How Many Dyck Words Do the Divisors of an Integer Make? II. Generic Rigidity and Persistent Ambiguity

Combinatorics on words · divisors of integers · line arrangements · Presburger arithmetic · computer-assisted proof

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At a glance

What the paper contributes

The language of the words $\langle\langle n \rangle\rangle_2$ and its questions are Rodríguez Caballero's; paper I (doi:10.5281/zenodo.24541288) counted the cells $R(a,b)$ of the pattern $r^a s^b$ and asked whether the excess $E(a,b) = R(a,b) - W(a,b)$ is linear in b for $a \geq 2$ (rigidity). This draft answers it for $a \leq 5$ and studies where the coincidences live, at three levels of status:

- **Level 1, general theory (every a; written proofs, re-derived by an independent AI agent, not checked by a human referee).** A dimension criterion (Proposition 4.1) and a finite reduction to affine observation vectors on pieces of the parameter space (Theorem 5.1). Crossing the wall $\Lambda + h\theta = \rho$ raises $\Sigma(c)$ by exactly $\omega(h,\rho) = (a+1-|h|)(b+1-|\rho|)$ (Proposition 3.1), so adjacent cells have different words, every fibre is an antichain of at most $4ab+1$ cells, $E \leq \mathrm{Amb} \leq 2E$, and boundedly many power sums of $Z(c)$ determine the word. From Alekseyev's count, $R(a,b) = C_a b(b+1) + (a+1)b + \pi_a(b)$ with π_a of period dividing P_a (Proposition 2.3; the count itself is classical).
- **Level 2, computer-assisted rigidity (program and reduction not checked by a human referee).** $E(a,b) = \Theta(b)$ and $W(a,b) = C_a b^2 + O(b)$ for $a = 2, 3, 4, 5$ (Theorem 6.1); the only affine candidates shift n by a non-integral multiple of $1/(a+1)$, explained by the smallest element $\tau(c)$ of $Z(c)$ (Proposition 7.1). For every $a \geq 2$ the all-positive chamber has no two-parameter family (Theorems 7.6 and 7.10, Corollary 7.11, written proofs; Theorem 7.10 and Proposition 7.4 are checked in Isabelle/HOL). Measured by length, ambiguity does not vanish: lower bounds on both sides of $\theta = 1/(a+1)$ give a mean ambiguous share with $\liminf \geq 3/(4a(a+1)|I|)$ (Proposition 6.3, Corollary 6.4). Lemma D for $a = 3$, $b \geq 317$ (Theorem 8.1).
- **Level 3, a conjectural classification for $a = 3$.** Nine one-parameter families of coincidences, each proved by hand (Theorems 9.2, 9.4, Proposition 9.1, Corollary 9.5), give $E(3,b)$ up to a periodic constant fitted to the data (Conjecture 9.8): it agrees with exact enumeration for $5 \leq b \leq 170$ and some $b \leq 400$; completeness is not proved.

Fifteen written results are checked, wholly or in part, in Isabelle/HOL (public: karlesmarin/formalizations, Isabelle/CaballeroDyckWords). What the paper claims as its own, as far as its searches could see: the finite reduction and its certificate, Lemma D, the all-positive-chamber analysis and the families of Section 9; the rest is known, standard or elementary and is labelled so in the paper. **Draft of 26 November 2026, not refereed, not published.**

🚀 **Start here: Proposition 3.1**: wall weights: crossing $\Lambda + h\theta = \rho$ upwards raises $\Sigma(c)$ by exactly $\omega(h,\rho) = (a+1-|h|)(b+1-|\rho|) > 0$.

Main results:

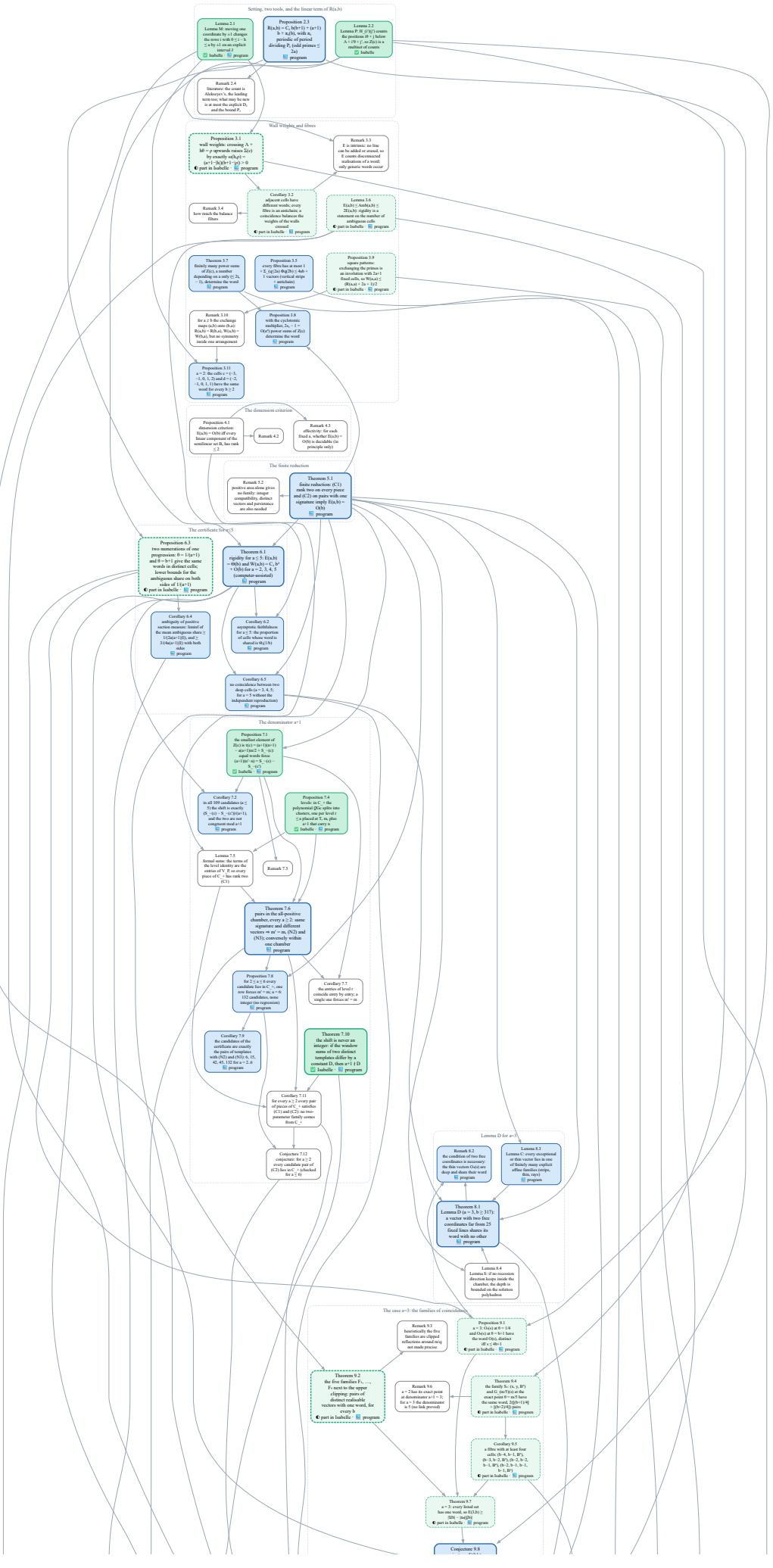
- **Proposition 2.3**: $R(a,b) = C_a\, b(b+1) + (a+1)\, b + \pi_a(b)$, with π_a periodic of period dividing P_a (odd primes $\leq 2a$)
- **Proposition 3.1**: wall weights: crossing $\Lambda + h\theta = \rho$ upwards raises $\Sigma(c)$ by exactly $\omega(h,\rho) = (a+1-|h|)(b+1-|\rho|) > 0$
- **Theorem 5.1**: finite reduction: (C1) rank two on every piece and (C2) on pairs with one signature imply $E(a,b) = O(b)$
- **Theorem 6.1**: rigidity for $a \leq 5$: $E(a,b) = \Theta(b)$ and $W(a,b) = C_a\, b^2 + O(b)$ for $a = 2, 3, 4, 5$ (computer-assisted)
- **Proposition 6.3**: two numerations of one progression: $\theta = 1/(a+1)$ and $\theta = b+1$ give the same words in distinct cells; lower bounds for the ambiguous share on both sides of $1/(a+1)$
- **Theorem 7.6**: pairs in the all-positive chamber, every $a \geq 2$: same signature and different vectors $\Rightarrow m' = m$, (N2) and (N3); conversely within one chamber
- **Theorem 7.10**: the shift is never an integer: if the window sums of two distinct templates differ by a constant D, then $a+1 \nmid D$
- **Theorem 8.1**: Lemma D ($a = 3$, $b \geq 317$): a vector with two free coordinates far from 25 fixed lines shares its word with no other
- **Theorem 9.2**: the five families $F_1, \ldots, F_5$ next to the upper clipping: pairs of distinct realisable vectors with one word, for every b
- **Conjecture 9.8**: conjecture: $E(3,b) = \mathrm{Efam}(b) + g^*_(b \bmod 4)$, slope 61/6 (checked by exact enumeration for $5 \leq b \leq 170$ and some $b \leq 400$)

