# The Pentagon, Machine-Checked: Abel's five-term relation for the dilogarithm in Lean 4

Special functions · the dilogarithm

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# At a glance

## What the paper contributes

A machine-checked proof of the identity that generates the weight-2 theory of the dilogarithm, on the dilogarithm built in Lean by the companion paper.

- **Abel's five-term relation** (Theorem 1), sorry-free in Lean 4: for x, y in (0,1), $L(x) + L(y) = L(xy) + L(x(1-y)/(1-xy)) + L(y(1-x)/(1-xy))$, with L the Rogers L-function.
- **The classical route, made formal**: the five-term difference has zero derivative in x (nine logarithms collapse into a basis of five), so it is constant; its value is read off as $x \to 0^+$, where $\log t \cdot \log(1-t) \to 0$ by a squeeze.
- **Three reusable blocks**: the derivative of L on (0,1), $L(t) \to 0$ as $t \to 0^+$, continuity of L on (0,1).
- The mathematics is classical (Spence, Abel, Rogers); the paper claims the formalization, after a documented search of proof-assistant libraries.

🚀 **Start here: Theorem 1**: Abel's five-term relation for the Rogers L-function on $(0,1)^2$.

**Main results:**

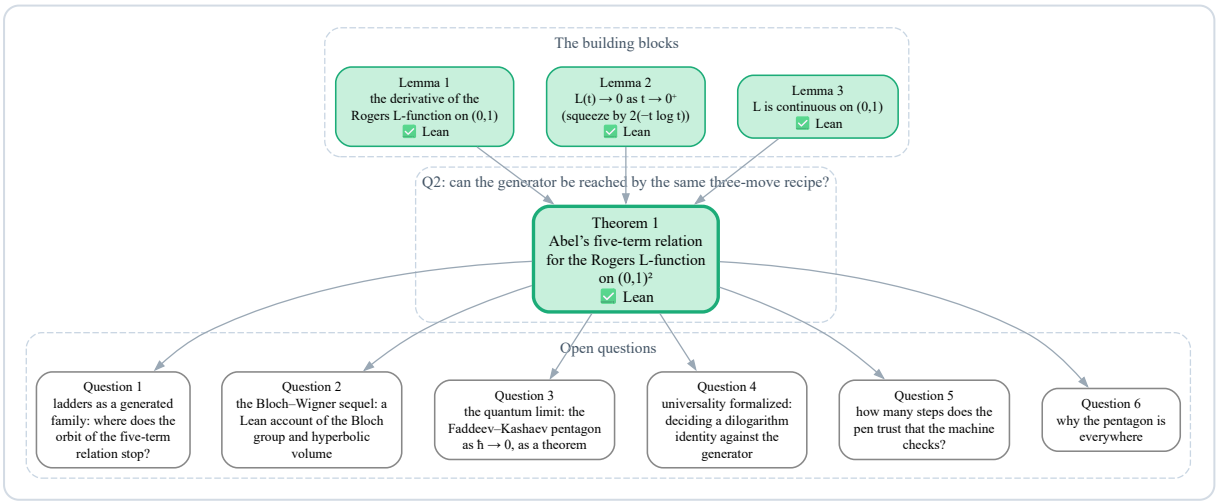
- **Theorem 1**: Abel's five-term relation for the Rogers L-function on $(0,1)^2$

**10** results | ✅ **4** entirely in Lean | ◐ **0** partly in Lean

💻 **0** checked by a program | written proof only | (a result can carry several tags)

### 🗺️ Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.




### 📐 Setting (copied from the paper, §1)

The cleanest statement uses the *Rogers* L-*function* $\mathrm{L}(x) := \mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x)$, which absorbs the stray logarithms of the raw $\mathrm{Li}_2$ form and turns the relation into a pure sum of five L values. *(§2)* Fix $y \in (0,1)$ and set

$$G(t) = \mathrm{L}(t) - \mathrm{L}(ty) - \mathrm{L}\Big(\tfrac{t(1-y)}{1-ty}\Big) - \mathrm{L}\Big(\tfrac{y(1-t)}{1-ty}\Big).$$

