m+2)$ odd the absolute values are known (Ayyer–Kumari, Albion). New: the factor κ for $\gcd(k, 2m+2)$ even and $q \geq 3$.

↳ Used in: Theorem D (p. 12), Theorem 10.1 (p. 21), Corollary 10.2 (p. 23)

Proof

By Theorem A, $|\chi_\lambda(a_P^k)| = Z_q(\ell)/Z_q(\rho)$. The roots α with $q \mid (\ell, \alpha)$ join two indices of the same folded class, or are $2e_i$ with ℓ_i in a fixed class. We compute Z_q class by class.

A non-fixed class c with n entries. For two of its entries exactly one of $(\ell, e_i - e_j)$, $(\ell, e_i + e_j)$ is divisible by q , and its absolute value is $|\varepsilon_i \ell_i - \varepsilon_j \ell_j| = q|y_i - y_j|$; no $2e_i$ contributes. The y_i are distinct, and with $y_1 > \dots > y_n$ the Weyl dimension formula for GL_n gives $\prod_{i < j} (y_i - y_j) = V_n \dim V_{\nu^{(c)}}^{\mathrm{GL}_n}$, where $V_n = \prod_{j < n} j!$. The contribution is $q^{\binom{n}{2}} V_n \dim V_{\nu^{(c)}}^{\mathrm{GL}_n}$.

The class 0 with n entries $\ell_i = qz_i$, $z_1 > \dots > z_n \geq 1$. All of $e_i \pm e_j$ and $2e_i$ contribute, with values $q(z_i \pm z_j)$ and $2qz_i$. Since $z = \nu^{(0)} + \rho_{C_n}$, the Weyl dimension formula for Sp_{2n} gives the contribution $q^{n^2} C_n \dim V_{\nu^{(0)}}^{\mathrm{Sp}_{2n}}$, where $C_n = 2^n \prod_{i < j} ((n - i + 1)^2 - (n - j + 1)^2) n! = 2^n \prod_{j \leq n} (2j - 1)!$.

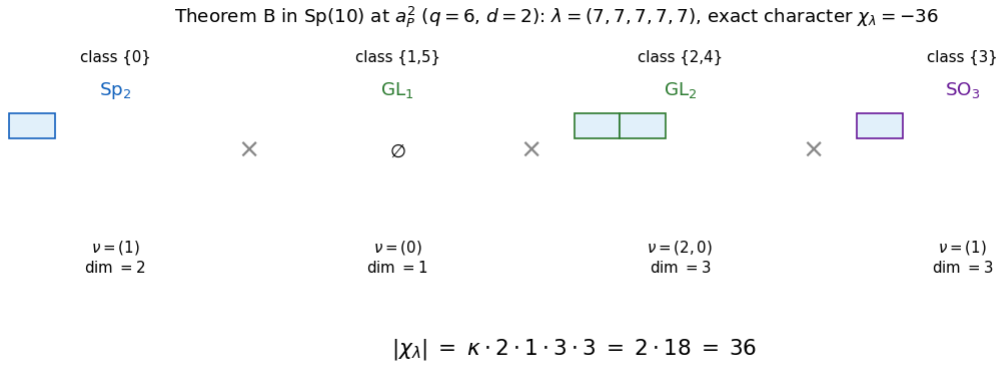
The class $q/2$ with n entries $\ell_i = qz_i$, $z_i \in \frac{1}{2} + \mathbb{Z}$. Again all of $e_i \pm e_j$ and $2e_i$ contribute, and since $z = \nu^{(q/2)} + \rho_{B_n}$ with $\rho_{B_n} = (n - \frac{1}{2}, \dots, \frac{1}{2})$, the Weyl dimension formula for SO_{2n+1} gives $q^{n^2} B_n \dim V_{\nu^{(q/2)}}^{\mathrm{SO}_{2n+1}}$, where $B_n = \prod_{j \leq n} (2j - 1)!$.

The same computation applies to ρ , whose entries in each class form an unbroken run (the class c receives $c, q + c, 2q + c, \dots$ and $q - c, 2q - c, \dots$, so the y_i are consecutive integers; the class 0 receives $q, 2q, \dots$; the class $q/2$ receives $q/2, 3q/2, \dots$), so all its partitions are empty and all its dimensions are 1. The powers of q agree on the two sides, since $N_q(\ell) = N_q(\rho)$. Hence $Z_q(\ell)/Z_q(\rho)$ is the product of the dimensions in the statement times the ratio of the constants $V_{n_c}, C_{n_0}, B_{n_{q/2}}$ for the occupancies of ℓ and of ρ . If d is odd the occupancies agree (Section 4) and the ratio is 1. If d is even, put $h = d/2$; the class 0 of ρ holds $h - 1$ entries and every other class is full; if the class of ℓ that falls short is 0, the occupancies agree again; otherwise that class holds one entry less and the class 0 one entry more than for ρ , and the ratio is

$$\frac{C_h/C_{h-1}}{V_{2h}/V_{2h-1}} = \frac{2(2h-1)!}{(2h-1)!} = 2 \quad \text{or} \quad \frac{C_h/C_{h-1}}{B_h/B_{h-1}} = 2$$

according as that class is non-fixed or is $q/2$.

Figure



Theorem B at a glance (commentary, not part of the paper): each folded class becomes a factor (Sp for the class 0, SO for $q/2$, GL otherwise) with its partition $\nu^{(c)}$; the absolute value of the character is κ times the product of their dimensions. The example is chosen by the script, which computes κ by the rule of the theorem ($\kappa = 1$ if d is odd or the class 0 holds fewer than $d/2$ entries, else 2), not as $|\chi|/\Delta$, and checks $|\chi| = \kappa \Delta$ against the exact character in $\mathbb{Z}[\zeta_q]$ on all 1232 weights it examines (`figuras/fig_factorizacion_B.py`, a check in the blueprint's source, published with the blueprint).

In plain words (commentary, not part of the paper)

By Theorem A, $|\chi_\lambda(a_P^k)| = Z_q(\ell)/Z_q(\rho)$, and the proof computes Z_q one folded class at a time. A root enters Z_q only if it joins two entries of the same class, or is $2e_i$ with ℓ_i in a fixed class. In a non-fixed class the values that enter are q times differences of the y_i : the Weyl dimension formula of GL_n . In the class 0 they are $q(z_i \pm z_j)$ and $2qz_i$: the formula of Sp_{2n} . In the class $q/2$, with half-integer z_i : the formula of SO_{2n+1} . For ρ every partition is empty. What is left is a ratio of normalising constants, which is 1 unless d is even and the class that falls short is not 0; then it is 2, and that is κ . The groups are not centralisers: at $m = 6, k = 7$ the factorisation group is $\mathrm{Sp}_6 \times \mathrm{SO}_7$, the centraliser $\mathrm{Sp}_6 \times \mathrm{Sp}_6$ (Question 12.1).

Evidence & scope

- **Not formalized in Lean.** Proof on paper.
- **Checked by program `dims_recuento.py`:** 11 972 surviving weights, $m = 2, \dots, 7$, every k , $\lambda_1 \leq 8$ ($m \leq 4$), ≤ 6 ($m \geq 5$), box not stated in the paper. Exact (rational arithmetic, integer determinant); 11 972 of 11 972 (`dims_recuento_OUT.txt`). Like `signo_Fq.py`, it builds the spectrum from Proposition 2.1. It has no negative control for κ : its column “kappa=1 mal” is not a mutation. A $\kappa \equiv 1$ mutation exists in `residual_factorizacion.py`, in the setting of Theorem 10.1 (floating point; breaks 371 of 1145).
- **Example 7.1:** `cifras_alco.py` E2, $Z_3(\ell)/Z_3(\rho) = 419904/52488 = 8$ and $E = 17$ exactly, value -8 in double precision (tolerance 10^{-9}). The partitions and the dimensions 4 and 2 are not printed.
- **End of Section 7:** `certifica_testigos.py`, $\chi_{(1,1,0,0,0)}(a_P^7) = -7$ exactly; `cifras_alco.py` E5, $d = 7, q = 2$, eigenvalues ± 1 six times each. The traces 0 and

−6 and the absence of a 7-dimensional irreducible of $\mathrm{Sp}_6 \times \mathrm{Sp}_6$ are not checked by any program.

- **No program:** $\chi_{jq\rho}(a_P^k) = (jq + 1)^{r_q}$ (introduction), and the constants C_n, B_n, V_n with the ratio 2 (proof on paper).

Read this theorem in another mathematical language

Read this theorem in:

Commentary, not part of the paper. The absolute value of the character is $\kappa(\lambda) \in \{1, 2\}$ times the dimension of an irreducible representation of the product group $\mathrm{Sp}_{2n_0} \times \mathrm{SO}_{2n_{q/2}+1} \times \prod_{0 < c < q/2} \mathrm{GL}_{n_c}$, one factor for each folded class c holding n_c of the entries of $\ell = \lambda + \rho$, with highest weights the partitions $\nu^{(c)}$. This group is not the centraliser of a_P^k (the paper gives an example), and Question 12.1 asks for a group that explains the product.