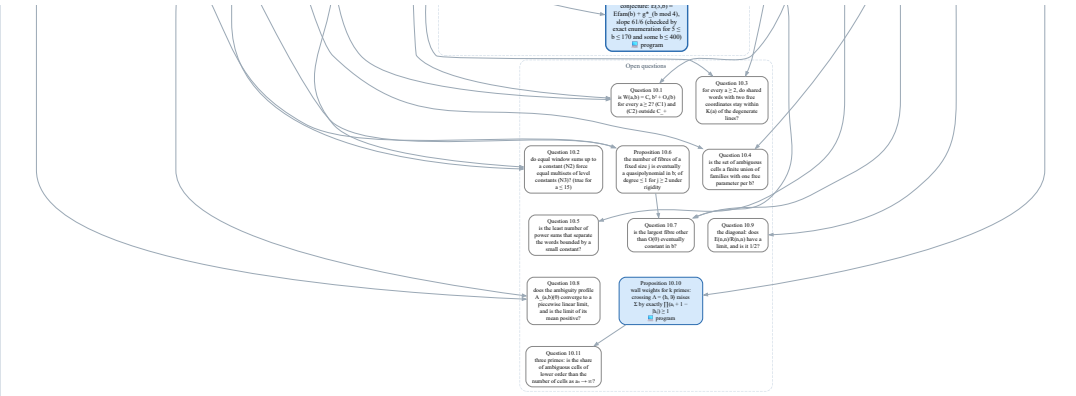
60 results | ✅ **0** entirely in Lean | ◐ **10** partly in Lean | ✅ **5** in Isabelle

🖥️ **33** checked by a program | written proof only | (a result can carry several tags)

🗺️ Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.


