The Pentagon, Machine-Checked: Abel's five-term relation for the dilogarithm in Lean 4

Special functions · the dilogarithm

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At a glance

What the paper contributes

A machine-checked proof of the identity that generates the weight-2 theory of the dilogarithm, on the dilogarithm built in Lean by the companion paper.

- **Abel's five-term relation** (Theorem 1), sorry-free in Lean 4: for x, y in (0,1), $L(x) + L(y) = L(xy) + L(x(1-y)/(1-xy)) + L(y(1-x)/(1-xy))$, with L the Rogers L-function.
- **The classical route, made formal**: the five-term difference has zero derivative in x (nine logarithms collapse into a basis of five), so it is constant; its value is read off as $x \to 0^+$, where $\log t \cdot \log(1-t) \to 0$ by a squeeze.
- **Three reusable blocks**: the derivative of L on (0,1), $L(t) \to 0$ as $t \to 0^+$, continuity of L on (0,1).
- The mathematics is classical (Spence, Abel, Rogers); the paper claims the formalization, after a documented search of proof-assistant libraries.

🚀 **Start here: Theorem 1**: Abel's five-term relation for the Rogers L-function on $(0,1)^2$.

Main results:

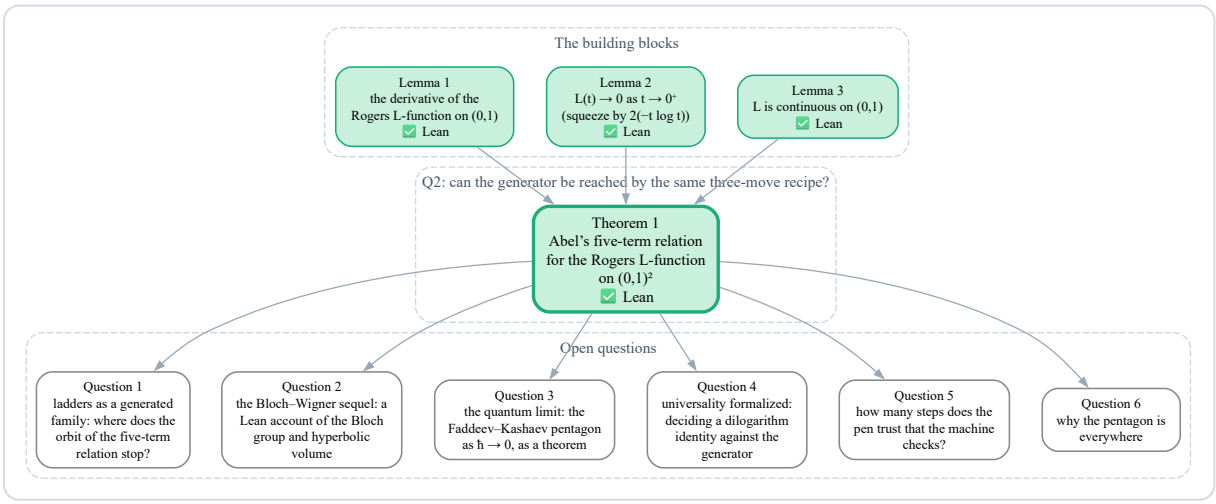
- **Theorem 1**: Abel's five-term relation for the Rogers L-function on $(0,1)^2$

10 results | ✅ **4** entirely in Lean | ◐ **0** partly in Lean

💻 **0** checked by a program | written proof only | (a result can carry several tags)

🗺️ Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



📐 Setting (copied from the paper, §1)

The cleanest statement uses the *Rogers* L*-function* $\mathrm{L}(x) := \mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x)$, which absorbs the stray logarithms of the raw Li_2 form and turns the relation into a pure sum of five L values. *(§2)* Fix $y \in (0,1)$ and set

$$G(t) = \mathrm{L}(t) - \mathrm{L}(ty) - \mathrm{L}\Big(\tfrac{t(1-y)}{1-ty}\Big) - \mathrm{L}\Big(\tfrac{y(1-t)}{1-ty}\Big).$$

The theorem is $G(x) = -\,\mathrm{L}(y)$.

Abstract.