Commentary, not part of the paper. With $\ell_i = \lambda_i + m - i + 1$ grouped by folded residue class modulo q :

$$|sp_\lambda(z^{\pm 1})| = \kappa \cdot sp_{\nu^{(0)}}(1^{n_0}) \cdot so_{\nu^{(q/2)}}(1^{n_{q/2}}) \cdot \prod_{0 < c < q/2} s_{\nu^{(c)}}(1^{n_c}),$$

a product of principal specialisations at 1 of smaller symplectic, odd orthogonal and Schur functions. For a class $0 < c < q/2$: choose signs with $\varepsilon_i \ell_i \equiv c$, list $y_i = (\varepsilon_i \ell_i - c)/q$ decreasingly and put $\nu_i^{(c)} = y_i - y_n - (n - i)$; then $s_{\nu^{(c)}}(1^{n_c})$ is the number of semistandard Young tableaux of shape $\nu^{(c)}$ with entries at most n_c . For the class 0, $\nu_i^{(0)} = \ell_i/q - (n_0 - i + 1)$; for $q/2$, $\nu_i^{(q/2)} = \ell_i/q - (n_{q/2} - i + \frac{1}{2})$.

Commentary, not part of the paper. The character value is a rational integer (Proposition 8.1), and Theorem B factors its absolute value: $\kappa \in \{1, 2\}$ times a product of integers, one per residue class of the ℓ_i modulo q up to sign. Each factor is a Weyl dimension, an explicit product of ratios of integers built from the quotients ℓ_i/q (classes 0 and $q/2$) or from the shifted quotients $(\pm \ell_i - c)/q$ (other classes). So the size of the value is read off from how the ℓ_i distribute among the residues modulo q .

 **The lab under Theorem A** also shows this factorisation: for a surviving character it lists the factor of each class (Sp, SO or GL, its partition and dimension), κ , and checks the product against the exact character.

Theorem C. p. 3

Let $m \geq 2$ and $q \geq 2$. The identity

$$\chi_\lambda(g_{p/q}) = (-1)^{E_{p,q}(\lambda)} \frac{Z_q(\lambda + \rho)}{Z_q(\rho)}$$

holds for every dominant weight λ with $N_q(\lambda + \rho) = N_q(\rho)$ and every p prime to q if and only if $q \mid m$, $q \mid 2m + 1$, $q \mid 2m + 2$, or $q = 6$ and $m \equiv 1 \pmod{6}$. Otherwise there is such a λ for which it fails at $p = 1$ even up to sign.

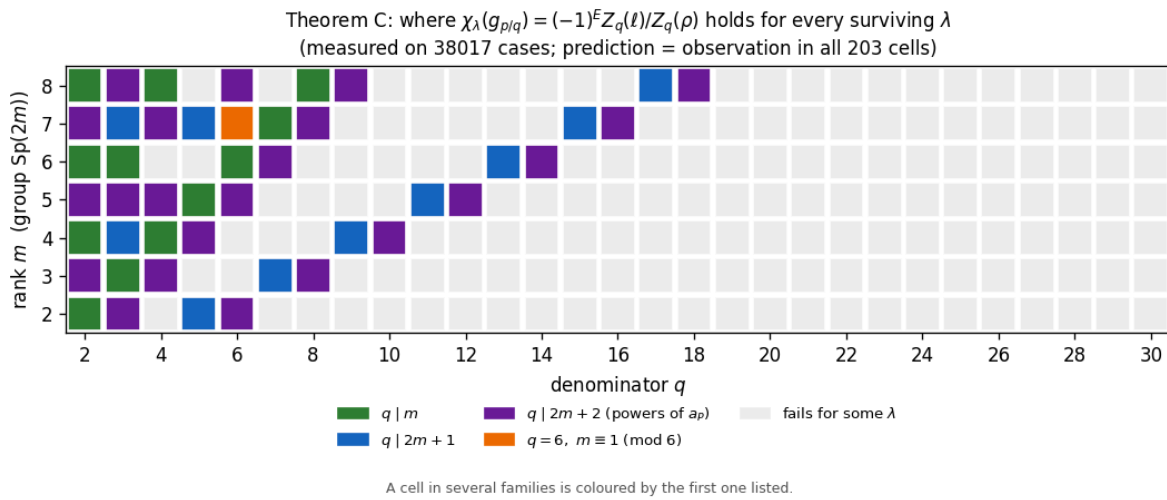
↳ Used in: [Theorem D](#) (p. 12)

Proof

The four families give the identity. If $q \mid m$, $q \mid 2m + 1$ or $q \mid 2m + 2$, then $v \in \{0, a, q - 1\}$ and $R_p(b) = 1$ by [Theorem 10.1](#). If $q = 6$ and $m \equiv 1 \pmod{6}$, then $v = 1 \leq a = 2$, $Y = C_1$, $\rho_Y = (1)$ and $b = (b_1)$ with $b_1 \in \{1, 2\}$, so by (9) $R_p(b) = |\sin(\pi p b_1 / 3)| / |\sin(\pi p / 3)| = 1$, since $p \equiv \pm 1 \pmod{6}$.

No other point does. Suppose that the identity holds up to sign at $p = 1$ for every surviving λ . Then every $\chi_\lambda(g_{1/q})$ is rational, the non-surviving ones being 0 by [Proposition 3.1](#), and an algebraic integer, hence an integer; by [Proposition 8.1](#), q divides $2m$, $2m + 1$ or $2m + 2$, or $q = 6$ and $m \equiv 1 \pmod{3}$. The cases not listed in [Theorem C](#) are (i) $q \geq 4$ even, $q \mid 2m$ and $q \nmid m$, that is $m \equiv q/2 \pmod{q}$, and (ii) $q = 6$ and $m \equiv 4 \pmod{6}$; for $q = 2$ either $q \mid m$ or $q \mid 2m + 2$. In both we take the natural representation, $\lambda = (1, 0, \dots, 0)$, and $\xi = e^{2\pi i/q}$. Its character is $\sum_{j=1}^m (\xi^j + \xi^{-j})$, and full turns contribute 0, so it equals $\sum_{j=1}^v (\xi^j + \xi^{-j})$. In case (i), $v = q/2$ and this is $\sum_{j=1}^{q-1} \xi^j + \xi^{q/2} = -2$; in case (ii), $v = 4$ and it is $2 \sum_{j=1}^5 \xi^j - (\xi^5 + \xi^{-5}) = -2 - 1 = -3$. In particular λ is surviving. On the other hand $\ell = \rho + e_1$, the roots not involving the index 1 take the same values on ℓ and on ρ , and those involving it take the values $\{1, \dots, 2m\} \setminus \{m\}$ on ρ and $(\{2, \dots, 2m\} \setminus \{m + 1\}) \cup \{2m + 2\}$ on ℓ . None of $1, m, m + 1, 2m + 2$ is divisible by q : in case (i) they are congruent to $1, q/2, q/2 + 1, 2$, and in case (ii) to $1, 4, 5, 4$. So the two sides have the same multiples of q , $Z_q(\ell) = Z_q(\rho)$, and the identity would give ± 1 . The failure holds at every p , not only at $p = 1$: since $\chi_\lambda(g_{p/q}) = \sigma_p(\chi_\lambda(g_{1/q}))$ for the automorphism σ_p of $\mathbb{Q}(\xi_q)$, a rational value at p forces a rational value at 1, and the natural representation gives -2 , respectively -3 , at every p .

Figure



Theorem C at a glance (commentary, not part of the paper): each cell (q, m) is coloured when the closed form holds for every surviving weight of the program's box ($\lambda_1 \leq 30, 16, 11, 8, 6, 5, 5$ for $m = 2, \dots, 8$), by the family that explains it (a cell in several families takes the first one in the legend). Measured, not drawn from the statement: the colours come from the archived output of `formula_pura.py` (203 cells; 38017 cases, counted up to the first counterexample in the grey cells), and the script checks that prediction and observation agree in every cell (`figuras/fig_mapa_C.py`, a check in the blueprint's source, published with the blueprint).

In plain words (commentary, not part of the paper)

The closed form of Theorem A(2) can be written at any rational point $g_{p/q}$. Theorem C says where it is right for every surviving weight and every p : $q \mid m$, $q \mid 2m + 1$, $q \mid 2m + 2$ (the powers of a_p), or $q = 6$ with $m \equiv 1 \pmod{6}$. By equation (8) of the paper the true value is the closed form times a residual factor $R_p(b)$, so the question is when $R_p(b) = 1$; in the four families it is. Conversely, if the closed form held up to sign at $p = 1$, every character at $g_{1/q}$ would be an integer, so q is one of the points of Proposition 8.1. That leaves two integer families outside the list, $q \geq 4$ even with $q \mid 2m$, $q \nmid m$, and $q = 6$ with $m \equiv 4 \pmod{6}$; there the natural representation has value -2 , resp. -3 , while the closed form gives ± 1 . The paper states that the list and the exclusion are new.