Setting, two tools, and the linear term of R(a,b)
Lemma 2.1
Lemma M: moving one coordinate by ±1 changes the zeros t with |t| ≤ 1 − b/a by ≤ a by ±1 on an explicit interval I.
Isabelle program
Proposition 2.3
R(a,b) = C_b(b+1) + (a+1) b − π_a(b), with π_a periodic of period dividing P_a (odd primes ≤ 2a)
program
Lemma 2.2
Lemma P. If ⌊j'/q'⌋ counts the positions iN − j below A = ⌊N·j / m⌋ on Z/q'Z is a multiset of counts
Isabelle
Remark 2.4
literature: the count is Aleksseyev's, the leading term too; what may be new is at most the explicit D, and the bound P.
Wall weights and fibres
Proposition 3.1
wall weights: crossing A + bN ↦ ρ upwards raises Σ(c) by exactly ord_q(ρ) = (a^{-1} b) (b+1 −|ρ|) = 0
part in Isabelle program
Remark 3.3
E is intrinsic: no line can be added or erased, so E counts disconnected realisations of a word; only generic words occur
Remark 3.4
how much the balance filters
Corollary 3.2
adjacent cells have different words; every fibre is an antichain; a coincidence balances the weights of the walls crossed
part in Isabelle program
Lemma 3.6
E(a,b) ≥ Ant(b,a) ≤ 2E(a,b): rigidity is a statement on the number of ambiguous cells
part in Isabelle program
Theorem 3.7
finitely many power sums of Z(c), a number depending on a only (≤ 2a − 1), determine the word
program
Proposition 3.5
every fibre has at most 1 + Σ_{c∈Δ} 0·q·|Δ| ≤ 4ab + 1 vectors (vertical strips + antichain)
program
Proposition 3.9
square patterns: exchanging the primes is an involution with 2a+1 final cells, so W(a,a) ≤ (R(a,a) + 2a + 1)/2
part in Isabelle program
Remark 3.10
for a ≥ b the exchange maps (a,b) onto (b,a): R(a,b) − R(b,a), W(a,b) − W(b,a), but no symmetry inside one arrangement
Proposition 3.8
with the cyclotomic multiplier, 2_k ⋅ 1 ≡ O(a²) power sums of Z(c) determine the word
program
Proposition 3.11
a = 2: the cells c = {−3, −1, 0, 1, 2} and c' = {−2, −1, 0, 1, 1} have the same word for every b ≥ 2
program
The dimension criterion
Proposition 4.1
dimension criterion: E(a,b) = O(b) iff every linear component of the semilinear set B_a has rank ≤ 2
Remark 4.2
Remark 4.3
effectivity: for each fixed a, whether E(a,b) = O(b) is decidable (in principle only)
The finite reduction
Remark 5.2
positive area alone gives no finitely: integer compatibility, distinct vectors and persistence are also needed
Theorem 5.1
finite reduction: (C1) rank two on every piece and (C2) on pairs with one signature: imply E(a,b) = O(b)
program
The certificate for a<5
Proposition 6.3
two numerations of one progression: 0 = 1·(a+1) and 0 = b+1 give the same words in distinct cells; lower bounds for the ambiguous share on both sides of 1/(a+1)
part in Isabelle program
Theorem 6.1
rigidity for a ≤ 5: E(a,b) = Θ(b) and W(a,b) = C_b b² + O(b) for a = 2, 3, 4, 5 (computer-assisted)
program
Corollary 6.4
ambiguity of positive section measure: liminf of the mean ambiguous share ≥ 1/(2a(a+1)|), and ≥ 3/(4a(a+1)|) with both sides
program
Corollary 6.2
asymptotic faithfulness for a ≤ 5: the proportion of cells whose word is shared is Θ(1/b)
program
Corollary 6.5
no coincidence between two deep cells (a = 3, 4, 5; for a = 5 without the independent reproduction)
program
The denominator a=1
Proposition 7.1
the smallest element of Z(c) is τ(c) = ⌊a·(1/(a+1) − a)a−1)/m2 + S_−(c)⌋ − equal words fibres (a+1)(a' −a) + S_−(c) − S_+(c)
Isabelle program
Corollary 7.2
in all 189 candidates (a ≤ 5) the shift is exactly (S_−(c) − S_+(c))/(a+1), and the two are not congruent mod a+1
program
Proposition 7.4
levels: in C_+ the polynomial |Gc·q|/m splits into clusters, one per level r ≤ a placed at T_m, plus a+1 that carry a
Isabelle program
Lemma 7.5
formal sums: the terms of the level identity are the entries of V. If, so every piece of C_+ has rank two (C1)
Remark 7.3
Theorem 7.6
pairs in the all-positive chamber, every a ≥ 2: same signature and different vectors ⇔ m' = m, (N2) and (N3); conversely within one chamber
program
Proposition 7.8
for 2 ≤ a ≤ 6 every candidate lies in C_+, one row forces m' = m; a = 6: 132 candidates, none integer (no regression)
program
Corollary 7.7
the entries of level r coincide entry by entry; a single one forces m' = m
Corollary 7.9
the candidates of the certificate are exactly the pairs of templates with (N2) and (N3): 6, 15, 42, 45, 132 for a = 2..6
program
Theorem 7.10
the shift is never an integer: if the window sums of two distinct templates differ by a constant D, then a+1 ∤ D
Isabelle program
Corollary 7.11
for every a ≥ 2 every pair of pieces of C_+ satisfies (C1) and (C2): no two-parameter family comes from C_+
Conjecture 7.12
conjecture: for a ≥ 2 every candidate pair of (C2) lies in C_+ − echecked for a ≤ 6)
Lemma D for a=3
Remark 8.2
the condition of two free coordinates is necessary: the thin vectors O(a) are deep and share their word
program
Lemma 8.3
Lemma C: every exceptional or thin vector lies in one of finitely many explicit affine families (strips, thin, rays)
program
Theorem 8.1
Lemma D (a = 3, b ≥ 317): a vector with two free coordinates far from 25 fixed lines shares its word with no other
program
Lemma 8.4
Lemma S: if no recession direction keeps inside the chamber, the depth is bounded on the solution polyhedron
The case a=3: the families of coincidences
Remark 9.3
heuristically the five families are clipped reflections around m/q not made precise
Proposition 9.1
a = 3: O(a) at θ = 1/4 and O(a) at θ = b+1 have the word O(a), distinct iff c ≠ 4b+1
part in Isabelle program
Theorem 9.2
the five families F₁, ..., F₅ next to the upper clipping: pairs of distinct realisable vectors with one word, for every b
part in Isabelle program
Remark 9.6
a = 2 has its exact point at denominator a+1 = 3; for a = 3 the denominator is 5 (no link proved)
Theorem 9.4
the family S₋ (x, y, B²) and C₊ (m/5)(x) at the exact point θ = m/5 have the same word, for b ∈ [4/9 | 2/3] | [2/3 | 4/5]
part in Isabelle program
Corollary 9.5
a fibre with at least four cells: (b−4, b−1, B²), (b−1, b−2, B²), (b−2, b−2, b−1, B²), (b−2, b−1, b−1, b−1, B²)
part in Isabelle program
Theorem 9.7
a = 3: every listed set has one word, or E(2,b) ≥ ⌊5b/2⌋ − ⌊m/2⌋
part in Isabelle program
Conjecture 9.8



📐 Setting (copied from the paper: §1, §2, §3, §5 and §7)

For an integer $n \geq 1$, Rodríguez Caballero's word $\langle\langle n \rangle\rangle_2$ records how the divisors of n interleave with the doubled divisors: one reads $D_n \triangle 2D_n$ in increasing order and writes an opening bracket for elements of D_n and a closing bracket for elements of $2D_n$. (§1)

$$W(a,b) = \#\{\langle\langle n \rangle\rangle_2 : \ n = 2^{e-1} r^a s^b, \ r \neq s \text{ odd primes}, \ e \geq 1\}, \qquad R(a,b) = \#\{\text{cells}\}.$$

(§1) So the excess $E(a,b) = R(a,b) - W(a,b)$, the number of cells lost to coincidences of words. (§1)

Fix $a, b \geq 1$. In units of $\log s$ put $\Lambda = \log t / \log s$ and $\theta = \log r / \log s$, where $t = 2^e$. For (Λ, θ) in the open quadrant, off the lines $\Lambda + h\theta = \rho$ ($|h| \leq a$, $\rho \in \mathbb{Z} \cap [-b, b]$), put

$$c_h = \operatorname{clip}\lfloor \Lambda + h\theta \rfloor \quad (-a \leq h \leq a), \qquad \operatorname{clip}(v) = \min(b, \max(-b-1, v)),$$

$$f(v) = \min(b+1, \max(0, v+1)), \qquad H_i(j) = \sum_{l=0}^{a} f(c_{i-l} + j) \quad (0 \leq i \leq a, \ 0 \leq j \leq b),$$

and let $Z(c) = \sum_{i,j} \delta_{H_i(j)}$ be the multiset of the $K = (a+1)(b+1)$ values $H_i(j)$, written as a formal sum (δ_v is the multiset $\{v\}$). By [I, Section 6, Steps 1 and 2], the word of n is determined by $Z(c)$ and conversely; a vector c is *realisable* if it comes from a point of the open quadrant off the lines, and every realisable vector is realised by primes [I, Lemma 2.3]. So $R(a,b)$ is the number of realisable vectors, $W(a,b)$ the number of distinct multisets $Z(c)$, and two realisable vectors are *equivalent* if $Z(c) = Z(d)$. A *witness* for c is a point (Λ, θ) with $\Lambda, \theta > 0$ and $\Lambda + h\theta \notin \mathbb{Z}$ for $|h| \leq a$ whose vector is c; it proves that c is realisable. A coordinate c_h is *free* if $-b \leq c_h \leq b-1$, and *clipped* otherwise. We write $T = K = (a+1)(b+1)$ for the largest value. (§2)

$$C_a = \sum_{q=1}^{2a} (2a + 1 - q) \frac{\varphi(q)}{q}.$$

(§1)

For a realisable c put

$$\Sigma(c) = \sum_{v \in Z(c)} v = \sum_{i=0}^{a} \sum_{j=0}^{b} H_i(j), \qquad \omega(h,\rho) = (a+1-|h|)(b+1-|\rho|) \quad (|h| \leq a, \ |\rho| \leq b),$$

and order vectors coordinatewise: $c \leq d$ if $c_h \leq d_h$ for all h. The *fibre* of a word is the set of realisable vectors with that word, and e_h is the unit vector of coordinate h. (§3)

$$\operatorname{Amb}(a,b) = \#\{c \text{ realisable} : \ Z(d) = Z(c) \text{ for some realisable } d \neq c\}.$$