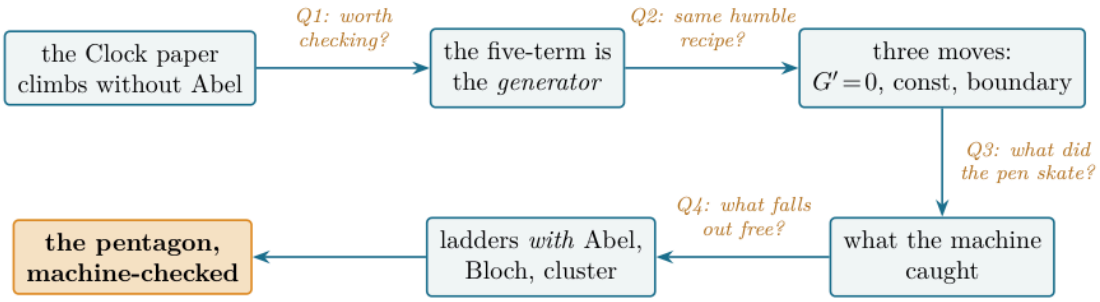
I give a machine-checked, `sorry`-free proof, in Lean 4 over Mathlib, of *Abel's five-term relation* for the dilogarithm. With L the Rogers L-function it reads, for $x, y \in (0,1)$,

$$\mathrm{L}(x) + \mathrm{L}(y) = \mathrm{L}(xy) + \mathrm{L}\Big(\tfrac{x(1-y)}{1-xy}\Big) + \mathrm{L}\Big(\tfrac{y(1-x)}{1-xy}\Big).$$

This single identity is the whole weight-2 theory in one line: by theorems of Wojtkowiak and de Jeu, *every* functional equation of Li_2 with rational arguments is a consequence of it; it is the defining relation of the Bloch group and the pentagon of cluster algebras and of the quantum dilogarithm. A companion paper, *The Clock That Never Ticks*, built a dilogarithm in Lean from scratch and climbed the golden-ratio ladder *without* this relation, leaving it explicitly open; this paper closes that thread. The proof is the classical differentiation argument made fully rigorous: fixing y, the five-term difference has identically zero derivative on $(0,1)$—nine logarithmic terms collapse into a basis of five and the rest is `ring`—so it is constant, and its boundary value as $x \to 0^+$ is computed by squeezing $\log x \cdot \log(1-x) \to 0$. I am exact about the boundary: the mathematics is entirely classical (Spence, Abel, Rogers; Wojtkowiak's and de Jeu's universality); the contribution is the formalization—to the best of my knowledge the first machine-checked five-term relation, on the first machine-checked dilogarithm, in any proof assistant—together with three reusable building blocks. The headline theorem and its blocks depend only on `propext`, `Classical.choice` and `Quot.sound`.

How to read this paper. It is the story of four questions, each opening a section and handing the reader the next, walking from a puzzle to a closed thread. The puzzle: a companion paper reached the dilogarithm's deepest evaluations *without* the identity everyone calls central—so is the identity worth machine-checking at all (Q1)? It is, because it is not one more identity but the *generator* of all of them; that earns it the proof (Q2), a proof the machine sharpened in one place a pen would skate over (Q3); and once it is certified, the showpieces the companion reached the long way round fall out as corollaries (Q4). If you read only one thing, read Theorem 1 and Figure 2; the Lean inventory is Table 2; the honest limits are in Section 6; reproduction is three commands.

Figure 1



The reader's map. The thread starts where the companion *Clock* paper left it (the golden ladder climbed without Abel) and closes at the machine-checked pentagon; each arrow is the question its section answers. Every node is a block of `sorry`-free Lean.

1 Q1: why machine-check an identity the companion paper did without?

Some identities matter for what they assert; this one matters for where it keeps reappearing. Take the five numbers that sit at the corners of a pentagon, apply the dilogarithm to each, and the five values are not free: they obey one fixed relation, the same one whatever pentagon you start from. That single fact is, in disguise, the defining relation of the Bloch group in hyperbolic geometry, the pentagon of mutations in cluster algebra, and the pentagon identity of the quantum dilogarithm in mathematical physics—one relation wearing three masks. It is two centuries old and has been rediscovered under a dozen names; and yet, after a decade of one of the most ambitious formalization efforts ever attempted, no proof assistant had ever been told it is true. This paper tells one—and then asks why such a thing should be everywhere at once.

A taste of what the relation does, before any theory. Put $x = y = \frac{1}{2}$. After $\frac{(1/2)(1/2)}{1-1/4} = \frac{1}{3}$ the relation below collapses to

$$2\,\mathrm{L}\big(\tfrac{1}{2}\big) = \mathrm{L}\big(\tfrac{1}{4}\big) + 2\,\mathrm{L}\big(\tfrac{1}{3}\big), \qquad \text{i.e.} \qquad \mathrm{L}\big(\tfrac{1}{4}\big) + 2\,\mathrm{L}\big(\tfrac{1}{3}\big) = \frac{\pi^2}{6},$$

since $\mathrm{L}(\frac{1}{2}) = \pi^2/12$. That a sum of three dilogarithm values at $\frac{1}{4}$ and $\frac{1}{3}$ is *exactly* $\pi^2/6$ is not obvious; it is one of infinitely many shadows of a single two-variable law—and that law is what we machine-check here.