Evidence & scope

- **Not formalized in Lean.** Proof on paper.
- **Checked by program `formula_pura.py`:** the 203 pairs (m, q) , $m = 2, \dots, 8$, $2 \leq q \leq 30$, characters computed exactly in $\mathbb{Z}[X]/(\Phi_q)$. In every pair the prediction (the pair is in the list) equals the observation (the closed form holds on all surviving (λ, p) of the box); discrepancies: none (`formula_pura_OUT.txt`). Box $\lambda_1 \leq 30, 16, 11, 8, 6, 5, 5$ for $m = 2, \dots, 8$, in the program, not in the paper. The 38 017 cases are counted up to the first counterexample in each pair outside the list, not over the whole box.
- **Natural representation** in cases (i) and (ii) of the proof, 56 pairs with $m \leq 30$: survives, Z -quotient 1, value -2 , resp. -3 ; exact. For the other pairs outside the list the

program prints a counterexample but does not test the clause “fails even up to sign”.

- **One exact implementation:** the principal specialisation in `evlib.py`, which is equation (2), the starting point of the proof. A second, floating-point evaluator (`carac.py`) confirms the closed form only in the families $q \mid m$ and $q \mid 2m + 1$, on the 312 cases of `valor_todo_q.py casos`; the converse and the family $q = 6$, $m \equiv 1 \pmod{6}$ have no second evaluator.
- **Section 8 examples:** `cifras_alco.py` E4, $\text{Sp}(4)$, $\lambda = (1, 0)$: -2 at $q = 4$ and $0.8019377358 = 2 \cos \frac{2\pi}{7} + 2 \cos \frac{4\pi}{7}$ at $q = 7$, Z -quotient 1 in both; double precision. The value $\chi_{(7,7)}(g_{1/7}) = 1$ is not checked by any program.

Theorem D. p. 3

Let $m \geq 2$, let $q \geq 2$ be one of the denominators of Proposition 8.1, and let λ be dominant with $\chi_\lambda(g_{p/q}) \neq 0$. Then

$$|\chi_\lambda(g_{p/q})| = c_\lambda \Delta_\lambda, \quad c_\lambda \in \{1, 2, 3, 4, 6\}.$$

More precisely, $c_\lambda = 1$ if $q \mid m$, $q \mid 2m + 1$, or $q = 6$ and $m \equiv 1 \pmod{6}$; $c_\lambda = \kappa(\lambda) \in \{1, 2\}$ if $q \mid 2m + 2$; $c_\lambda \in \{1, 2\}$ if $q \geq 4$ is even, $q \mid 2m$ and $q \nmid m$; and $c_\lambda \in \{1, 2, 3, 4, 6\}$ if $q = 6$ and $m \equiv 4 \pmod{6}$. In the last two cases every listed value occurs.

NEW **Classical and new:** the families $q \mid m$ and $q \mid 2m+1$, and $q \mid 2m+2$ with $\gcd(k, 2m+2)$ odd, are known through the references of the paper (Ayyer–Kumari, Albion). New: κ for $\gcd(k, 2m+2)$ even and $q \geq 3$, and the constants for $q \geq 4$ even with $q \mid 2m$, $q \nmid m$, and for $q = 6$ with $m \equiv 4 \pmod{6}$.

↳ Not used by another result of the paper.

Proof

By Proposition 8.1, $g_{p/q}$ is conjugate to $g_{1/q}$, so in (8) the value $|\chi_\lambda|$ and the factor $\kappa \Delta_\lambda$ do not depend on p , and $c_\lambda = \kappa R_1(b)$. If $q \mid m$, $q \mid 2m + 1$, or $q = 6$ and $m \equiv 1 \pmod{6}$, then $v \leq a$, so $\kappa = 1$, and $R_1(b) = 1$ by the proof of Theorem C. If $q \mid 2m + 2$ this is Theorem B. Two cases remain.

$q \geq 4$ even, $q \mid 2m$, $q \nmid m$. Put $h = q/2$. Then $v = h > a$, $s = q - v = h$ and $Y = D_h$. The folded classes are $0, 1, \dots, h$, and exactly one of them, say j , holds its full number of entries; b lists the other h , while $\rho_Y = (h - 1, \dots, 0)$ omits $j = h$. With $p = 1$ the numbers $t_k = \cos(\pi k/h)$, $0 \leq k \leq h$, are distinct, and with $\omega(x) = \prod_{k=0}^h (x - t_k)$ the product of $|t_i - t_k|$ over the pairs not containing j is the product over all pairs divided by $|\omega'(t_j)|$. By (9),

$$R_1(b) = \frac{|\omega'(t_h)|}{|\omega'(t_j)|}.$$

The roots of ω are ± 1 and the zeros of the Chebyshev polynomial U_{h-1} , so ω is a constant multiple of $(x^2 - 1)U_{h-1}(x)$, which equals $-\sin \theta \sin h\theta$ at $x = \cos \theta$. At an interior node $\theta = \pi k/h$, $0 < k < h$, its derivative in x has absolute value h ; at $x = \pm 1$ it

is $\pm 2U_{h-1}(\pm 1)$, of absolute value $2h$, since $U_{h-1}(\pm 1) = (\pm 1)^{h-1}h$. Hence $R_1(b) = 1$ if $j \in \{0, h\}$ and $R_1(b) = 2$ otherwise. Since $\kappa = 2$ exactly when the class 0 is full, that is $j = 0$, one gets $c_\lambda = 1$ for $j = h$ and $c_\lambda = 2$ for every other j .

$q = 6$, $m \equiv 4 \pmod{6}$. Here $v = 4 > a = 2$, $s = 2$, $Y = D_2$ and $\rho_Y = (1, 0)$; two of the classes 0, 1, 2, 3 fall one entry short, and $b = (b_1, b_2)$ lists them. With $p = 1$, (9) gives $R_1(b) = 2 |\cos(\pi b_1/3) - \cos(\pi b_2/3)|$, and $\kappa = 2$ exactly when $0 \notin b$:

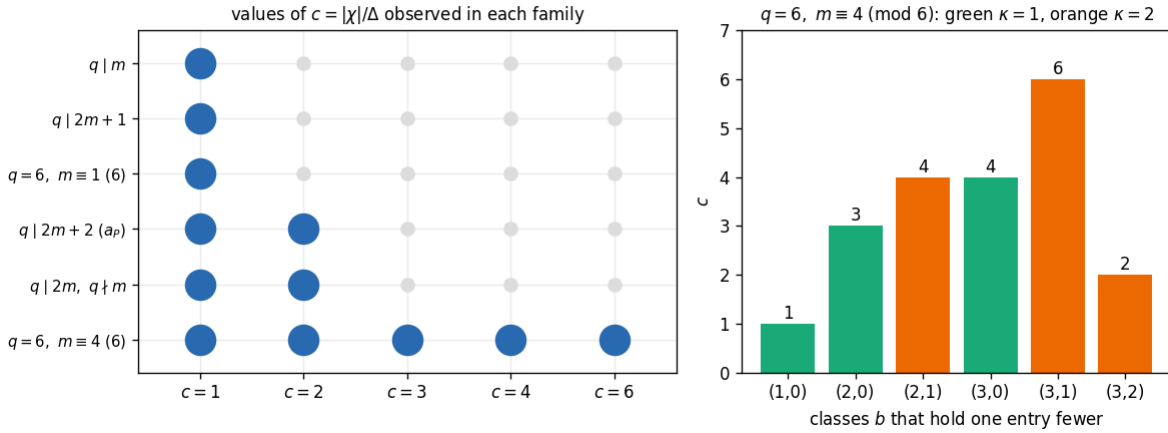
b	(1, 0)	(2, 0)	(3, 0)	(2, 1)	(3, 1)	(3, 2)
$R_1(b)$	1	3	4	2	3	1
κ	1	1	1	2	2	2
c_λ	1	3	4	4	6	2
$\lambda \ (m = 4)$	0	(1)	(1, 1)	(2)	(2, 1)	(2, 2)
$\chi_\lambda(g_{1/6})$	1	-3	4	4	-6	2

The last two rows give, for $m = 4$, a weight realising each column, with $\Delta_\lambda = 1$; the values are computed exactly in $\mathbb{Z}[\zeta_6]$ (`certifica_testigos.py`).

In the last two cases every occupancy pattern of a surviving weight, that is every choice of the classes that fall short, is the pattern of some $\lambda + \rho$, by the argument in the proof of Corollary 4.1; so every listed value of c_λ occurs.