(§3)

Write a point of the quadrant off the lines $\Lambda + h\theta \in \mathbb{Z}$ as $\Lambda = n + u$, $\theta = m + v$ with $n, m \in \mathbb{Z}_{\geq 0}$ and $u, v \in (0,1)$. Then $\lfloor \Lambda + h\theta \rfloor = n + hm + \varepsilon_h$ with $\varepsilon_h = \lfloor u + hv \rfloor$, so

$$c = g(n, m, \varepsilon) := \big(\mathrm{clip}(n + hm + \varepsilon_h)\big)_{|h| \le a}.$$

The vector $\varepsilon = (\varepsilon_h)$, the *template*, depends only on the cell of (u, v) in the arrangement of the lines $u + hv = k$ ($|h| \le a$, $k \in \mathbb{Z}$) in the open unit square. (§5) A *piece* $P = (C_P, \varepsilon)$ is a chamber together with a template. (§5)

For a template put $S_-(\varepsilon) = \sum_{h=-a}^{0} \varepsilon_h$. (§7) Let C_+ be the chamber $0<\lambda + h\mu<1$ ($|h| \le a$). (§7.1)

1 Introduction

No numbered statement in this section.

Its figures and tables are below; its text is in the paper, p. 1.

result	status
Level 1	
Proposition 2.3 (linear term D_a of $R(a,b)$; period divides P_a)	written proof, a consequence of the published count (checked exactly, $a \le 6$, every class mod $\mathrm{lcm}(1,\dots,2a)$); minimal period $= P_a$: exact enumeration $a \le 6$
Proposition 3.1 (wall weights)	Isabelle/HOL (a), (c) and $\omega \ge 1$; the crossing argument (b): written proof
Corollary 3.2 (antichains)	Isabelle/HOL (ii), (iii); (i): written proof
Proposition 3.5 (fibres $\le 4ab + 1$)	written proof (finite checks, $a \le 3$, $b \le 15$)
Lemma 3.6 ($E \le \mathrm{Amb} \le 2E$)	Isabelle/HOL (the counting lemma)
Theorem 3.7, Proposition 3.8 (finitely many power sums)	written proof; the minimal number: exact enumeration $a \le 3$, $b \le 20$
Proposition 3.9 ($W(a,a) \le (R(a,a) + 2a + 1)/2$)	Isabelle/HOL the counting step ($2W \le R + f$); the geometric step: written proof
Proposition 3.11 (one pair of $a = 2$, every $b \ge 2$)	written proof (checked $b \le 300$)
Proposition 4.1 (dimension criterion)	written proof
Theorem 5.1 (finite reduction)	written proof; the encoding tested on exact data, $a \le 3$, $b \le 200$ (Appendix A)
Level 2	
Theorem 6.1: $E(a,b) = O(b)$, $W(a,b) = C_a b^2 + O(b)$ for $a = 2, 3, 4, 5$	computer-assisted, conditional on the correctness of the program and of Theorem 5.1
Corollary 6.2 ($\mathrm{Amb}/R = \Theta_a(1/b)$ for $a = 2, \dots, 5$)	computer-assisted: a written consequence of Theorem 6.1 and Lemma 3.6 (data in Table 3)
Proposition 6.3 (two numerations of one progression)	Isabelle/HOL (i), and (ii) as a line of the arrangement that separates the two points; (iii), (iv): written proof
Corollary 6.4 (ambiguity of positive section measure, every $a \ge 2$)	written proof (segments checked in exact arithmetic, $a \le 6$, $b \le 60$)
Corollary 6.5 (no coincidence between two deep cells, $a = 3, 4, 5$)	computer-assisted
Proposition 7.1 (the identity for τ)	Isabelle/HOL; Corollary 7.2 (the 109 candidates): computer-assisted
Proposition 7.4 (levels of C_+, every a)	Isabelle/HOL

Lemma 7.5, Theorem 7.6 (all-positive chamber, every $a \geq 2$)	written proof (checked: Corollary 7.9, $a \leq 6$; all templates, $a \leq 15$)

Table 1: Status of the results, row by row (Levels 1 and 2; the categories are explained at the end of the table).

Colour: the paper's own colours (see the caption).

result	**status**
Proposition 7.8 (candidates only in C_+; one row forces $m' = m$)	computer-assisted ($a \leq 6$; $a = 6$ without regression)
Theorem 7.10 (the shift is never an integer)	Isabelle/HOL (the whole statement)
Corollary 7.11 ((C1), (C2) inside C_+) written proof, from written and formalised results	
Conjecture 7.12 (candidates only in C_+) conjecture (checked $a \leq 6$)	
Theorem 8.1 (Lemma D for $a = 3$, $b \geq 317$) computer-assisted (Lemmas 8.3 and 8.4: written proof)	
Level 3	
Proposition 9.1 ($O(s)$ for $a = 3$) Isabelle/HOL	
Theorem 9.2 (families $F_1, \ldots, F_5$) Isabelle/HOL the identities and the witnesses; the counts: written proof	
Theorem 9.4 (family S_1) Isabelle/HOL the witnesses; the words, their equality and the count: written proof	
Corollary 9.5 (a fibre with at least four cells) Isabelle/HOL the equality of the four multisets; that the fibre has exactly four cells: observed	
Theorem 9.7 (lower bound by the coincidence graph) Isabelle/HOL the counting implication; its premises are the results above	
Conjecture 9.8: the list is complete ($\Rightarrow$), formula for $E(3, b)$ conjecture exact enumeration $b \leq 170$, some $b \leq 400$	
Section 10	
Proposition 10.6 (fibre counts of fixed size are quasipolynomials of degree ≤ 1) written proof, conditional on rigidity (so computer-assisted for $a = 2, \ldots, 5$)	
Proposition 10.10 (wall weights for k primes)	

written proof (checked on the shape $(2,2,3)$)	

Table 1: Status of the results (continued). *Written proof*: a mathematical argument given in the paper. *Isabelle/HOL*: the stated part is checked by the proof assistant (Section 1, Formalisation). *Computer-assisted*: a written reduction plus a finite exact computation in rational arithmetic that the result needs; no proof assistant was used for these, and the programs have not been checked by a human referee. *Exact enumeration*: exact data, with no conclusion for all b. *Conjecture*: not proved. The finite checks in parentheses, and the independent re-computations by an AI agent, are described in Appendix A. Row colours: written proof, written proof with a part checked in Isabelle/HOL, computer-assisted, conjecture (two shades alternate within a category).