History bead. The dilogarithm is Euler's (1760s). Its functional equations were charted by Landen (1780), Spence (1809), Abel (1827, published 1881 [1]) and Kummer; Rogers gave the symmetric form in 1907 [2] . The five-term relation's depth became structural only later, with Bloch's regulators [7] and the group that bears his name.

There is a reason to hesitate. A companion paper in this line, *The Clock That Never Ticks* [16] , built a dilogarithm in Lean from scratch—its series, its derivative $-\log(1-x)/x$, Euler's

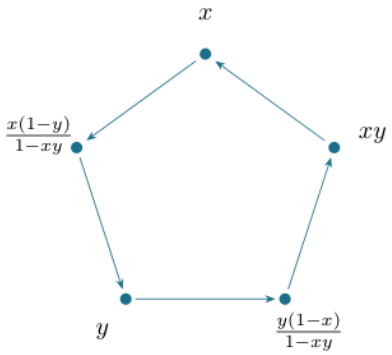
reflection, Landen's transformation, the duplication formula—and then reached the subject's classical showpieces, the golden-ratio evaluations $\mathrm{Li}_2(1/\varphi)$ and $\mathrm{Li}_2(1/\varphi^2)$, by a three-by-three linear system, with *no five-term relation at all*. Every textbook reaches those values through Abel; the formalization found a side door. So the honest first question is not how to prove the five-term relation but whether to bother: if the showpieces are reachable without it, what does the identity earn its keep on?

The answer is that the five-term relation is not one identity among several. It is the *generator* of the whole subject. Wojtkowiak proved that every one-variable functional equation of Li_2 with rational arguments is a $\mathbb{Z}$-linear combination of instances of the five-term relation; de Jeu extended this to any number of variables [5 , 6] . The same relation *defines* the Bloch group, hence governs scissors-congruence invariants and the volumes of hyperbolic 3-manifolds [7 , 8 , 9] ; in cluster algebras it is the *pentagon*, the relation of five mutations [11] ; and in its quantum form it is the pentagon identity of the Faddeev–Kashaev quantum dilogarithm [12] . The companion's side door climbs one ladder; the five-term relation is the staircase. That is what is worth certifying. *Can the same humble toolkit that proved reflection and Landen also reach the generator?*

The cleanest statement uses the *Rogers* L*-function* $\mathrm{L}(x) := \mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x)$, which absorbs the stray logarithms of the raw Li_2 form and turns the relation into a pure sum of five L values:

$$\mathrm{L}(x) + \mathrm{L}(y) = \mathrm{L}(xy) + \mathrm{L}\Big(\tfrac{x(1-y)}{1-xy}\Big) + \mathrm{L}\Big(\tfrac{y(1-x)}{1-xy}\Big), \qquad x, y \in (0,1).$$

Figure 2



The five arguments on a cycle. The five-fold symmetry of this pentagon is the combinatorial shadow shared by the Bloch-group relation, the cluster-algebra mutation pentagon, and the quantum-dilogarithm pentagon identity.

2 Q2: can the generator be reached by the same three-move recipe?

History bead. The differentiation proof is the oldest one: fix one variable, show the derivative in the other vanishes, evaluate the constant at an endpoint. It is essentially Abel's, repeated in

Lewin [3] and Zagier [4] . The whole question is whether “the derivative vanishes” and “evaluate at the endpoint” survive being made fully formal.

Theorem 1. ✅ Lean (`Dilog.rogersL_fiveterm`)✓ 📕 p. 3

For all $x, y \in (0,1)$,

$$\mathrm{L}(x) + \mathrm{L}(y) = \mathrm{L}(xy) + \mathrm{L}\Big(\tfrac{x(1-y)}{1-xy}\Big) + \mathrm{L}\Big(\tfrac{y(1-x)}{1-xy}\Big).$$

Lean · ours show

- `Dilog.rogersL_fiveterm`
- `Dilog.fiveterm_hasDerivAt_zero`

✅ Checked when this page was built: FiveTerm.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — rogersL_fiveterm, fiveterm_hasDerivAt_zero: propext, Classical.choice, Quot.sound. Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean FiveTerm.lean` (SHA-256 of the compiled file: `ce29fa490a366ddb…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only. Lean: 2 results, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Theorem 1, p. 3, doi:10.5281/zenodo.20682716

🆕 **Classical and new:** classical (Spence, Abel, Rogers); the paper claims the formalization: «to the best of my knowledge the first machine-checked five-term relation, on the first machine-checked dilogarithm, in any proof assistant» (§6), from a documented search of June 2026.

↳ Used in: 1 (p. 13), 2 (p. 13), 3 (p. 13), 4 (p. 13), 5 (p. 14), 6 (p. 14)

Proof

Formalized in Lean (see the linked declarations).