Figure

Theorem D: $|\chi_\lambda(g_{p/q})| = c_\lambda \Delta_\lambda$ with $c_\lambda \in \{1, 2, 3, 4, 6\}$ (measured exactly; no other value occurs)



Theorem D at a glance (commentary, not part of the paper): left, the values of $c = |\chi|/\Delta$ observed in each family, for $m \in \{2, \dots, 8, 10, 13, 16\}$ with a box on λ per m ; the points that lie in two families ($q \mid 2m+2$ together with $q \mid m$, or with $q \mid 2m, q \nmid m$) are not shown; right, the exceptional case $q = 6, m \equiv 4 \pmod{6}$, where c depends on which two classes hold one entry fewer. Data from the archived output of `prefactores.py`, exact fractions; the script checks it against the table as the program prints it, transcribed from the paper (`figuras/fig_constantes_D.py`, a check in the blueprint's source, published with the blueprint).

In plain words (commentary, not part of the paper)

At the integer points of Proposition 8.1 every non-vanishing character is c_λ times the product of dimensions Δ_λ of Theorem B, with $c_\lambda \in \{1, 2, 3, 4, 6\}$. Since $g_{p/q}$ is conjugate to $g_{1/q}$

there, $c_\lambda = \kappa R_1(b)$. In the families of Theorem C other than $q \mid 2m + 2$, $\kappa = 1$ and $R_1(b) = 1$; for $q \mid 2m + 2$ it is Theorem B. Two cases are left. For $q \geq 4$ even, $q \mid 2m$, $q \nmid m$, the residual group is $D_{q/2}$ and $R_1(b)$ is a ratio of derivatives of $(x^2 - 1)U_{h-1}(x)$ at the nodes $\cos(\pi k/h)$, $h = q/2$: 1 at the end nodes, 2 inside. So $c_\lambda = 1$ when the full class is $q/2$, and 2 otherwise. For $q = 6$, $m \equiv 4 \pmod{6}$, the group is D_2 and the six choices of the two short classes give the table of the proof: $c_\lambda = 1, 3, 4, 4, 6, 2$.

Evidence & scope

- **Not formalized in Lean.** Proof on paper.
- **Checked by program** `prefactores.py`: integrality at the points of Proposition 8.1 on 52 928 pairs (λ, q) , exact in $\mathbb{Z}[X]/(\Phi_q)$, 0 non-integers; the set of $c = |\chi|/\Delta$ in each family, as exact fractions, equal to the statement, with every listed value occurring in the last two families; 0 values outside $\{1, 2, 3, 4, 6\}$ (`prefactores_OUT.txt`). Range $m \in \{2, \dots, 8, 10, 13, 16\}$ with a box on λ per m , in the program, not in the paper.
- **The table:** same program, (κ, c) for each pair b , printed next to the row c_λ as transcribed in the program, and equal to it; the end-node/interior rule, (κ, c) by omitted node, for the even $q \leq 32$ that occur in its range (the end node 0 only for $q \leq 12$). Both are printed, not asserted. `certifica_testigos.py`: the last two rows ($m = 4$, values $1, -3, 4, 4, -6, 2$, $\Delta_\lambda = 1$) exactly in $\mathbb{Z}[\zeta_6]$. The map from λ to b in the table is not checked by any program.
- **Partial second implementation:** `teorema_II.py` (b) and `residual_factorizacion.py` (V2, V3) find the same set $\{1, 2, 3, 4, 6\}$ and the six rows of the table, in floating point, on a smaller range.
- No negative control in `prefactores.py`.

2 The element and the criterion

Proposition 2.1. p. 5

Let $t = 2m + 2$, $d = \gcd(k, t)$ and $q = t/d$. As a multiset, the spectrum $A = \{\zeta^{\pm jk} : j = 1, \dots, m\}$ of a_P^k on $\mathbb{C}^{2m}$, with ζ of order t , satisfies

$$A = d \cdot \mu_q - 2 \cdot \{1\} \quad (d \text{ even}), \quad A = d \cdot \mu_q - \{1\} - \{-1\} \quad (d \text{ odd}).$$

↳ Not used by another result of the paper.

Proof

Since $t = 2m + 2$, the exponents $\{\pm j : j = 1, \dots, m\}$ are exactly $\mathbb{Z}/t \setminus \{0, t/2\}$. The map $e \mapsto \zeta^{ke}$ factors through $\mathbb{Z}/q$, because ζ^k has order q , and every residue mod q is the image of exactly d elements of $\mathbb{Z}/t$; hence $\{\zeta^{ke} : e \in \mathbb{Z}/t\} = d \cdot \mu_q$. It remains to delete the two excluded exponents. For $e = 0$ the element is 1. For $e = t/2 = dq/2$: if d is even this is $\equiv 0 \pmod{q}$ and the element is again 1; if d is odd then q is even, since $dq = t$ is, and $dq/2 \equiv q/2$, so the element is -1 .

In plain words (commentary, not part of the paper)

The exponents $\pm j$, $j = 1, \dots, m$, are all of $\mathbb{Z}/t$ except 0 and $t/2$, where $t = 2m + 2$. Raising to the k -th power maps $\mathbb{Z}/t$ onto $\mathbb{Z}/q$, d to one, so the whole of $\mathbb{Z}/t$ gives d copies of

every q -th root of unity. Removing the two missing exponents: 0 gives 1, and $t/2$ gives 1 if d is even and -1 if d is odd. So the spectrum depends on d and q only, $\chi_\lambda(a_P^k) = \chi_\lambda(a_P^d)$, and one may take $p = 1$. The full multiset $d \cdot \mu_q$ is the alphabet of Prasad's character relationship; the paper says the two deleted letters change the occupancies of the fixed classes and produce the factor κ of Theorem B.

Evidence & scope

- **Not formalized in Lean.** Proof on paper.
- **No direct check.** `signo_Fq.py`, `valor_signo_recuento.py` and `dims_recuento.py` build the spectrum from this proposition, so they assume it. `formula_pura.py` and `valor_todo_q.py` evaluate at the eigenvalues of $g_{p/q}$ directly; their agreement with Theorem A is an indirect check.
- `valor_signo_recuento.py` V2: the spectra of a_P^k and a_P^d agree as multisets in 32 pairs ($m \leq 5$), compared numerically with rounding, not in exact arithmetic.
- `cifras_alco.py` E5: $m = 6$, $k = 7$ gives $d = 7$, $q = 2$, eigenvalues 1 and -1 six times each.

Proposition 3.1. p. 5

Let G be simple and simply connected, λ dominant, $\ell = \lambda + \rho$, $\theta = p/q$ in lowest terms, and $N_q(x) = \#\{\alpha > 0 : q \mid (x, \alpha)\}$, $r_q = N_q(\rho)$. Then $N_q(\ell) \geq r_q$, and

$$\chi_\lambda(g_{p/q}) \neq 0 \iff N_q(\ell) = r_q. \quad (1)$$

↳ Used in: Theorem A (p. 4), Theorem C (p. 9), Corollary 4.1 (p. 16), Theorem 9.1 (p. 20)

Proof

Each factor $1 - z^j$ of (2) has a simple zero at a primitive q -th root of unity exactly when $q \mid j$, so at $z = e^{2\pi i p/q}$ the numerator vanishes to order $N_q(\ell)$ and the denominator to order r_q . A character has no poles, so the first order is at least the second; and the limit is non-zero exactly when they agree.

In plain words (commentary, not part of the paper)

Notation: $r_q = N_q(\rho)$. The statement holds in every simple simply connected group and at every rational point $g_{p/q}$, not only at the powers of a_P . In the product formula (equation (2) of the paper) each factor $1 - z^j$ has a simple zero at a primitive q -th root of unity exactly when $q \mid j$, so the numerator vanishes to order $N_q(\ell)$ and the denominator to order r_q . A character has no poles, so $N_q(\ell) \geq r_q$, and the limit is non-zero exactly when the two orders agree. When ρ is regular, $r_q = 0$ and this is the condition of Cellini–Möseneder Fradjria–Papi; on the line of $\rho^\vee$, with coroots in place of roots, the same count is due to Nadimpalli–Pattanayak–Prasad, and the paper proves it by the same argument.

Evidence & scope

- **Not formalized in Lean.** Proof on paper. Programs check type C only; the other types are not checked.
- **Checked by program `rho_line_any_q.py`:** 27 183 dominant weights, $m = 2, \dots, 6$, $q = 2, \dots, 14$, $p = 1$, $\lambda_1 \leq 8$ ($m \leq 4$), ≤ 6 ($m = 5, 6$), box not stated in the paper; 0

failures, 19 986 of them with $q \nmid 2m + 2$ (`rho_line_any_q_OUT.txt`). **Floating point:** `mpmath`, 40 digits, χ counted as non-zero when $|\chi| \geq 10^{-6}$, 0 cases in the ambiguous zone; no rigorous error bound. Negative control: the naive rule (“some root value divisible by q ”) fails on 7968 weights.

- **Exact:** `formula_pura.py` checks the criterion in $\mathbb{Z}[X]/(\Phi_q)$ on every weight it visits, 0 failures: the whole box at the 45 pairs of the list of Theorem C, and up to the first counterexample of Theorem C at the other 158 pairs; and `comprueba_traduccion_aP.py` (a check in the blueprint’s source, published with the blueprint, not part of the supplement) on 2336 cases with $q \mid 2m + 2$.
- `valor_todo_q.py`: 0 violations in the range of Theorem 10.1 (floating point).