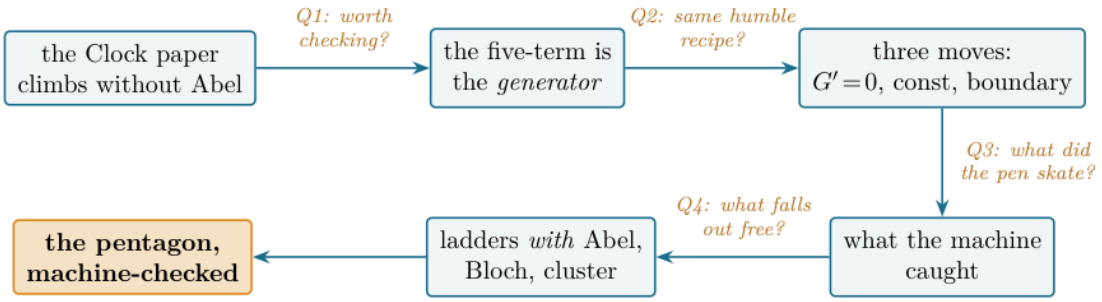
The theorem is $G(x) = -\mathrm{L}(y)$.

# Introduction

No numbered statement in this section.

Its figures and tables are below; its text is in the paper (PDF).

**Figure 1**



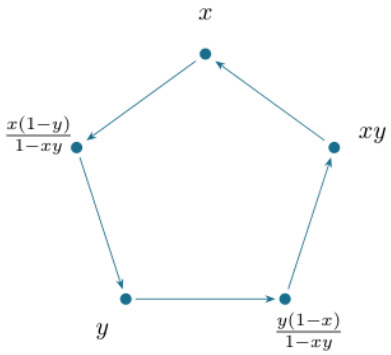

The reader's map. The thread starts where the companion *Clock* paper left it (the golden ladder climbed without Abel) and closes at the machine-checked pentagon; each arrow is the question its section answers. Every node is a block of `sorry`-free Lean.

# 1 Q1: why machine-check an identity the companion paper did without?

No numbered statement in this section.

Its figures and tables are below; its text is in the paper, p. 2.

🖼️ **Figure 2**




The five arguments on a cycle. The five-fold symmetry of this pentagon is the combinatorial shadow shared by the Bloch-group relation, the cluster-algebra mutation pentagon, and the quantum-dilogarithm pentagon identity.

# 2 Q2: can the generator be reached by the same three-move recipe?

**Theorem 1.** ✅ Lean (`Dilog.rogersL_fiveterm`)✓ 📕 p. 3

*For all* $x, y \in (0,1)$,

$$\mathrm{L}(x)+\mathrm{L}(y)=\mathrm{L}(xy)+\mathrm{L}\Big(\tfrac{x(1-y)}{1-xy}\Big)+\mathrm{L}\Big(\tfrac{y(1-x)}{1-xy}\Big).$$

**Lean · ours** show

- `Dilog.rogersL_fiveterm`
- `Dilog.fiveterm_hasDerivAt_zero`

✅ Checked when this page was built: FiveTerm.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — rogersL_fiveterm, fiveterm_hasDerivAt_zero: propext, Classical.choice, Quot.sound. Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean FiveTerm.lean` (SHA-256 of the compiled file: `ce29fa490a366ddb…`; a source page on this site may show it with comments edited, under its own hash).

**Trust base:** 🔒 kernel only. Lean: 2 results, standard axioms only

**Proof:** in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Theorem 1, p. 3, doi:10.5281/zenodo.20682716

🆕 **Classical and new:** classical (Spence, Abel, Rogers); the paper claims the formalization: «to the best of my knowledge the first machine-checked five-term relation, on the first

machine-checked dilogarithm, in any proof assistant» (§6), from a documented search of June 2026.

↳ Used in: 1 (p. 8), 2 (p. 8), 3 (p. 9), 4 (p. 9), 5 (p. 9), 6 (p. 9)

**Proof**

Formalized in Lean (see the linked declarations).

💡 **In plain words (commentary, not part of the paper)**

1. Take two numbers $x, y$ between 0 and 1 and form the three numbers $xy$, $\frac{x(1-y)}{1-xy}$ and $\frac{y(1-x)}{1-xy}$. The Rogers function L of the first two adds up to L of the other three. For $x = y = \frac{1}{2}$ the three numbers are $\frac{1}{4}, \frac{1}{3}, \frac{1}{3}$, and since $\mathrm{L}(\frac{1}{2}) = \pi^2/12$ the relation says $\mathrm{L}(\frac{1}{4}) + 2\,\mathrm{L}(\frac{1}{3}) = \pi^2/6$. 2. Why it is true (the classical argument, here checked by Lean): fix $y$ and look at the difference $G(t)$ of the two sides as a function of $t$. Its derivative is zero on $(0, 1)$, because every logarithm that appears splits into the five logarithms of $t$, $y$, $1 - t$, $1 - y$, $1 - ty$ and all coefficients cancel. So $G$ is constant, and its value is read off as $t \to 0^+$, where $\mathrm{L}(t) \to 0$ by squeezing $\log t \cdot \log(1 - t)$ between 0 and $2(-t \log t)$. 3. New and not new, as the paper says: the identity is classical (Spence, Abel, Rogers) and the proof is the standard differentiation proof; the paper claims only the formalization, to the author's knowledge the first machine-checked five-term relation in any proof assistant, after a documented search (§6), on the dilogarithm built in Lean by the companion paper.