💡 **In plain words (commentary, not part of the paper)**

1. Take two numbers x, y between 0 and 1 and form the three numbers xy, $\frac{x(1-y)}{1-xy}$ and $\frac{y(1-x)}{1-xy}$. The Rogers function L of the first two adds up to L of the other three. For $x = y = \frac{1}{2}$ the three numbers are $\frac{1}{4}, \frac{1}{3}, \frac{1}{3}$, and since $\mathrm{L}(\frac{1}{2}) = \pi^2/12$ the relation says $\mathrm{L}(\frac{1}{4}) + 2\,\mathrm{L}(\frac{1}{3}) = \pi^2/6$. 2. Why it is true (the classical argument, here checked by Lean): fix y and look at the difference $G(t)$ of the two sides as a function of t. Its derivative is zero on $(0,1)$, because every logarithm that appears splits into the five logarithms of t, y, $1-t$, $1-y$, $1-ty$ and all coefficients cancel. So G is constant, and its value is read off as $t \to 0^+$, where $\mathrm{L}(t) \to 0$ by squeezing $\log t \cdot \log(1-t)$ between 0 and $2(-t \log t)$. 3. New and not new, as the paper says: the identity is classical (Spence, Abel, Rogers) and the proof is the standard differentiation proof; the paper claims only the formalization, to the author’s knowledge the

first machine-checked five-term relation in any proof assistant, after a documented search (§6), on the dilogarithm built in Lean by the companion paper.

The recipe is the one the companion paper used for every elementary identity: differentiate, conclude constant, evaluate at the boundary. Fix $y \in (0,1)$ and set

$$G(t) = \mathrm{L}(t) - \mathrm{L}(ty) - \mathrm{L}\Big(\tfrac{t(1-y)}{1-ty}\Big) - \mathrm{L}\Big(\tfrac{y(1-t)}{1-ty}\Big).$$

The theorem is $G(x) = -\,\mathrm{L}(y)$. Three movements.

1. The derivative vanishes.

For $x, y \in (0,1)$ all three composite arguments again lie in $(0,1)$, so every term is differentiable. With $\mathrm{L}'(t) = -\tfrac{1}{2}\big(\tfrac{\log(1-t)}{t} + \tfrac{\log t}{1-t}\big)$, the chain rule writes $G'(t)$ as four such expressions weighted by the derivatives of the composite arguments. Every logarithm that occurs—of t, of ty, of the two fractional arguments and their complements—expands, by $\log(ab) = \log a + \log b$ and $\log(a/b) = \log a - \log b$, into the five-element basis $\{\log t, \log y, \log(1-t), \log(1-y), \log(1-ty)\}$. In that basis $G'(t)$ is a rational expression whose every coefficient cancels: $G' \equiv 0$. In Lean this is the one delicate step (`fiveterm_hasDerivAt_zero`): the nine logarithms are rewritten into the basis and the rational remainder is cleared and closed by `ring`.

2. Hence G is constant.

A function with vanishing derivative on the open connected interval $(0,1)$ is constant there—Mathlib's `IsOpen.is_const_of_deriv_eq_zero`, the mean-value theorem in disguise. So $G(x) = G(t)$ for every $t \in (0,1)$.

3. The boundary value.

As $t \to 0^+$, $ty \to 0$, $\tfrac{t(1-y)}{1-ty} \to 0$ and $\tfrac{y(1-t)}{1-ty} \to y$, so $G(t) \to \mathrm{L}(0) - \mathrm{L}(0) - \mathrm{L}(0) - \mathrm{L}(y) = -\,\mathrm{L}(y)$—*provided* L is right-continuous at 0 with $\mathrm{L}(0) = 0$. This is the one analytic subtlety, the same one the companion paper isolated. $\mathrm{L}(t) = \mathrm{Li}_2(t) + \tfrac{1}{2}\log t \log(1-t)$, and while Li_2 is continuous, $\log t \cdot \log(1-t)$ is an indeterminate $(-\infty)\cdot 0$. It is tamed by a squeeze: on $(0, \tfrac{1}{2}]$ one has $-\log(1-t) \le 2t$ (from $\log z \ge 1 - z^{-1}$), so

$$0 \;\le\; \log t \cdot \log(1-t) = (-\log t)\,(-\log(1-t)) \;\le\; 2t\,(-\log t) = 2\,(-t\log t) \;\longrightarrow\; 0,$$

because $t \log t \to 0$ (Mathlib's `Real.continuous_mul_log`). Thus $G \to -\,\mathrm{L}(y)$; being constant and equal to $G(x)$ near 0, it *equals* $-\,\mathrm{L}(y)$, which is Theorem 1. □

So the answer to Q2 is yes: the generator needs no machinery the elementary identities did not—only that the nine-into-five cancellation actually close. *And there the pen and the machine parted company.*

3 Q3: what did the machine catch that the pen skates over?

History bead. "$G' \equiv 0$ by a short calculation" is how every textbook dispatches step 1. The phrase hides a clearing of denominators that, on paper, one trusts. A proof assistant does not trust it; it executes it, and execution has an order.