2 Setting, two tools, and the linear term of $R(a,b)$

Lemma 2.1. Isabelle **(Lemma M: elementary move)** p. 9

> *Let c' agree with c except $c'_h = c_h + \sigma$, $\sigma \in \{1,-1\}$, with $c'_h \in [-b-1, b]$. Put $J = \{j \in [0,b] : 0 \le c_h + 1 + j \le b\}$ if $\sigma = 1$ and $J = \{j \in [0,b] : 0 \le c_h + j \le b\}$ if $\sigma = -1$. Then $H'_i(j) = H_i(j) + \sigma$ if $0 \le i - h \le a$ and $j \in J$, and $H'_i(j) = H_i(j)$ otherwise. If several coordinates move, the changes of one row add, computed on the original vector.*

Isabelle · ours **show**

`lem_M` · `Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy`, line 142

```
theorem lem_M:
  assumes "sg = 1 \<or> sg = -1" and "c' = c(h := c h + sg)" and "0 \<le> j" "j \<le>
b"
  shows "Hval a b c' i j
         = Hval a b c i j + (if 0 \<le> i - h \<and> i - h \<le> a \<and> j \<in> Jset
b sg (c h) then sg else 0)"
```

`lem_M_multi` · `Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy`, line 87

```
theorem lem_M_multi:
  assumes fin: "finite M" and s: "\<And>h. h \<in> M \<Longrightarrow> s h = 1 \<or> s
h = -1"
    and cc: "\<And>h. c' h = c h + (if h \<in> M then s h else 0)"
    and j: "0 \<le> j" "j \<le> b"
  shows "Hval a b c' i j = Hval a b c i j
          + (\<Sum>h \<in> M. if 0 \<le> i - h \<and> i - h \<le> a \<and> j \<in>
Jset b (s h) (c h) then s h else 0)"
```

Trust base: kernel only. Isabelle: 2 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** the whole statement

↳ Used in: Proposition 3.1 (p. 12), Proposition 3.11 (p. 18), Theorem 9.2 (p. 43)

Proof

> c_h enters row i exactly when $0 \le i - h \le a$, and $f(v+1) - f(v) = [-1 \le v \le b-1]$. Different coordinates enter a row through different terms.

Evidence & scope

- **Program:** a = 3: every non-decreasing clipped c, every h and sign, b = 3..5: 594594/594594. working script `pruebas_a3.py`, not published

Lemma 2.2. Isabelle **(Lemma P: exact point)** p. 9

> *Let $\Lambda, \theta{>}0$ with $\Lambda + h\theta \notin \mathbb{Z}$ for $|h| \le a$, let c be its vector and $\mathcal{P}$ the multiset of the positions $i\theta + j$ $(0 \le i \le a,\ 0 \le j \le b)$. Then $H_{i'}(j') = \#\{p \in \mathcal{P} : p{<}\Lambda + i'\theta + j'\}$, so $Z(c)$ is the multiset of the counts $\#\{p \in \mathcal{P} : p{<}p_0 + \Lambda\}$, $p_0 \in \mathcal{P}$.*

Isabelle · ours **show**

`lem_P_count` · `Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy`, line 606

```
theorem lem_P_count:
  fixes Lam th :: real
  assumes noint: "\<And>h. -a \<le> h \<Longrightarrow> h \<le> a \<Longrightarrow>
Lam + real_of_int h * th \<notin> \<int>"
    and i': "0 \<le> i'" "i' \<le> a" and j': "0 \<le> j'" "j' \<le> b"
  shows "Hval a b (cpt b Lam th) i' j'
        = int (size (filter_mset (\<lambda>p. p < Lam + real_of_int i' * th +
real_of_int j') (Pms a b th)))"
```

`lem_P` · `Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy`, line 646

```
theorem lem_P:
  fixes Lam th :: real
  assumes noint: "\<And>h. -a \<le> h \<Longrightarrow> h \<le> a \<Longrightarrow>
Lam + real_of_int h * th \<notin> \<int>"
  shows "Zms a b (cpt b Lam th)
        = image_mset (\<lambda>p0. int (size (filter_mset (\<lambda>p. p < p0 + Lam)
(Pms a b th)))) (Pms a b th)"
```

Trust base: 🔒 kernel only. Isabelle: 2 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** the whole statement

↳ Used in: 3.3 (p. 14), Proposition 6.3 (p. 24), Proposition 9.1 (p. 42), Theorem 9.4 (p. 46)

Proof

> For one row i of positions and $h = i' - i$, $\#\{j : i\theta + j{<}\Lambda + i'\theta + j'\} = f(\lfloor \Lambda + h\theta \rfloor + j')$, and clipping does not change $f(v + j')$ for $0 \le j' \le b$ [I, eq. (2)].

Proposition 2.3. p. 10

> *Let $a \ge 1$ and let P_a be the product of the odd primes $p \le 2a$ $(P_1 = 1)$.*
>
> 1. *For all $b \ge 1$, $R(a,b) = C_a b^2 + D_a b + \pi_a(b)$ with π_a periodic and*
>
> $$D_a = C_a + a + 1, \qquad \text{that is,} \qquad R(a,b) = C_a\, b(b+1) + (a+1)\, b + \pi_a(b).$$
>
> *In particular $D_1, \dots, D_6 = \frac{9}{2}, \frac{31}{3}, \frac{93}{5}, \frac{2059}{70}, \frac{8917}{210}, \frac{44707}{770}$.*
>
> 2. *The period of π_a (equivalently, of the quasipolynomial $R(a,\cdot)$) divides P_a.*

Trust base: ⚠️ not checked. Written proof (Möbius inversion on Alekseyev's formula), not checked by a human referee, not machine-checked; checked exactly for a ≤ 6 in every residue class mod lcm(1..2a); the minimal period = P_a for a ≤ 6 is an exact computation

Proof: in this note · **Covers:** not formalized

🆕 **Classical and new:** the paper: «This is a computation from the closed formula of Acketa–Žunić and Alekseyev; we claim no novelty beyond writing out the linear term and the period, which we did not find stated in the literature we read» (Remark 2.4).

↳ Used in: 2.4 (p. 11), Theorem 6.1 (p. 22), 9.8 (p. 50)

Proof

By Möbius inversion $\Phi_q(x) = \sum_{d\mid q} \mu(d)\lfloor x/d\rfloor$, so $S_q(b) = \sum_{d\mid q} \mu(d) G_d(2b)$ with $G_d(X) = \sum_{x=1}^{X} \lfloor x/d\rfloor$. Write $X = dk + r$, $0 \le r < d$. Since $\lfloor x/d\rfloor = (x - (x \bmod d))/d$ and the residues of $1, \dots, X$ sum to $k\,d(d-1)/2 + r(r+1)/2$,

$$G_d(X) = \frac{X^2 - (d-2)X}{2d} + \gamma_d(r), \qquad \gamma_d(r) = \frac{r(d-2-r)}{2d}, \qquad r = X \bmod d,$$

for every $X \ge 0$. Only squarefree d occur ($\mu(d) \neq 0$). For $X = 2b$, $r = 2b \bmod d$ depends only on b modulo $d/\gcd(d,2)$, which for squarefree $d \mid q \le 2a$ is the product of the odd primes dividing d, a divisor of P_a. So each $S_q(b)$, hence by (1) $R(a,b)$, is a polynomial of degree 2 in b plus a function of period dividing P_a, for every $b \ge 1$; this is (b). The coefficients of b^2 and b in this decomposition are unique (a periodic function is bounded); the constant term is not, unless the periodic part is required to have mean zero, and it is not used.