Corollary 4.1.

Let $q \mid 2m + 2$ and $d = (2m + 2)/q$. Then $N_q(\ell) = r_q$ if and only if each folded class modulo q contains at most d of the ℓ_i , and each fixed class at most $\lfloor d/2 \rfloor$.

↳ Used in: Theorem A (p. 4), Theorem D (p. 12), Proposition 5.1 (p. 17), Theorem 9.1 (p. 20)

Proof

By (5) the marginal cost of one more entry is $0, 1, 2, \dots$ in a non-fixed class and $1, 3, 5, \dots$ in a fixed one, so the places costing less than d number exactly d per non-fixed class and $\lfloor d/2 \rfloor$ per fixed one. There are $\lfloor q/2 \rfloor + 1$ folded classes, of which one is fixed when q is odd and two when q is even; since $m = dq/2 - 1$, the places of cost less than d number exactly m if d is odd and $m + 1$ if d is even. By Proposition 3.1 the minimum of N_q over the vectors $\ell = \lambda + \rho$ is r_q , and it is attained at ρ . It is also the minimum over all of $\mathbb{Z}^m$, hence over all placements of m entries in the folded classes: every $x \in \mathbb{Z}^m$ is congruent modulo q to a vector $\lambda + \rho$ with λ dominant, namely $x_i + (m - i + 1)Mq$ for a large integer M , whose entries are positive and strictly decreasing. If d is odd, reaching the minimum forces occupying all m cheap places, and any overflow replaces a place of cost $< d$ by one of cost $\geq d$. If d is even, the last admissible place of every class costs $d - 1$, so omitting any one of them gives the same minimum, while exceeding a capacity again costs strictly more. So $N_q(\ell) = r_q$ if and only if no capacity is exceeded.

In plain words (commentary, not part of the paper)

Equation (5) of the paper counts the vanishing factors class by class:

$N_q(\ell) = \sum_{c \text{ not fixed}} \binom{n_c}{2} + \sum_{f \text{ fixed}} n_f^2$, with n_c the number of ℓ_i in the class. So one more entry costs $0, 1, 2, \dots$ in a non-fixed class and $1, 3, 5, \dots$ in a fixed one. The places costing less than d are d per non-fixed class and $\lfloor d/2 \rfloor$ per fixed class: m in all if d is odd, $m + 1$ if d is even. The minimum of N_q is r_q , reached at ρ , and every placement of residues comes from some $\lambda + \rho$; so a character survives exactly when no class is over capacity. When d is odd every survivor has the occupancies of ρ ; when d is even exactly one class is one entry short, which for ρ is the class 0.

Evidence & scope

- **Not formalized in Lean.** Proof on paper.
- **Additional check, not part of the supplement:** `comprueba_traduccion_aP.py` (a check in the blueprint’s source, published with the blueprint): the occupancy rule

against $\chi \neq 0$, exact, 2336 cases, 0 failures; with the bound of the fixed classes changed to d it fails in 567 cases.

- `rho_line_any_q.py`, column D2: the capacity rule agrees with $\chi \neq 0$, computed numerically as for Proposition 3.1, on 7197 weights with $q \mid 2m + 2$. The program labels it a negative control, but it is measured only where it must agree: a positive check of Theorem A(1) in its capacity form.
- `signo_Fq.py` O2: in all 10 230 surviving weights with d even, exactly one class is one entry short (the remark after the corollary).

3 The residue lemma

Proposition 5.1. ■ p. 6

Work in $\mathbb{Z}[X]/(X^q - 1)$. For $x = (x_1, \dots, x_m) \in \mathbb{Z}^m$ put

$$P_x(X) = \sum_{i=1}^m (X^{x_i} + X^{-x_i}), \quad \mathcal{R}_x(X) = \sum_{\alpha \in \Phi(C_m)} X^{(x, \alpha)}.$$

Then

$$\mathcal{R}_x(X) = \frac{P_x(X)^2 + P_x(X^2)}{2} - m, \quad (2)$$

and, for $q \mid 2m + 2$ and every λ satisfying the capacities of Corollary 4.1, $\mathcal{R}_{\lambda+\rho} = \mathcal{R}_\rho$.

↳ Used in: Theorem A (p. 4), Theorem 10.1 (p. 21)

Proof

Expanding $P_x(X)^2$ collects $X^{\pm x_i \pm x_j}$ over all ordered pairs. The off-diagonal terms give each root $\pm e_i \pm e_j$ twice; the diagonal terms give $X^{\pm 2x_i}$ once each and $2m$ copies of X^0 ; and $P_x(X^2)$ supplies the second copy of each $X^{\pm 2x_i}$. Halving therefore leaves every root of C_m exactly once, together with m copies of X^0 , and subtracting m removes those. That is (6).

For the second claim write $T_q(X) = 1 + X + \dots + X^{q-1}$, so that $T_q(X) X^c = T_q(X)$ for every c . If d is odd the capacities are saturated, the folded occupancies of ℓ are those of ρ , and $P_\ell = P_\rho$ outright. If d is even, one class c falls a single unit below its capacity, and

$$P_\ell(X) = dT_q(X) - X^c - X^{-c}, \quad P_\rho(X) = dT_q(X) - 2,$$

so that $P_\ell(X)^2 - P_\rho(X)^2 = X^{2c} + X^{-2c} - 2$ while

$P_\ell(X^2) - P_\rho(X^2) = -X^{2c} - X^{-2c} + 2$. The two corrections cancel in (6), and

$\mathcal{R}_\ell = \mathcal{R}_\rho$.

 **In plain words (commentary, not part of the paper)**

Notation: $P_x = \sum_i (X^{x_i} + X^{-x_i})$ in $\mathbb{Z}[X]/(X^q - 1)$, and $\mathcal{R}_x$ the sum of $X^{(x,\alpha)}$ over all roots of C_m . The identity $\mathcal{R}_x = (P_x^2 + P_x(X^2))/2 - m$ is the adjoint character minus the rank, since in type C the adjoint representation is the symmetric square of the natural one (Remark 5.2). The second claim: under the capacities of Corollary 4.1, $\mathcal{R}_\ell = \mathcal{R}_\rho$. For d odd the occupancies agree, so $P_\ell = P_\rho$. For d even one class c is one entry short; then $P_\ell^2 - P_\rho^2 = X^{2c} + X^{-2c} - 2$ and $P_\ell(X^2) - P_\rho(X^2)$ is its negative, so they cancel. In words: the root values of ℓ and of ρ coincide modulo q , up to sign, as multisets. This replaces Weyl conjugacy in the proof of Theorem A(2) (Example 6.1); outside type C it is the missing step (Question 12.2).

Evidence & scope

- **Not formalized in Lean.** Proof on paper.
- **Checked by program** `valor_signo_recuento.py` V1: the identity in 48 669 cases and $\mathcal{R}_\ell = \mathcal{R}_\rho$ on 11 972 surviving weights, $m = 2, \dots, 7$, every $k, \lambda_1 \leq 8$ ($m \leq 4$), ≤ 6 ($m \geq 5$), box not stated in the paper; exact integer vectors, 0 failures (`valor_signo_recuento_OUT.txt`). No negative control (no mutation of the identity).

4 Every rational point

Proposition 8.1.

Let $m \geq 2$ and $\gcd(p, q) = 1$. Then $\chi_\lambda(g_{1/q}) \in \mathbb{Z}$ for every dominant weight λ of $\mathrm{Sp}(2m)$ if and only if at least one of

$$q \mid 2m, \quad q \mid 2m + 1, \quad q \mid 2m + 2, \quad q = 6 \text{ and } m \equiv 1 \pmod{3}$$

holds; and in that case $g_{p/q}$ is conjugate to $g_{1/q}$ for every p coprime to q , so no character value depends on the numerator.

↳ Used in: [Theorem C](#) (p. 9), [Theorem D](#) (p. 12)

Proof

Suppose every $\chi_\lambda(g_{1/q})$ is an integer. The exterior powers $\Lambda^j \mathbb{C}^{2m}$ are direct sums of irreducibles, so their characters are $\mathbb{Z}$ -combinations of the χ_λ and are integers as well; those characters are the coefficients of the characteristic polynomial of $g_{1/q}$ on $\mathbb{C}^{2m}$, so that polynomial lies in $\mathbb{Z}[X]$, and the multiplicities of its roots are constant along Galois orbits. Conversely, if the spectrum is carried to itself by every $\xi_q \mapsto \xi_q^p$, then $g_{p/q}$ and $g_{1/q}$ have the same spectrum and are conjugate in $\mathrm{Sp}(2m)$; the character values are then algebraic integers fixed by the Galois group of $\mathbb{Q}(\xi_q)$, hence rational integers.