The cancellation was checked numerically to thirty digits before it was formalized—and the formalization still caught a subtlety the eye misses. When `field_simp` clears the denominators of $G'(t)$ it silently reorders the product $x \cdot y$ to $y \cdot x$, and then needs $1 - yx \neq 0$, in *that* term order and already in context, to discharge the inverse it produces. The fix is one line—state the non-vanishing fact for both orders, before the simplification—but finding it cost more than the rest of the proof.

What a proof assistant earns its keep on On paper, "clear denominators and the rest cancels" is one phrase. Formally it is an ordered rewrite: `field_simp` may reorder a product ($xy \rightsquigarrow yx$) and then demand the matching non-vanishing fact ($1 - yx \neq 0$) *in that order, already in context*. The diagnostic, banked for the next weight-2 development: if `field_simp` leaves an uncleared $(1-a)^{-1}$, the missing hypothesis is $(1-a) \neq 0$ in the term order `field_simp` produced—supply both $1 - xy \neq 0$ and its commute $1 - yx \neq 0$.

This is the whole of the new mathematics, and it is honestly small: not a theorem but a discipline. It is also exactly the kind of step that justifies machine-checking an identity whose informal proof everyone already believes. *With the generator certified, what does the companion paper's long way round now buy us for free?*

4 Q4: with the pentagon certified, what falls out for free?

History bead. The golden-ratio evaluations are Landen's (1780) and Rogers' (1907 [2]); their modern role as conformal central charges belongs to the thermodynamic Bethe ansatz of the 1990s [14 , 15 , 13] . Every classical route to them passes through the five-term relation; the companion paper took a detour, and now the main road is open.

The five-term relation is a machine that turns a choice of (x, y) into a closed relation among five L values. Feed it the right points and the catalogue of the subject prints out (Table 1). Two instances are especially clean because their arguments *coincide*. At $(x, y) = (\tfrac{1}{2}, \tfrac{1}{2})$ the two fractional arguments are both $\tfrac{1}{3}$, giving $2\,\mathrm{L}(\tfrac{1}{2}) = \mathrm{L}(\tfrac{1}{4}) + 2\,\mathrm{L}(\tfrac{1}{3})$. At $(x, y) = (1/\varphi, 1/\varphi)$, the three composite arguments *all* collapse to $1/\varphi^2$ (because $1 - 1/\varphi^2 = 1/\varphi$ and $1 - 1/\varphi = 1/\varphi^2$), so the five-term relation reads $2\,\mathrm{L}(1/\varphi) = 3\,\mathrm{L}(1/\varphi^2)$—and with the ladder values $\mathrm{L}(1/\varphi) = \pi^2/10$, $\mathrm{L}(1/\varphi^2) = \pi^2/15$ this is the exact identity

$\pi^2/5 = \pi^2/5$. The golden ladder the companion reached by a side door is, from here, a single substitution.

(x, y)	arguments	the relation reads	exact value
$(\frac{1}{2}, \frac{1}{2})$	$\frac{1}{4}, \frac{1}{3}, \frac{1}{3}$	$2\,\mathrm{L}(\frac{1}{2}) = \mathrm{L}(\frac{1}{4}) + 2\,\mathrm{L}(\frac{1}{3})$	$\mathrm{L}(\frac{1}{4}) + 2\,\mathrm{L}(\frac{1}{3}) = \frac{\pi^2}{6}$
$(\frac{1}{\varphi}, \frac{1}{\varphi})$	$\frac{1}{\varphi^2}, \frac{1}{\varphi^2}, \frac{1}{\varphi^2}$	$2\,\mathrm{L}(\frac{1}{\varphi}) = 3\,\mathrm{L}(\frac{1}{\varphi^2})$	$\frac{\pi^2}{5} = \frac{\pi^2}{5}$

Table 1: The generator at work: two choices of (x, y) whose composite arguments coincide, and the exact identity each prints. Both rows are re-verified to thirty digits; the first is hand-checkable from $\mathrm{L}(\frac{1}{2}) = \pi^2/12$, the second from the ladder values $\mathrm{L}(1/\varphi) = \pi^2/10$, $\mathrm{L}(1/\varphi^2) = \pi^2/15$. Euler's reflection $\mathrm{L}(x) + \mathrm{L}(1-x) = \pi^2/6$ is *not* an instance of the five-term relation but its companion—the symmetrization that makes the Rogers form clean—and is proved separately.