Its coefficients follow from the polynomial parts. For $q = 1$, $S_1(b) = b(2b+1) = 2b^2 + b$. For $q \ge 2$, the polynomial part of $\sum_{d\mid q} \mu(d) \frac{(2b)^2 - (d-2)2b}{2d}$ is $\frac{\varphi(q)}{q}\, 2b^2 - \sum_{d\mid q} \mu(d) \frac{d-2}{d}\, b = \frac{\varphi(q)}{q}(2b^2 + 2b)$, because $\sum_{d\mid q} \mu(d) = 0$ and $\sum_{d\mid q} \mu(d)/d = \varphi(q)/q$. Hence the polynomial part of $\frac{1}{2}V(2b, 2a)$ is

$$\tfrac{1}{2} \cdot 2a(2b^2 + b) + \sum_{q=2}^{2a} (2a + 1 - q)\frac{\varphi(q)}{q}(b^2 + b) = 2ab^2 + ab + (C_a - 2a)(b^2 + b) = C_a b^2$$

and adding $1 + a + b + 2ab$ gives the coefficient $C_a - a + 1 + 2a = C_a + a + 1$ of b.

💡 **In plain words (commentary, not part of the paper)**

Paper I counted the cells exactly; by its Proposition 5.3 the count is Alekseyev's number of threshold functions in disguise. Here the second term is written out: $R(a,b) = C_a b^2 + (C_a + a + 1)b + \pi_a(b)$, where π_a repeats with a period dividing the product P_a of the odd primes up to $2a$ ($1, 3, 15, 105, 105, 1155$ for $a = 1, \dots, 6$, and these are exactly the minimal periods for $a \le 6$, by computation). For instance $D_3 = \frac{93}{5}$.

Why it is true: Möbius inversion turns the coprimality condition into sums of $\lfloor x/d\rfloor$ for squarefree $d \mid q \le 2a$, and each such sum is a quadratic polynomial plus a term that depends only on $2b \bmod d$. The paper claims no novelty for the count or its quasipolynomial structure, only possibly for the explicit linear term and the period bound, which it did not find stated in the literature it read. Every lab of this site recomputes $R(a,b)$ from Alekseyev's formula in exact arithmetic and compares it with the enumerated cells.

🛡️ **Evidence & scope**

- **Program:** block D: for a = 1..6 and every residue class b mod lcm(1..2a) (2, 12, 60, 840, 2520, 27720 classes), R(a,b+L) − R(a,b) − C_a(2bL + L²) = D_a L exactly; control with H_a in place of H_⌊a/q⌋ fails in every class. working script `revision_mayor.py`, not published
- **Program:** π_a periodic from b = 1 with minimal period 1, 3, 15, 105, 105, 1155 = P_a for a = 1..6 (b ≤ 60, 60, 200, 800, 800, 3600); with linear coefficient C_a + a the remainder is not periodic in the six cases. working scripts `V376_propuestas.py`, `comprobar_periodo_moebius.py`, not published
- **Program:** the labs of this site recompute R(a,b) from Alekseyev's formula in BigInt and compare it with every exact enumeration of the paper's logs (a ≤ 5). `labjs/nucleo.js`, `labjs/control_js.js`

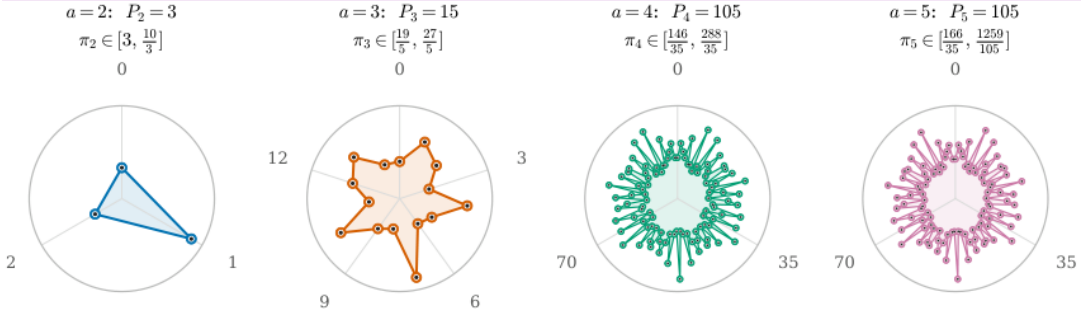
Remark 2.4. (literature, and what is checked) p. 10

The exact count (1) is Alekseyev's [1 , Theorem 3] , through [I, Proposition 5.3], and the quasipolynomial structure follows from it by the elementary argument above; we claim no novelty for either. The leading term is in [1 , Theorem 25] ([I, Remark 5.4]). The explicit linear term $D_a = C_a + a + 1$ and the period bound P_a are not stated in [1] , which we read in

full, nor in the OEIS entries cited in [I]; we have not read [2] or [3] in full; the closed formula of [2 , Theorem 3] , $t(m,n) = f(m,n) + 2$ with $f(m,n) = \sum_{-m<i<m,\ -n<j<n,\ (i,j)=1} (m-|i|)(n-|j|)$, is seen as quoted in [13 , eq. (1)] . It is a sum over coprime pairs of exactly the kind treated above, so the Möbius computation of Proposition 2.3 is a routine consequence of it. Haukkanen and Merikoski's asymptotics of the threshold-function count (read in full) give $t(m,n) = \frac{6}{\pi^2}(mn)^2 + O(mn^2)$ for $m \le n$, which with one side fixed leaves the linear term inside the error. So what may be new here is at most this explicit form. The bound P_a sharpens the period $\operatorname{lcm}\big(\operatorname{lcm}(1,\dots,a),\ \text{odd } q \le 2a\big)$ given in the proof of [I, Corollary 5.2]: through (1) the boundary terms of [I, Theorem 5.1], which carry the factor $\operatorname{lcm}(1,\dots,a)$, add up to $2ab$. By exact computation the minimal period of π_a is exactly P_a for $a \le 6$: $1, 3, 15, 105, 105, 1155$ (checked for $b \le 60, 60, 200, 800, 800, 3600$). With the linear coefficient $C_a + a$ in place of D_a, the remainder is not periodic in any of the six cases. Equality for every a is not proved. Figure 1 shows the periodic part π_a for $a = 2, \dots, 5$.

↳ Not used by another result of the paper.

Figure 1



The periodic part of the count of cells. For $a = 2, 3, 4, 5$ each clock has P_a hours and shows $\pi_a(b) = R(a,b) - C_a b^2 - D_a b$ for $b = 1, \dots, 3P_a$, at angle $2\pi b / P_a$ and at a radius that is an affine function of $\pi_a(b)$ (scaled per panel; the range of π_a is printed above each clock). The three laps, $b \le P_a$ (large coloured rings), $P_a < b \le 2P_a$ (white rings) and $2P_a < b \le 3P_a$ (black dots), fall on the same points, as Proposition 2.3(b) predicts, and no proper divisor of P_a is a period. Values of $R(a,b)$ from (1), which agree with the exact enumerations of cells used in Figure 4.