Write $m = uq + v$ with $0 \leq v < q$ and $a = \lfloor (q-1)/2 \rfloor$. The spectrum $\{\xi_q^{\pm j} : j = 1, \dots, m\}$ is, as a multiset of μ_q ,

$$2u \cdot \mu_q + \{\pm 1, \dots, \pm v\},$$

and the first summand is Galois-stable. So the question is when the segment $\{\pm 1, \dots, \pm v\}$ is. It is stable for $v = 0$, $v = a$, $v = q - 1$ and, when q is even, $v = q/2$.

For necessity write $A_v = \sum_{j=1}^v (\xi_q^j + \xi_q^{-j})$; if the segment is stable then A_v is fixed by the whole Galois group. A full turn sums to zero, whence $A_v + A_{q-1-v} = -2$ and $\mathbb{Q}(A_v) = \mathbb{Q}(A_w)$ for $w = \min(v, q-1-v)$; the four segments just listed are $w = 0$ and $w = a$, so only $1 \leq w < a$ is left. There set $\beta_w = 1 + A_w = \sum_{j=-w}^w \xi_q^j$, whose conjugates are

$$\sigma_p(\beta_w) = \frac{\sin((2w+1)\pi p/q)}{\sin(\pi p/q)}.$$

For $q \geq 4$, $2 \leq N \leq q-2$ and $1 \leq r \leq q-1$ with $r \neq 1, q-1$, one has the strict inequality

$$\left| \frac{\sin(N\pi r/q)}{\sin(\pi r/q)} \right| < \frac{\sin(N\pi/q)}{\sin(\pi/q)}.$$

Indeed, replacing N by $n = \min(N, q-N)$ changes no modulus; on $0 < x < \pi/n$ the function $f(x) = \sin(nx)/\sin x$ is positive and strictly decreasing, since $f'/f = n \cot(nx) - \cot x < 0$ because $t \cot t$ decreases on $(0, \pi)$; and on $\pi/n \leq x \leq \pi/2$ one has $|f(x)| \leq 1/\sin(\pi/n) < 1/\sin(\pi/2n) = f(\pi/2n) \leq f(\pi/q)$. Taking $N = 2w+1$, which lies in that range precisely when $1 \leq w < a$, the only automorphisms fixing β_w are the identity and complex conjugation; so $\mathbb{Q}(\beta_w) = \mathbb{Q}(\xi_q + \xi_q^{-1})$, which is $\mathbb{Q}$ only when $\varphi(q) = 2$. Inside this range that leaves $q = 6$, $w = 1$, that is $(q, v) = (6, 1)$ and $(6, 4)$. Translating the surviving residues into divisibilities gives the four conditions.

In plain words (commentary, not part of the paper)

When are all the characters at $g_{1/q}$ integers? If they are, the exterior powers show that the characteristic polynomial on $\mathbb{C}^{2m}$ has integer coefficients, so the spectrum is stable under Galois. Conversely a Galois-stable spectrum makes every $g_{p/q}$ conjugate to $g_{1/q}$, and the values are rational integers. Writing $m = uq + v$, the spectrum is $2u$ full turns of μ_q plus the segment $\{\pm 1, \dots, \pm v\}$, so everything rests on the segment. It is stable for $v = 0$, $v = a$ with $a = \lfloor (q-1)/2 \rfloor$, $v = q-1$ and, for q even, $v = q/2$. For the other v , the sum over the segment has a strictly largest conjugate, so it generates $\mathbb{Q}(\xi_q + \xi_q^{-1})$, which is $\mathbb{Q}$ only when $\varphi(q) = 2$; in that range this leaves $q = 6$. The key step is a lemma of Gannon–Schopieray; sufficiency for $q \mid 2m+1$ and $q \mid 2m+2$ also follows from Kac. Irrational θ never gives integers (Remark 8.2).

Evidence & scope

- **Not formalized in Lean.** Proof on paper.
- **Checked by program `galois.py`:** Galois stability of the spectrum against the four conditions on 11 800 pairs (m, q) , $q \leq 60$, $m \leq 200$ (the range includes $m = 1$, outside the statement); exact combinatorics, 0 disagreements (`galois_OUT.txt`). This compares two conditions, not character values.
- Against the characters themselves, `galois.py` part (2): 44 pairs, 0 disagreements, **floating point** (mpmath, 40 digits, limit taken at a perturbed point, tolerance 10^{-12} ,

$$\lambda_1 \leq 5).$$

- **Sufficiency, exact:** `prefactores.py`, integrality in $\mathbb{Z}[X]/(\Phi_q)$ on 52 928 pairs (λ, q) at the four families.
- Necessity against actual character values is checked only by the 44-pair floating-point run.

Theorem 9.1. ■ p. 10

$$S_{C_m}(q) = \begin{cases} \frac{m! 2^{2ua+v} \binom{a}{v}}{((2u+1)!)^v ((2u)!)^{a-v} (u!)^f}, & v \leq a, \\ \frac{m! 2^{a(2u+1)} \binom{a+f}{v-a}}{((2u+1)!)^a (u!)^f (u+1)^{v-a}}, & v > a. \end{cases}$$

↳ Used in: [Corollary 10.2](#) (p. 23)

Proof

By (5) the cost of an extra entry is $0, 1, 2, \dots$ in a non-fixed class and $1, 3, 5, \dots$ in a fixed one, so the minimum of N_q is attained by filling the cheapest places, and $\rho = (m, \dots, 1)$ attains it, its residues being u full turns of $\mathbb{Z}/q$ followed by $1, \dots, v$; by Proposition 3.1, and since every placement of entries in classes is the residue of some $\lambda + \rho$ (proof of Corollary 4.1), that minimum is r_q . When $v \leq a$, each fixed class carries u entries, v non-fixed classes carry $2u + 1$ and the remaining $a - v$ carry $2u$; the binomial $\binom{a}{v}$ chooses the v classes, the multinomial coefficient distributes the m coordinates, and each entry of a non-fixed class has two possible residues, which gives the power of 2. When $v > a$ one starts from $2u + 1$ in every non-fixed class and u in every fixed one and places the remaining $v - a$ entries in distinct classes, chosen in $\binom{a+f}{v-a}$ ways; adding an entry multiplies the count by $1/(u+1)$ whether the class is fixed or not.

 In plain words (commentary, not part of the paper)

$S_X(q)$ counts the residues of weights modulo qP at which the criterion of Proposition 3.1 allows a non-zero character. Notation: $f = 1$ for q odd and 2 for q even (the number of fixed classes), $a = (q - f)/2$, $m = uq + v$. By the costs of Corollary 4.1, the minimum is reached by filling the cheapest places. When $v \leq a$: u entries in each fixed class, $2u + 1$ in v of the non-fixed classes, $2u$ in the rest. The binomial chooses the v classes, the multinomial distributes the coordinates, and each entry in a non-fixed class has two residues, hence the power of 2; $v > a$ is similar. Special values: $S_{C_m}(2m + 2) = 2^m m!$, $S_{C_3}(5) = 72$, $S_{C_3}(6) = 96$. It is not a quasi-polynomial: $S_{C_2}(q) = (q - 1)(q - 3)$ for large odd q , but $S_{C_2}(3) = 8$.

 Evidence & scope

- **Not formalized in Lean.** Proof on paper.
- **Checked by program `valor_signo_recuento.py`** V4: formula against exhaustive enumeration of $(\mathbb{Z}/q)^m$ on 40 pairs (m, q) , $m \leq 4$, $q \leq 12$; exact, 0 discrepancies (`valor_signo_recuento_OUT.txt`).

- `cifras_alco.py` E6: $S_{C_3}(5) = 72$, $S_{C_3}(6) = 96$, $S_{C_2}(3) = 8$, and formula equal to enumeration on 36 pairs ($m \leq 4$, $q \leq 10$). `signo_Fq.py` O3 (the case $q > 2m$) and `dims_recuento.py` part (2) (the case $q \mid 2m + 2$) agree with a direct count. All exact.
- “ $(q - 1)(q - 3)$ for every large odd q ” follows from the formula; enumeration covers odd $q \leq 11$.
- No negative control.

Theorem 10.1. ■ p. 11

Let λ be a dominant weight of $\mathrm{Sp}(2m)$ with $N_q(\lambda + \rho) = r_q$. Then $b - \rho_Y$ is a dominant weight of Y , the element $g_{p/q}^Y$ is regular, and

$$\chi_\lambda(g_{p/q}) = (-1)^{E_{p,q}(\lambda)} \kappa \Delta_\lambda |\chi_{b-\rho_Y}^Y(g_{p/q}^Y)|.$$

For $v \in \{0, a, q - 1\}$, that is, for $q \mid m$, $q \mid 2m + 1$ or $q \mid 2m + 2$, the last factor is 1 for every such λ ; for $q \mid 2m + 2$ the theorem is Theorems A(2) and B.