The recipe: read off a π^2-evaluation in three lines Pick any $x, y \in (0, 1)$ at which the four arguments are nice. (1) Form xy, $\frac{x(1-y)}{1-xy}$, $\frac{y(1-x)}{1-xy}$. (2) Apply $\mathrm{L}(x) + \mathrm{L}(y) =$ their sum, with a machine-checked guarantee. (3) Substitute any L values you know.
Worked: take $x = y = \frac{1}{2}$. Then $xy = \frac{1}{4}$ and both fractions are $\frac{1}{3}$, so $2\,\mathrm{L}(\frac{1}{2}) = \mathrm{L}(\frac{1}{4}) + 2\,\mathrm{L}(\frac{1}{3})$. Since $\mathrm{L}(\frac{1}{2}) = \mathrm{Li}_2(\frac{1}{2}) + \frac{1}{2}\log^2 2$ and $\mathrm{Li}_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{1}{2}\log^2 2$, the log terms cancel and $\mathrm{L}(\frac{1}{2}) = \frac{\pi^2}{12}$; hence $\mathrm{L}(\frac{1}{4}) + 2\,\mathrm{L}(\frac{1}{3}) = \frac{\pi^2}{6}$—a closed form for three dilogarithm values, certified.

So the thread the companion paper left hanging is closed. It climbed the golden ladder without Abel and asked whether the side door was a curiosity or a necessity; the answer is that it was a shortcut, and the staircase it bypassed is now machine-checked. The four questions end where they began—at a sum of dilogarithm values equal to $\pi^2/6$—but the identity behind it is no longer a shadow on the wall: it is a theorem in the library, with the whole weight-2 subject hanging from it.

5 The building blocks

The headline rests on three reusable pieces, each `sorry`-free with the standard three axioms (Table 2).

Lemma 1. Lean (`Dilog.hasDerivAt_rogersL`)✓ p. 6

> *On* $(0, 1)$, $\mathrm{L}'(x) = -\frac{1}{2}\left(\frac{\log(1-x)}{x} + \frac{\log x}{1-x}\right)$.

Lean · ours **show**

- `Dilog.hasDerivAt_rogersL`

✅ Checked when this page was built: FiveTerm.lean compiles with Lean 4.35.0, no errors, no sorry.
Axioms — hasDerivAt_rogersL: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean FiveTerm.lean` (SHA-256 of the compiled file: `ce29fa490a366ddb…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only. Lean: 1 result, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Lemma 1, p. 6, doi:10.5281/zenodo.20682716

🆕 **Classical and new:** the paper: one of «three reusable building blocks» of the formalization (§5); «the same derivative scheme the companion used for reflection and Landen».

↳ Used in: Theorem 1 (p. 6)

Proof

> Formalized in Lean (see the linked declarations).

The same derivative scheme the companion used for reflection and Landen: combine $\mathrm{Li}_2'(x) = -\log(1-x)/x$ with the product rule for $\frac{1}{2}\log x \log(1-x)$.

Lemma 2. ✅ Lean (`Dilog.tendsto_rogersL_nhdsGT_zero`)✓ p. 6

> $\mathrm{L}(t) \to 0$ *as* $t \to 0^+$.

Lean · ours show

- `Dilog.tendsto_rogersL_nhdsGT_zero`

✅ Checked when this page was built: FiveTerm.lean compiles with Lean 4.35.0, no errors, no sorry.
Axioms — tendsto_rogersL_nhdsGT_zero: propext, Classical.choice, Quot.sound.
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Trust base: 🔒 kernel only. Lean: 1 result, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Lemma 2, p. 6, doi:10.5281/zenodo.20682716

🆕 **Classical and new:** the paper: a reusable block; «the only place the proof leaves pure algebra for analysis» (§5).

↳ Used in: Theorem 1 (p. 6)

Proof

> Formalized in Lean (see the linked declarations).

The Li_2 part is continuous on $[-1,1]$; the log-product is squeezed to 0 by $2(-t\log t)$. This is the only place the proof leaves pure algebra for analysis.

Lemma 3. Lean (`Dilog.continuousAt_rogersL`)✓ p. 6

> L *is continuous at every* $x \in (0,1)$.

Lean · ours show

- `Dilog.continuousAt_rogersL`

✅ Checked when this page was built: FiveTerm.lean compiles with Lean 4.35.0, no errors, no sorry.
Axioms — continuousAt_rogersL: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean FiveTerm.lean` (SHA-256 of the compiled file: `ce29fa490a366ddb…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only. Lean: 1 result, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Lemma 3, p. 6, doi:10.5281/zenodo.20682716

🆕 **Classical and new:** the paper: a reusable block, «needed for the one surviving boundary term» (§5).

↳ Used in: Theorem 1 (p. 6)

Proof

> Formalized in Lean (see the linked declarations).

Needed for the one surviving boundary term, $\mathrm{L}\left(\frac{y(1-t)}{1-ty}\right) \to \mathrm{L}(y)$.