# 3 Q3: what did the machine catch that the pen skates over?

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper, p. 4.

# 4 Q4: with the pentagon certified, what falls out for free?

No numbered statement in this section.

Its figures and tables are below; its text is in the paper, p. 5.

| $(x, y)$ | arguments | the relation reads | exact value |
|---|---|---|---|
| $(\frac{1}{2}, \frac{1}{2})$ | $\frac{1}{4}, \frac{1}{3}, \frac{1}{3}$ | $2\,\mathrm{L}(\frac{1}{2}) = \mathrm{L}(\frac{1}{4}) + 2\,\mathrm{L}(\frac{1}{3})$ | $\mathrm{L}(\frac{1}{4}) + 2\,\mathrm{L}(\frac{1}{3}) = \frac{\pi^2}{6}$ |

| | | | |
|---|---|---|---|
| $(\frac{1}{\varphi}, \frac{1}{\varphi})$ | $\frac{1}{\varphi^2}, \frac{1}{\varphi^2}, \frac{1}{\varphi^2}$ | $2\,\mathrm{L}(\frac{1}{\varphi}) = 3\,\mathrm{L}(\frac{1}{\varphi^2})$ | $\frac{\pi^2}{5} = \frac{\pi^2}{5}$ |

**Table 1:** The generator at work: two choices of $(x, y)$ whose composite arguments coincide, and the exact identity each prints. Both rows are re-verified to thirty digits; the first is hand-checkable from $\mathrm{L}(\frac{1}{2}) = \pi^2/12$, the second from the ladder values $\mathrm{L}(1/\varphi) = \pi^2/10$, $\mathrm{L}(1/\varphi^2) = \pi^2/15$. Euler's reflection $\mathrm{L}(x) + \mathrm{L}(1-x) = \pi^2/6$ is *not* an instance of the five-term relation but its companion—the symmetrization that makes the Rogers form clean—and is proved separately.

# 5 The building blocks

**Lemma 1.** Lean (`Dilog.hasDerivAt_rogersL`)✓ p. 6

> *On* $(0,1)$, $\mathrm{L}'(x) = -\frac{1}{2}\left(\frac{\log(1-x)}{x} + \frac{\log x}{1-x}\right)$.

Lean · ours **show**

- `Dilog.hasDerivAt_rogersL`

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Axioms — hasDerivAt_rogersL: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean FiveTerm.lean` (SHA-256 of the compiled file: `ce29fa490a366ddb…`; a source page on this site may show it with comments edited, under its own hash).

**Trust base:** 🔒 kernel only. Lean: 1 result, standard axioms only

**Proof:** in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Lemma 1, p. 6, doi:10.5281/zenodo.20682716

🆕 **Classical and new:** the paper: one of «three reusable building blocks» of the formalization (§5); «the same derivative scheme the companion used for reflection and Landen».

↳ Used in: Theorem 1 (p. 4)

**Proof**

Formalized in Lean (see the linked declarations).

**Lemma 2.** Lean (`Dilog.tendsto_rogersL_nhdsGT_zero`)✓ p. 6

> $\mathrm{L}(t) \to 0$ *as* $t \to 0^+$.

Lean · ours **show**

- `Dilog.tendsto_rogersL_nhdsGT_zero`

✅ Checked when this page was built: FiveTerm.lean compiles with Lean 4.35.0, no errors, no sorry.
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**Proof:** in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Lemma 2, p. 6, doi:10.5281/zenodo.20682716

🆕 **Classical and new:** the paper: a reusable block; «the only place the proof leaves pure algebra for analysis» (§5).

↳ Used in: Theorem 1 (p. 4)

**Proof**

Formalized in Lean (see the linked declarations).