3 Wall weights and fibres

Proposition 3.1. (wall weights) p. 12

1. *Let $c \in [-b-1, b]^{2a+1}$ (realisable or not), $|h| \le a$ and $c_h \le b-1$. Then*

$$\Sigma(c + e_h) - \Sigma(c) = (a + 1 - |h|)(b + 1 - |c_h + 1|).$$

2. *Crossing the wall $\ell_{h,\rho} : \Lambda + h\theta = \rho$ ($|\rho| \le b$) at a point on no other line, from $\Lambda + h\theta < \rho$ to $\Lambda + h\theta > \rho$, changes only c_h, from $\rho - 1$ to ρ, and raises Σ by exactly $\omega(h,\rho) \ge 1$.*
3. *For realisable c, d, $\Sigma(d) - \Sigma(c) = \sum_{|h| \le a} (a + 1 - |h|)\big(F(d_h) - F(c_h)\big)$, where $F(v) = \sum_{j=0}^{b} f(v + j)$ and $F(v) - F(u) = \sum_{\rho=u+1}^{v} (b + 1 - |\rho|)$ for $u \le v$: the signed sum of the weights of the walls that separate the two cells, independent of the path.*

Isabelle · ours · part **show**

Only a part is in Isabelle: (a), (c) and the weight ω ≥ 1 of (b); the geometric part of (b), that crossing a wall at a point on no other line changes only c_h, is not formalized

`prop_wall_a` · `Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy`, line 257

```
theorem prop_wall_a:
  assumes "-a \<le> h" "h \<le> a" "-b - 1 \<le> c h" "c h \<le> b - 1"
```

```
  shows "Sig a b (c(h := c h + 1)) - Sig a b c = (a + 1 - \<bar>h\<bar>) * (b + 1 - \
<bar>c h + 1\<bar>)"
```

`prop_wall_c` · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy, line 274

```
theorem prop_wall_c:
  "Sig a b d - Sig a b c = (\<Sum>h \<in> {-a..a}. (a + 1 - \<bar>h\<bar>) * (Fsum b (d
h) - Fsum b (c h)))"
```

`prop_wall_F` · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy, line 284

```
theorem prop_wall_F:
  assumes "-b - 1 \<le> u" "u \<le> v" "v \<le> b"
  shows "Fsum b v - Fsum b u = (\<Sum>\<rho> \<in> {u + 1..v}. b + 1 - \<bar>\<rho>\
<bar>)"
```

`wall_omega_pos` · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy, line 308

```
theorem wall_omega_pos:
  fixes a b h \<rho> :: int
  assumes "\<bar>h\<bar> \<le> a" "\<bar>\<rho>\<bar> \<le> b"
  shows "1 \<le> (a + 1 - \<bar>h\<bar>) * (b + 1 - \<bar>\<rho>\<bar>)"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Isabelle: 4 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: (a), (c) and the weight ω ≥ 1 of (b); the geometric part of (b), that crossing a wall at a point on no other line changes only c_h, is not formalized

🆕 **Classical and new:** the paper lists it among the elementary or routine results, no novelty claimed (Section 1, Literature status).

↳ Used in: Corollary 3.2 (p. 14), Proposition 10.10 (p. 54)

Proof

The coordinate c_h enters row i exactly when $0 \le i - h \le a$, that is, in $a + 1 - |h|$ rows, always through the term $f(c_h + j)$. So $\Sigma(c) = \sum_h (a + 1 - |h|) F(c_h)$. Since $f(v+1) - f(v) = [-1 \le v \le b - 1]$ (proof of Lemma 2.1), $F(v+1) - F(v) = \#\{j \in [0, b] : -1 \le v + j \le b - 1\}$, which is $b - v = b + 1 - |v + 1|$ if $v \ge -1$ and $v + b + 2 = b + 1 - |v + 1|$ if $v \le -2$; this is (a), and (c) follows by summing. For (b): at such a point only the floor $\lfloor \Lambda + h\theta \rfloor$ changes, from $\rho - 1$ to ρ, and since $-b - 1 \le \rho - 1 < \rho \le b$ clipping leaves both values unchanged; (a) with $c_h = \rho - 1$ gives $\omega(h, \rho)$, which is ≥ 1 because $|h| \le a$ and $|\rho| \le b$.

💡 **In plain words (commentary, not part of the paper)**

Add up all the numbers of the multiset $Z(c)$ that encodes the word: that is $\Sigma(c)$. Each coordinate c_h enters $a + 1 - |h|$ rows, so moving across one line $\Lambda + h\theta = \rho$ of the arrangement (the only coordinate that changes is c_h, from $\rho - 1$ to ρ) raises Σ by the explicit positive number $\omega(h, \rho) = (a + 1 - |h|)(b + 1 - |\rho|)$. So Σ is a "height" that strictly increases upwards through every wall.

Why it matters: two cells next to each other differ in height, so they never have the same word; two comparable vectors $c \le d$, $c \ne d$, differ in height too, so every fibre (the set of cells with one word) is an antichain; and a coincidence must cross walls of equal total weight upwards and downwards. Combined with a decomposition into vertical strips (Proposition 3.5), this bounds every fibre by $4ab + 1$ cells. The paper calls these results elementary and claims no novelty for them; parts (a), (c) and the positivity of the weight are checked in Isabelle/HOL. The lab draws every cell as a terrace at height $\Sigma(c)$, as in the paper's Figure 2.

🛡️ **Evidence & scope**

- **Program:** a = 1, 2, 3, b ∈ 1, 2, 3, 5, 8: the step formula on 32984 steps, 0 failures; Σ(d) − Σ(c) = signed sum of ω over the walls crossed, 61002 pairs, 0 failures; control with (b+1−|c_h|): 859 failures. working script `V377_propuestas.py`, not published
- **Program:** the lab «Figure 2 live» checks the step formula (a) on every cell and coordinate, in exact integers, for the a and b you choose. `labjs/nucleo.js`

Corollary 3.2. p. 12

> 1. *Two cells separated by a single wall (adjacent cells) have different words.*
> 2. *If $c \le d$ and $c \neq d$ then $\Sigma(c)<\Sigma(d)$. In particular every fibre is an antichain for $\le$.*
> 3. *If $Z(c) = Z(d)$, the weights of the walls crossed upwards (towards $\Lambda + h\theta>\rho$) and downwards between the two cells balance: $\sum_h (a+1-|h|)\big(F(d_h) - F(c_h)\big) = 0$.*

Isabelle · ours · part **show**

Only a part is in Isabelle: (ii) and (iii); (i) rests on the geometric part of Proposition 3.1(b) and is not formalized

`cor_antichain_ii` · `Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy`, line 338

```
theorem cor_antichain_ii:
  assumes rng: "\<And>h. -a \<le> h \<Longrightarrow> h \<le> a \<Longrightarrow> -b -
1 \<le> c h \<and> c h \<le> b \<and> -b - 1 \<le> d h \<and> d h \<le> b"
    and le: "\<And>h. -a \<le> h \<Longrightarrow> h \<le> a \<Longrightarrow> c h \
<le> d h"
    and ne: "\<exists>h. -a \<le> h \<and> h \<le> a \<and> c h \<noteq> d h"
  shows "Sig a b c < Sig a b d"
```

`cor_antichain_fibre` · `Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy`, line 373

```
corollary cor_antichain_fibre:
  assumes "Zms a b c = Zms a b d"
    and "\<And>h. -a \<le> h \<Longrightarrow> h \<le> a \<Longrightarrow> -b - 1 \<le>
c h \<and> c h \<le> b \<and> -b - 1 \<le> d h \<and> d h \<le> b"
    and "\<And>h. -a \<le> h \<Longrightarrow> h \<le> a \<Longrightarrow> c h \<le> d
h"
  shows "\<forall>h. -a \<le> h \<and> h \<le> a \<longrightarrow> c h = d h"
```

`cor_antichain_iii` · `Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy`, line 387

```
corollary cor_antichain_iii:
  assumes "Zms a b c = Zms a b d"
  shows "(\<Sum>h \<in> {-a..a}. (a + 1 - \<bar>h\<bar>) * (Fsum b (d h) - Fsum b (c
h))) = 0"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Isabelle: 3 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: (ii) and (iii); (i) rests on the geometric part of Proposition 3.1(b) and is not formalized

↳ Used in: 3.3 (p. 14), 3.4 (p. 15), Proposition 3.5 (p. 15)

Proof

> Σ is a function of $Z(c)$, hence of the word. (i) is Proposition 3.1(b), (ii) follows from (a) because every step weight is ≥ 1, and (iii) is (c) with $\Sigma(c) = \Sigma(d)$.