Lean name	statement	where
`rogersL_fiveterm`	the five-term relation (Theorem 1)	§2
`fiveterm_hasDerivAt_zero`	$G' \equiv 0$: nine logs collapse to five	§2–3
`hasDerivAt_rogersL`	$\mathrm{L}'(x) = -\frac{1}{2}\left(\frac{\log(1-x)}{x} + \frac{\log x}{1-x}\right)$	§5
`tendsto_rogersL_nhdsGT_zero`	$\mathrm{L}(t) \to 0$ as $t \to 0^+$	§5
`continuousAt_rogersL`	L continuous on $(0,1)$	§5

Table 2: The load-bearing results, all `sorry`-free in `Dilog/FiveTerm.lean` of the repository, on the dilogarithm of `Dilog/Basic.lean` from the companion paper. Axioms of every entry: `propext`, `Classical.choice`, `Quot.sound` only.

6 The boundary of the contribution

Everything here that is mathematics is classical. The five-term relation is Spence's and Abel's; the Rogers L-function is Rogers's (1907); the universality statements are Wojtkowiak's and de Jeu's; the differentiation proof is standard. I claim no new identity and no new inequality. The contributions are three: the *formalization* (to my knowledge the first machine-checked five-term relation, on the first machine-checked dilogarithm, in any proof assistant); the *reusable infrastructure*—the Rogers-L derivative, the right-continuity at 0 via the $-t\log t$ squeeze, and the interior continuity—which is exactly what any further weight-2 development (ladders, the Bloch group, the Bloch–Wigner five-term) will need; and the *proof-engineering record* of Section 3, where the nine-into-five cancellation is the kind of step a formal check earns its keep on.

Prior formalizations. The priority claim rests on a documented search, not a feeling. On 2026-06-12/13 I searched, for the dilogarithm and for the five-term relation: Mathlib (Loogle and `grep`; no `dilog`/`polylog` declaration exists, only a comment referring to Li_2) and the wider Lean community repositories; the Isabelle Archive of Formal Proofs; the Coq/Rocq ecosystem (`coq-community`, `opam`); HOL Light; and Metamath (`set.mm`). None contains the dilogarithm as an analytic object with its functional equations, and none contains the five-term relation. The structural backstop is simpler than any search: the relation needs a formalized Li_2 to be *stated*, and no proof assistant had one until the companion paper built it—so a machine-checked five-term relation cannot predate that. As always, such a claim is made to the best of my knowledge.

Size of the formalization. The new work for this paper is `Dilog/FiveTerm.lean`: 243 lines, five theorems (Table 2), elaborating in 17 s once Mathlib and the companion's `Dilog/Basic.lean` are cached. It sits on `Dilog/Basic.lean` (581 lines: the dilogarithm, its derivative, reflection, Landen, duplication) from the companion paper, and on Mathlib for all the analysis (term-by-term differentiation, the mean-value constancy theorem, the $-t\log t$ limit). The proof reuses Mathlib's analytic library wholesale; what is original is the 243-line bridge and the three reusable blocks. Every theorem depends only on the three standard axioms (`propext`, `Classical.choice`, `Quot.sound`); there is no `native_decide`, no `sorry`, and no custom axiom.

On methodology. The formalization was done with an AI assistant (Claude, Anthropic), under the discipline of the companion paper: the central cancellation verified numerically to thirty digits before formalizing, and every named theorem checked by `#print axioms` to depend only on `propext`, `Classical.choice` and `Quot.sound`. Correctness does not rest on the assistant: the Lean kernel checks every proof. The mathematics and the claims are the author's responsibility.

7 Open questions

Question 1. p. 7

The pentagon is now a theorem in the library. Standing on it, six questions worth the next effort—the last one is the one I would keep.

1. *Ladders as a generated family.* Each row of Table 1 is the five-term relation evaluated at a point; mechanizing the substitution so that every ladder *prints* from the one theorem would turn a catalogue into a corollary. How much of the classical table of dilogarithm values is a finite orbit of the five-term relation under $\mathbb{Q}$-rational substitution—and where does that orbit stop?
2. *The Bloch–Wigner sequel.* The same relation holds for the single-valued Bloch–Wigner function on $\mathbb{C}$, where it underlies hyperbolic volume and scissors congruence [8 , 9 , 10] .

Cite: Question 1, p. 7, doi:10.5281/zenodo.20682716

↳ Not used by another result of the paper.

Question 2. p. 7

What is the smallest faithful Lean account of the Bloch group in which Vol of a hyperbolic 3-manifold becomes a computable, certified number? *The quantum limit.* The Faddeev–Kashaev quantum dilogarithm satisfies a pentagon identity [12] whose $\hbar \to 0$ limit is the relation proved here.

Cite: Question 2, p. 7, doi:10.5281/zenodo.20682716

↳ Not used by another result of the paper.