**Lemma 3.** ✅ Lean (`Dilog.continuousAt_rogersL`)✓ 📕 p. 6

L *is continuous at every* $x \in (0,1)$.

Lean · ours **show**

- `Dilog.continuousAt_rogersL`

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**Trust base:** 🔒 kernel only. Lean: 1 result, standard axioms only

**Proof:** in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Lemma 3, p. 6, doi:10.5281/zenodo.20682716

🆕 **Classical and new:** the paper: a reusable block, «needed for the one surviving boundary term» (§5).

↳ Used in: Theorem 1 (p. 4)

**Proof**

Formalized in Lean (see the linked declarations).

| Lean name | statement | where |
|---|---|---|
| `rogersL_fiveterm` | the five-term relation (Theorem 1) | §2 |
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| `hasDerivAt_rogersL` | $\mathrm{L}'(x) = -\frac{1}{2}\left(\frac{\log(1-x)}{x} + \frac{\log x}{1-x}\right)$ | §5 |
| `tendsto_rogersL_nhdsGT_zero` | $\mathrm{L}(t) \to 0$ as $t \to 0^+$ | §5 |
| `continuousAt_rogersL` | L continuous on $(0,1)$ | §5 |

**Table 2:** The load-bearing results, all `sorry`-free in `Dilog/FiveTerm.lean` of the repository, on the dilogarithm of `Dilog/Basic.lean` from the companion paper. Axioms of every entry: `propext`, `Classical.choice`, `Quot.sound` only.

# 6 The boundary of the contribution

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper, p. 6.

# 7 Open questions

**Question 1.** p. 7

The pentagon is now a theorem in the library. Standing on it, six questions worth the next effort—the last one is the one I would keep.

1. *Ladders as a generated family.* Each row of Table 1 is the five-term relation evaluated at a point; mechanizing the substitution so that every ladder *prints* from the one theorem would turn a catalogue into a corollary. How much of the classical table of dilogarithm values is a finite orbit of the five-term relation under $\mathbb{Q}$-rational substitution—and where does that orbit stop?
2. *The Bloch–Wigner sequel.* The same relation holds for the single-valued Bloch–Wigner function on $\mathbb{C}$, where it underlies hyperbolic volume and scissors congruence [ 8 , 9 , 10 ] .

**Cite:** Question 1, p. 7, doi:10.5281/zenodo.20682716

↳ Not used by another result of the paper.

**Question 2.** p. 7

What is the smallest faithful Lean account of the Bloch group in which Vol of a hyperbolic 3-manifold becomes a computable, certified number? *The quantum limit.* The Faddeev–Kashaev quantum dilogarithm satisfies a pentagon identity [ 12 ] whose $\hbar \to 0$ limit is the relation proved here.

**Cite:** Question 2, p. 7, doi:10.5281/zenodo.20682716

↳ Not used by another result of the paper.

**Question 3.** p. 7

Can a proof assistant make that limit a theorem rather than a heuristic—and would doing so expose, as the classical proof did, a step the physics literature skates over? *Universality, formalized.* Wojtkowiak and de Jeu reduce *every* rational functional equation of $\mathrm{Li}_2$ to the five-term relation [ 5 , 6 ] . That reduction is an algorithm.

**Cite:** Question 3, p. 7, doi:10.5281/zenodo.20682716

↳ Not used by another result of the paper.

**Question 4.** p. 7

What would it take to run it inside Lean, so that an arbitrary claimed dilogarithm identity is *decided* against the generator? *What the machine sees and the page hides.* Section 3 found one place where the formal proof was strictly sharper than the informal one—an ordered rewrite the pen trusts.

**Cite:** Question 4, p. 7, doi:10.5281/zenodo.20682716

↳ Not used by another result of the paper.

**Question 5.** p. 7

In a subject two centuries old and re-proved a dozen times, how many such silent steps remain, and is “the proof a human believes” ever quite the proof a machine will accept? *One to keep.* The five-term relation says five dilogarithm values, read around a pentagon, sum to nothing new. It is the defining relation of the Bloch group, of cluster mutation, of the quantum dilogarithm—one identity wearing the masks of geometry, algebra, and physics.

**Cite:** Question 5, p. 7, doi:10.5281/zenodo.20682716

↳ Not used by another result of the paper.

**Question 6.** p. 7

Now that it is machine-checked, the question is not whether it is true but why it is *everywhere*: what is it about a pentagon that so many parts of mathematics are, underneath, the same five steps that return you home?