↳ Used in: [Theorem C](#) (p. 9), [Corollary 10.2](#) (p. 23)

Proof

The class of ρ . The residues of ρ are u full turns of $\mathbb{Z}/q$ followed by $1, \dots, v$. If $v \leq a$ the classes with $2u + 1$ entries are $1, \dots, v$. If $v > a$ the classes with one entry fewer are 0 and the non-fixed $c < q - v$; the class $q/2$, for q even, is full, since $v \geq q/2$. In both cases these classes are the entries of ρ_Y .

Regularity. The entries of b are distinct and lie in $\{1, \dots, a\}$ if $Y = C_s$, and in $\{0, \dots, a\}$, together with $q/2$ when q is even, if $Y = D_s$. Hence every (b, α) with α a positive root of Y , namely $b_i \pm b_j$ and, for C_s , $2b_i$, lies strictly between 0 and q , since $2a < q$ and $a + q/2 < q$ for q even. The same holds for ρ_Y , which is one of the possible b . So $b - \rho_Y$ is dominant and $g_{p/q}^Y$ is regular. The computation that gives (3) in Section 3 is stated there for G simple and simply connected, but it uses only the Weyl character formula and the Weyl denominator identity, with ρ a weight of the group; it therefore applies verbatim to $\mathrm{Sp}(2s)$ and to $Y = \mathrm{SO}(2s)$, where ρ_Y is a weight, including D_2 , which is not simple, and D_1 , a torus, where both sides are 1. It gives for Y

$$|\chi_{b-\rho_Y}^Y(g_{p/q}^Y)| = \frac{\Pi^Y(b)}{\Pi^Y(\rho_Y)}, \quad \Pi^Y(x) = \prod_{\alpha \in \Phi^+(Y)} |\sin(\pi p(x, \alpha)/q)|.$$

Residues. In $\mathbb{Z}[X]/(X^q - 1)$, with P_x and T_q as in Proposition 5.1, a non-fixed class c holding n entries contributes $n(X^c + X^{-c})$ to P_ℓ , and a fixed class c contributes $2nX^c = n(X^c + X^{-c})$. Hence

$$P_\ell = 2uT_q + P_b \quad (v \leq a), \quad P_\ell = 2(u + 1)T_q - P_b \quad (v > a),$$

and the same holds for ρ with ρ_Y in place of b . Since $T_q^2 = qT_q$ and $T_qP_x = 2sT_q$ for $x \in \mathbb{Z}^s$, every term of (6) that contains $T_q(X)$ or $T_q(X^2)$ is the same for ℓ and for ρ , and

$$\mathcal{R}_\ell - \mathcal{R}_\rho = \frac{P_b^2 \pm P_b(X^2)}{2} - \frac{P_{\rho_Y}^2 \pm P_{\rho_Y}(X^2)}{2} = \mathcal{R}_b^Y - \mathcal{R}_{\rho_Y}^Y,$$

with the upper sign for $v \leq a$. Here $\mathcal{R}_x^Y = \sum_{\alpha \in \Phi(Y)} X^{(x, \alpha)}$, which is $(P_x^2 + P_x(X^2))/2 - s$ for C_s by (6), and $(P_x^2 - P_x(X^2))/2 - s$ for D_s , whose roots $\pm e_i \pm e_j$ are those of C_s without the $\pm 2e_i$.

The value. The limit in the proof of Theorem A uses only $N_q(\ell) = r_q$, so

$$\chi_\lambda(g_{p/q}) = (-1)^{E_{p,q}(\lambda)} \frac{Z_q(\ell)}{Z_q(\rho)} \cdot \frac{\Pi(\ell)}{\Pi(\rho)}, \quad \Pi(x) = \prod_{\alpha > 0, q \nmid (x, \alpha)} |\sin(\pi p(x, \alpha)/q)|.$$

Now $\Pi(x)^2$ is the product of $|\sin(\pi pc/q)|$ over the non-zero residues c of $\mathcal{R}_x$, counted with multiplicity, and $\Pi^Y(x)^2$ is the same product for $\mathcal{R}_x^Y$, none of whose residues is zero. The identity of residue multisets $\mathcal{R}_\ell + \mathcal{R}_{\rho_Y}^Y = \mathcal{R}_\rho + \mathcal{R}_b^Y$ therefore gives $\Pi(\ell)/\Pi(\rho) = \Pi^Y(b)/\Pi^Y(\rho_Y)$.

The dimensions. As in the proof of Theorem B, $Z_q(\ell)/Z_q(\rho)$ is Δ_λ times the ratio of the constants V_n, C_n, B_n for the occupancies of ℓ and of ρ , since the dimensions for ρ are 1 and the powers of q agree. If $v \leq a$ the two multisets of occupancies agree class type by class type and the ratio is 1. If $v > a$, the occupancies agree outside the classes that fall short, a class falling short multiplies the constant by $V_{2u+1}/V_{2u+2} = 1/(2u+1)!$, by $B_u/B_{u+1} = 1/(2u+1)!$ or, for the class 0, by $C_u/C_{u+1} = 1/(2(2u+1)!)$, and both ℓ and ρ have $q-v$ such classes, the class 0 among them for ρ . The ratio is 2 if the class 0 is full for ℓ and 1 otherwise, that is, κ .

The cases $v \in \{0, a, q-1\}$. If $v = 0$ then $s = 0$ and $Y = C_0$. If $v = a$ then $Y = C_a$, and the a distinct entries of b in $\{1, \dots, a\}$ give $b = (a, \dots, 1) = \rho_Y$. If $v = q-1$ then $Y = D_1$. In the three cases the last factor is 1. Now $v = 0$ means $q \mid m$; $v = a$ means $2m+1 = (2u+1)q$ for q odd and $2m+2 = (2u+1)q$ for q even; and $v = q-1$ means $q \mid m+1$. For $q \mid 2m+2$, with $d = (2m+2)/q$, the case d odd is $v = a$ with q even, where $\kappa = 1$, and the case d even is $v = q-1$, where $u+1 = d/2$; in both, κ is the $\kappa(\lambda)$ of Theorem B.

In plain words (commentary, not part of the paper)

At a general rational point the residue multisets of ℓ and ρ differ, and the quotient of sines in the proof of Theorem A is no longer 1. This theorem identifies it. Let b list the classes holding $2u+1$ entries when $v \leq a$, or, when $v > a$, the classes holding one entry fewer than the full occupancy ($2u+2$ for a non-fixed class, $u+1$ for a fixed one). Then $P_\ell = 2uT_q + P_b$ or $2(u+1)T_q - P_b$, every term containing T_q is common to ℓ and ρ , and $\mathcal{R}_\ell - \mathcal{R}_\rho = \mathcal{R}_b^Y - \mathcal{R}_{\rho_Y}^Y$ for a group $Y = C_s$ or D_s of rank $s \leq m$ (equal to m only for $q > 2m$, $Y = C_m$, and $q = 2m$, $Y = D_m$). So the quotient of sines is $|\chi_{b-\rho_Y}^Y(g_{p/q}^Y)|$, a character of $\text{Sp}(2s)$ or $\text{SO}(2s)$ at a regular element, and $\chi_\lambda = \pm \kappa \Delta_\lambda |\chi^Y|$. For $v \in \{0, a, q-1\}$ the last factor is 1. Known cases: $v = 0$ (Ayyer–Kumari), $q \mid 2m+1$ (Albion), $v = 1$ (related to Kumari). The paper states that the reduction for $q \leq 2m$, $2 \leq v \leq q-1$, $v \neq a$, and its residual factor of type D , are new.

Evidence & scope

- **Not formalized in Lean.** Proof on paper.
- **Checked by program `valor_todo_q.py`:** all 4507 surviving (λ, p) , $m = 2, \dots, 5$, $q = 2, \dots, 11$, p prime to q , $\lambda_1 \leq 4$ (box stated in the paper); 0 failures (`valor_todo_q_OUT.txt`). **Floating point:** characters of $\mathrm{Sp}(2m)$, $\mathrm{Sp}(2s)$, $\mathrm{SO}(2s)$ as Weyl determinant quotients at a perturbation of the element (`mpmath`, 700 digits), tolerance $10^{-6}(1 + |\chi|)$; no rigorous error bound. The evaluator `carac.py` uses no formula of the paper; b , Y , κ , E are transcribed from the statement, which is what is tested. Negative control: $\rho_{D_s} \rightarrow (s, \dots, 1)$ breaks 1224 cases.
- **`valor_todo_q.py` casos:** 600 cases at six listed (m, q) , among them 192 with $u = 1$, $Y = D_3$, $b \neq \rho_Y$; $\chi = \chi^{C_v}$ with $\Delta_\lambda = 1$ on 164 weights equal to their q -core with at most v parts. Same floating-point method; the control breaks 192.
- **`prefactores.py` (e):** 11 244 cases, $m \leq 6$, $q \leq 16$, 0 failures. Second evaluator, independent of `carac.py`: the character of $\mathrm{Sp}(2m)$ exact in `evlib.py`, then evaluated to 40 digits; χ^Y by the Weyl determinant; tolerance 10^{-25} . Floating point.
- **`residual_factorizacion.py`:** an earlier form of $|\chi|$, $p = 1$, 1814 weights, $m \leq 6$, $q \leq 12$, floating point; four mutations each break 355 to 390 of 1145 cases.
- **Gap:** away from the integer points of Proposition 8.1, every check of the residual factor for $v \notin \{0, a, q - 1\}$ is floating point. At the integer points with $v \notin \{0, a, q - 1\}$ ($q = 6$, $m \equiv 1, 4 \pmod{6}$; $q \geq 4$ even, $q \mid 2m$, $q \nmid m$) it is checked exactly at $p = 1$, through $c_\lambda = \kappa R_1(b)$, by `prefactores.py` and `formula_pura.py`. The constant ratios $1/(2u + 1)!$ and $1/(2(2u + 1)!)!$ have no program.