Question 3. p. 7

Can a proof assistant make that limit a theorem rather than a heuristic—and would doing so expose, as the classical proof did, a step the physics literature skates over? *Universality, formalized.* Wojtkowiak and de Jeu reduce *every* rational functional equation of Li_2 to the five-term relation [5 , 6] . That reduction is an algorithm.

Cite: Question 3, p. 7, doi:10.5281/zenodo.20682716

↳ Not used by another result of the paper.

Question 4. p. 7

What would it take to run it inside Lean, so that an arbitrary claimed dilogarithm identity is *decided* against the generator? *What the machine sees and the page hides.* Section 3 found one place where the formal proof was strictly sharper than the informal one—an ordered rewrite the pen trusts.

Cite: Question 4, p. 7, doi:10.5281/zenodo.20682716

↳ Not used by another result of the paper.

Question 5. p. 7

In a subject two centuries old and re-proved a dozen times, how many such silent steps remain, and is “the proof a human believes” ever quite the proof a machine will accept? *One to keep.* The five-term relation says five dilogarithm values, read around a pentagon, sum to nothing new. It is the defining relation of the Bloch group, of cluster mutation, of the quantum dilogarithm—one identity wearing the masks of geometry, algebra, and physics.

Cite: Question 5, p. 7, doi:10.5281/zenodo.20682716

↳ Not used by another result of the paper.

Question 6. p. 7

Now that it is machine-checked, the question is not whether it is true but why it is *everywhere*: what is it about a pentagon that so many parts of mathematics are, underneath, the same five steps that return you home?

Cite: Question 6, p. 7, doi:10.5281/zenodo.20682716

↳ Not used by another result of the paper.

Question

Reproducing

```
git clone https://github.com/karlesmarin/godsil-gutman-lean
cd godsil-gutman-lean && lake exe cache get
lake build Dilog.FiveTerm
```

Lean `v4.30.0-rc2`, Mathlib pin `701fb6e9`. The relation is `Dilog.rogersL_fiveterm` in `Dilog/FiveTerm.lean`; its supports are in Table 2. Each is checked by `#print axioms` to depend only on `propext`, `Classical.choice`, `Quot.sound`.

Tables

Every table of the paper, in its order: click one to unfold it, with where the paper cites it, how its numbers were checked, and its data as CSV. The typeset table is one click away, on its page of the PDF.

Table 1. The generator at work: two choices of (x, y) whose composite arguments coincide, and the exact identity each prints. Both rows are re-verified to thirty digits; the first is hand-checkable from $\mathrm{L}(\frac{1}{2}) = \pi^2/12$, the second from the ladder values $\mathrm{L}(1/\varphi) = \pi^2/10$, $\mathrm{L}(1/\varphi^2) = \pi^2/15$. Euler's reflection $\mathrm{L}(x) + \mathrm{L}(1-x) = \pi^2/6$ is *not* an instance of the five-term relation but its companion—the symmetrization that makes the Rogers form clean—and is proved separately.

(x, y)	arguments	the relation reads	exact value
$(\frac{1}{2}, \frac{1}{2})$	$\frac{1}{4}, \frac{1}{3}, \frac{1}{3}$	$2\,\mathrm{L}(\frac{1}{2}) = \mathrm{L}(\frac{1}{4}) + 2\,\mathrm{L}(\frac{1}{3})$	$\mathrm{L}(\frac{1}{4}) + 2\,\mathrm{L}(\frac{1}{3}) = \frac{\pi^2}{6}$
$(\frac{1}{\varphi}, \frac{1}{\varphi})$	$\frac{1}{\varphi^2}, \frac{1}{\varphi^2}, \frac{1}{\varphi^2}$	$2\,\mathrm{L}(\frac{1}{\varphi}) = 3\,\mathrm{L}(\frac{1}{\varphi^2})$	$\frac{\pi^2}{5} = \frac{\pi^2}{5}$

p. 5 in the PDF · ⬇ CSV

cited in §4 Q4: with the pentagon certified, what falls out for free?

cited in §7 Open questions

Table 2. The load-bearing results, all `sorry`-free in `Dilog/FiveTerm.lean` of the repository, on the dilogarithm of `Dilog/Basic.lean` from the companion paper. Axioms of every entry: `propext`, `Classical.choice`, `Quot.sound` only.

Lean name	statement	where
`rogersL_fiveterm`	the five-term relation (Theorem 1)	§2
`fiveterm_hasDerivAt_zero`	$G' \equiv 0$: nine logs collapse to five	§2–3
`hasDerivAt_rogersL`	$\mathrm{L}'(x) = -\frac{1}{2}\left(\frac{\log(1-x)}{x} + \frac{\log x}{1-x}\right)$	§5
`tendsto_rogersL_nhdsGT_zero`	$\mathrm{L}(t) \to 0$ as $t \to 0^+$	§5
`continuousAt_rogersL`	L continuous on $(0, 1)$	§5

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