**Cite:** Question 6, p. 7, doi:10.5281/zenodo.20682716

↳ Not used by another result of the paper.

**Question**

# Reproducing

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper (PDF).

# Tables

Every table of the paper, in its order: click one to unfold it, with where the paper cites it, how its numbers were checked, and its data as CSV. The typeset table is one click away, on its page of the PDF.

**Table 1. The generator at work: two choices of $(x, y)$ whose composite arguments coincide, and the exact identity each prints. Both rows are re-verified to thirty digits; the first is hand-checkable from $\mathrm{L}(\frac{1}{2}) = \pi^2/12$, the second from the ladder values $\mathrm{L}(1/\varphi) = \pi^2/10$, $\mathrm{L}(1/\varphi^2) = \pi^2/15$. Euler's reflection $\mathrm{L}(x) + \mathrm{L}(1-x) = \pi^2/6$ is *not* an instance of the five-term relation but its companion—the symmetrization that makes the Rogers form clean—and is proved separately.**

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p. 5 in the PDF ⬇ CSV

cited in §4 Q4: with the pentagon certified, what falls out for free?

cited in §7 Open questions

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| `continuousAt_rogersL` | L continuous on $(0,1)$ | §5 |

p. 6 in the PDF · ⬇ CSV · cited in the opening, before §1 · cited in §5 The building blocks · cited in §6 The boundary of the contribution · cited in §7 Open questions

# References


1

N. H. Abel, *Œuvres complètes* (L. Sylow, S. Lie, eds.), vol. II, Grøndahl, Christiania, 1881; the five-term relation is in the note on the dilogarithm. 🔎 find it

2

L. J. Rogers, On function sum theorems connected with the series $\sum x^n/n^2$, *Proc. London Math. Soc.* **4** (1907) 169–189. 🔎 find it

3

L. Lewin, *Polylogarithms and Associated Functions*, North-Holland, 1981. 🔎 find it

4

D. Zagier, The dilogarithm function, in *Frontiers in Number Theory, Physics, and Geometry II*, Springer, 2007, 3–65. 🔎 find it

5

Z. Wojtkowiak, The basic structure of polylogarithmic functional equations, in *Structural Properties of Polylogarithms* (L. Lewin, ed.), Math. Surveys Monogr. **37**, AMS, 1991, 205–231. 🔎 find it

6

R. de Jeu, Towards regulator formulae for the $K$-theory of curves and number fields, *Compositio Math.* **124** (2000) 137–194. 🔎 find it

7

S. Bloch, *Higher Regulators, Algebraic K-theory, and Zeta Functions of Elliptic Curves*, CRM Monogr. Ser. **11**, AMS, 2000 (Irvine lectures, 1978). 🔎 find it

8

A. A. Suslin, $K_3$ of a field, and the Bloch group, *Proc. Steklov Inst. Math.* **183** (1991) 217–239. 🔎 find it

9

W. D. Neumann, D. Zagier, Volumes of hyperbolic three-manifolds, *Topology* **24** (1985) 307–332. 🔎 find it

10

A. B. Goncharov, Geometry of configurations, polylogarithms, and motivic cohomology, *Adv. Math.* **114** (1995) 197–318. 🔎 find it

11

T. Nakanishi, Dilogarithm identities for conformal field theories and cluster algebras, in *Symmetries, Integrable Systems and Representations*, Springer Proc. Math. Stat. **40** (2013) 407–444. 🔎 find it

12

L. D. Faddeev, R. M. Kashaev, Quantum dilogarithm as a kernel of the quantum modular double, *Modern Phys. Lett. A* **9** (1994) 427–434. 🔎 find it

13

A. N. Kirillov, Dilogarithm identities, *Prog. Theor. Phys. Suppl.* **118** (1995) 61–142. 🔎 find it

14

Al. B. Zamolodchikov, Thermodynamic Bethe ansatz in relativistic models, *Nucl. Phys. B* **342** (1990) 695–720. 🔎 find it

15

W. Nahm, Conformal field theory and torsion elements of the Bloch group, in *Frontiers in Number Theory, Physics, and Geometry II*, Springer, 2007, 67–132. 🔎 find it

16

C. Marín, *The Clock That Never Ticks: a machine-checked path from the dilogarithm to quantum speed limits in Lean 4*, Zenodo, 2026. doi:10.5281/zenodo.20675271.

17

The mathlib Community, The Lean mathematical library, in *CPP 2020*, ACM, 2020, 367–381. 🔎 find it