Corollary 10.2. p. 13

Let $q \mid m$ or $q \mid 2m + 1$, let $\gcd(p, q) = 1$, and let λ be a dominant weight of $\mathrm{Sp}(2m)$. Then $\chi_\lambda(g_{p/q}) \neq 0$ if and only if each non-fixed folded class modulo q contains at most $2m/q$, respectively $(2m + 1)/q$, of the entries of $\ell = \lambda + \rho$, and the fixed classes at most m/q , respectively $(2m + 1 - q)/(2q)$; these bounds are then attained. In that case

$$\chi_\lambda(g_{p/q}) = (-1)^{E_{p,q}(\lambda)} \frac{Z_q(\lambda + \rho)}{Z_q(\rho)}, \quad |\chi_\lambda(g_{p/q})| = \Delta_\lambda,$$

where Δ_λ is the product of dimensions of Theorem B.

↳ Not used by another result of the paper.

Proof

The two cases are $v = 0$, with $u = m/q$, and $v = a$ with q odd, with $u = (2m + 1 - q)/(2q)$. In both $v \leq a$, so the occupancies of the surviving weights, listed before Theorem 10.1, are $2u$ or $2u + 1$ in every non-fixed class and u in every fixed class; they add up to m , so the bounds are attained, and a placement exceeding one of them costs more than r_q by (5) and the marginal costs in the proof of Theorem 9.1. Since $v \leq a$, $\kappa = 1$, and the last factor of Theorem 10.1 is 1. Its proof gives $\chi_\lambda(g_{p/q}) = (-1)^{E_{p,q}(\lambda)} (Z_q(\ell)/Z_q(\rho)) \Pi(\ell)/\Pi(\rho)$ with $\Pi(\ell)/\Pi(\rho)$ equal to that factor, and $Z_q(\ell)/Z_q(\rho) = \kappa \Delta_\lambda$.

In plain words (commentary, not part of the paper)

The two families $q \mid m$ ($v = 0$) and $q \mid 2m + 1$ ($v = a$, q odd) are cases of Theorem 10.1 where nothing is left over, as is $q \mid 2m + 2$ with d odd: $v \leq a$, so $\kappa = 1$, and the residual factor is 1. Survival becomes a pure capacity rule: at most $2m/q$, resp. $(2m + 1)/q$, entries of ℓ in each non-fixed class and m/q , resp. $(2m + 1 - q)/(2q)$, in each fixed class, and the bounds are then attained. On the survivors the closed form of Theorem A(2) holds and $|\chi_\lambda| = \Delta_\lambda$. The paper attributes these values to known results: the case $v = 0$ is Ayer–Kumari’s factorisation at $X = (1, \dots, 1)$, and $q \mid 2m + 1$ follows from Albion. The rest of the family $q \mid 2m$ behaves differently: at $\text{Sp}(4)$, $q = 4$, $\lambda = (1, 0)$ has Z -quotient 1 and value -2 .

Evidence & scope

- **Not formalized in Lean.** Proof on paper.
- **Exact:** `formula_pura.py` finds the closed form on every surviving (λ, p) of its box at each pair of these two families with $m \leq 8$, $q \leq 30$; `prefactores.py` finds $c = 1$, that is $|\chi_\lambda| = \Delta_\lambda$, in both families.
- `valor_todo_q.py` casos: 312 of 312 cases at $(m, q) = (3, 3), (5, 5), (2, 5), (4, 3), (6, 13)$; floating point, as for Theorem 10.1.
- The capacity bounds are not checked separately; they are covered by the checks of Proposition 3.1.

References

- 1 A. Ayer and N. Kumari, *Factorization of classical characters twisted by roots of unity*, J. Algebra **609** (2022), 437–483. → [doi:10.1016/j.jalgebra.2022.06.015](https://doi.org/10.1016/j.jalgebra.2022.06.015)
- 2 A. Ayer and N. Kumari, *Further results for classical and universal characters twisted by roots of unity*, J. Algebraic Combin. **63** (2026), Paper No. 2; [arXiv:2501.00275](https://arxiv.org/abs/2501.00275). Statements are cited by their numbering in version 2 of the arXiv preprint.
- 3 S. Albion, *Universal characters twisted by roots of unity*, Algebr. Comb. **6** (2023), 1653–1676.  [find it](#)
- 4 S. Albion, *Character factorisations, z -asymmetric partitions and plethysm*, [arXiv:2501.18520](https://arxiv.org/abs/2501.18520).
- 5 P. Cellini, P. Möseneder Frajria and P. Papi, *The $\widehat{W}$ -orbit of ρ , Kostant’s formula for powers of the Euler product and affine Weyl groups as permutations of $\mathbb{Z}$* , J. Pure Appl. Algebra **208** (2007), 1103–1119.  [find it](#)

6

T. Gannon and A. Schopieray, *Algebraic number fields generated by dimensions in fusion rings*, Commun. Number Theory Phys. **18** (2024), no. 4, 705–743; [arXiv:1912.12260](#). Statements are cited by their numbering in version 1 of the arXiv preprint.

7

V. G. Kac, *Simple Lie groups and the Legendre symbol*, in: Algebra, Carbondale 1980, Lecture Notes in Math. **848**, Springer, 1981, 110–123. → [doi:10.1007/bfb0090559](#)

8

C. Karmakar, *Character values at elements of order 2*, Proc. Indian Acad. Sci. Math. Sci. **135** (2025), Paper No. 41. → [doi:10.1007/s12044-025-00847-0](#)

9

B. Kostant, *On Macdonald's η -function formula, the Laplacian and generalized exponents*, Adv. Math. **20** (1976), 179–212. → [doi:10.1016/0001-8708\(76\)90186-9](#)

10

B. Kostant, *Powers of the Euler product and commutative subalgebras of a complex simple Lie algebra*, Invent. Math. **158** (2004), 181–226. → [doi:10.1007/s00222-004-0370-7](#)

11

S. Kumar and D. Prasad, *Character of irreducible representations restricted to finite order elements: an asymptotic formula*, J. Algebra **676** (2025), 69–81. → [doi:10.1016/j.jalgebra.2025.03.030](#)

12

N. Kumari, *Factorization of classical characters twisted by roots of unity: II*, J. Pure Appl. Algebra **228** (2024), no. 11, Paper No. 107714; [arXiv:2212.12477](#). Statements are cited by their numbering in the arXiv preprint.

13

C. Lecouvey, *Parabolic Kazhdan–Lusztig polynomials, plethysm and generalized Hall–Littlewood functions for classical types*, European J. Combin. **30** (2009), 157–191. → [doi:10.1016/j.ejc.2008.02.007](#)

14

D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, Oxford University Press, 1940; 2nd ed. 1950.  [find it](#)

15

C. Marín, *Schur polynomials twisted by roots of unity and reciprocal pairs: exactly three factors, and where they vanish*, [arXiv:2608.09619](#). Statements are cited by their numbering in version 3.

16

R. V. Moody and J. Patera, *Characters of elements of finite order in Lie groups*, SIAM J. Algebraic Discrete Methods **5** (1984), 359–383.  [find it](#)

17

H. Mizukawa, *Factorization of Schur functions of various types*, J. Algebra **269** (2003), 215–226. → [doi:10.1016/s0021-8693\(03\)00527-1](https://doi.org/10.1016/s0021-8693(03)00527-1)

18

S. Nadimpalli, S. Pattanayak and D. Prasad, *Character theory at a torsion element*, [arXiv:2504.14684](https://arxiv.org/abs/2504.14684).

19

A. Pianzola, *On the arithmetic of the representation ring and elements of finite order in Lie groups*, J. Algebra **108** (1987), 1–33.  [find it](#)

20

A. Pianzola and A. Weiss, *The rationality of elements of prime order in compact connected simple Lie groups*, J. Algebra **144** (1991), 510–521.  [find it](#)

21

D. Prasad, *A character relationship on $\mathrm{GL}_n(\mathbb{C})$* , Israel J. Math. **211** (2016), 257–270.  [find it](#)

22

M. Reeder, *Weyl group characters afforded by zero weight spaces*, Transform. Groups **29** (2024), 1161–1198; [arXiv:1910.02463](https://arxiv.org/abs/1910.02463). Statements are cited by their numbering in version 2 of the arXiv preprint.