Evidence & scope

- **Program:** a ≤ 3, b ≤ 15 (45 cases): adjacent pairs with the same word 0 (e.g. 6187 adjacent pairs at a = 3, b = 15); comparable pairs inside a fibre 0; control: 1966 comparable pairs among arbitrary cells. working script `V376_propuestas.py`, not published

Remark 3.3. (what E counts: the arrangement, and generic words) p. 12

(a) *E does not depend on a choice of presentation.* A word is a function of the cell (Lemma 2.2: the counts depend only on the signs of $\Lambda + h\theta - \rho$), so no line is missing; and by Corollary 3.2(i) every wall between two cells separates two different words, so no line can be erased, and no two cells of one fibre are adjacent. The cells are convex, and two cells meet at most along walls and at vertices, which are off the set of generic parameters. Hence the connected components of the open set of generic parameters whose word is w are exactly the cells of the fibre of w, and $E = \sum_w (\#\pi_0(U_w) - 1)$ where U_w is that set: there is no excess from over-subdivision, only from words that have several disconnected realisations. (If one adds lines that change no word, R and E grow together; here none is superfluous.) The proof is the one of the cited results; it is stated here because it is what makes E an intrinsic count of the word map. (b) *Only generic words occur.* No integer lies on a line: $\Lambda + h\theta = \rho$ means $t\,r^h = s^\rho$ with $t = 2^e$, $e \geq 1$, and r, s odd primes, which is impossible because the 2-adic valuation of the left side is $e \geq 1$ and that of the right side is 0. So $W(a, b)$ counts the words of integers, all of which lie in open cells, and the words of singular parameters (where two projected values coincide and brackets cancel) never occur.

↳ Not used by another result of the paper.

Remark 3.4. (how much the balance filters) p. 13

Corollary 3.2(iii) is necessary, not sufficient, and is cheap to test before comparing observation vectors. For $a \leq 3$ and $b \leq 15$ we measured it on all pairs of cells. Few pairs pass it: 4011 of the 6.36×10^6 pairs for $a = 3$, $b = 15$ (0.06%). Among those that pass, the share of genuine coincidences falls with b: from 75% ($b = 2$) to 40% ($b = 15$) for $a = 1$, from 74% to 21% for $a = 2$, and from 57% ($b = 3$) to 17% for $a = 3$.

↳ Not used by another result of the paper.

Proposition 3.5. p. 13

> *Every fibre has at most $1 + \sum_{q=1}^{2a} \Phi_q(2b) \leq 4ab + 1$ vectors.*

Trust base: ⚠ not checked. Written proof, not checked by a human referee, not machine-checked; checked for a ≤ 3, b ≤ 15

Proof: in this note · **Covers:** not formalized

↳ Used in: 10.7 (p. 53)

Proof

Two lines $\ell_{h,\rho}$, $\ell_{h',\rho'}$ with $h \neq h'$ meet at $\theta = (\rho - \rho')/(h - h')$, and lines with the same h are parallel. So every vertex of the arrangement in the open quadrant has θ in the set $\mathcal{T}$ of positive fractions p/q in lowest terms with $q \leq 2a$ and $p \leq 2b$, and $|\mathcal{T}| = \sum_{q \leq 2a} \Phi_q(2b)$. The set $\mathcal{T}$ also contains the abscissa $\theta = \rho/h$ at which a line $\ell_{h,\rho}$ crosses the boundary line $\ell_{0,0} = \{\Lambda = 0\}$, so inside each strip below every line lies entirely above or entirely below $\Lambda = 0$. The vertical lines $\theta \in \mathcal{T}$ cut the quadrant into $|\mathcal{T}| + 1$ open vertical strips. Inside one strip no two lines cross, so the lines are graphs $\Lambda = \rho - h\theta$ over its θ-interval, totally ordered, and every cell that meets the strip meets each vertical line $\theta = \theta_0$ of it. Fix such a θ_0 : along it every coordinate clip$\lfloor \Lambda + h\theta_0 \rfloor$ is non-decreasing in Λ, so any two cells meeting the strip have comparable vectors. By Corollary 3.2(ii) a fibre meets each strip in at most one cell, and every cell, being open, meets some strip. Finally $\Phi_q(2b) \leq 2b$.

Evidence & scope

- **Program:** a ≤ 3, b ≤ 15: no fibre above 4ab+1 nor above 1 + Σ_q≤2a Φ_q(2b); the largest fibre is that of O(0) in every case (34 at a = 3, b = 15, against 115 and 181). working script `V376_propuestas.py`, not published

Lemma 3.6. p. 14

> $E(a,b) \le \mathrm{Amb}(a,b) \le 2E(a,b)$. *Hence* $E(a,b) = O_a(b)$ *if and only if* $\mathrm{Amb}(a,b) = O_a(b)$.

Isabelle · ours · part **show**

Only a part is in Isabelle: the counting argument, for any finite set with a word map; that the set is the set of cells is the paper's setting

`lem_amb` · `Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy`, line 684

```
theorem lem_amb:
  assumes "finite X"
  shows "Ecount X w \<le> int (Amb X w)" and "int (Amb X w) \<le> 2 * Ecount X w"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Isabelle: 1 result, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the counting argument, for any finite set with a word map; that the set is the set of cells is the paper's setting

↳ Used in: Corollary 6.2 (p. 23), 10.4 (p. 52), Proposition 10.6 (p. 53)

Proof

> A fibre of size $s \ge 2$ contributes $s-1$ to E and s to Amb, and $s-1 \le s \le 2(s-1)$; fibres of size 1 contribute to neither.

Evidence & scope

- **Program:** 42 cases ($a \le 3$, $2 \le b \le 15$), all $E \le \mathrm{Amb} \le 2E$, Amb/E $\in [1.750, 1.920]$; and the 240 cases of Table 3. working scripts `V377_propuestas.py`, `V385_resumen.py`, not published

Theorem 3.7. p. 14

> *Let* $L_a = \mathrm{lcm}(1,\dots,a+1)$ *and* $t_a = 2(a+1)(2a+3)L_a$. *If* c, d *are realisable for the same* b *and* $I_j(c) = I_j(d)$ *for* $0 \le j \le 2t_a - 2$, *then* $Z(c) = Z(d)$. *So the number of power sums needed depends on* a *only.*

Trust base: ⚠️ not checked. Written proof (Hajós's lemma on the sparse form of [I, §6, Step 3]), not checked by a human referee; the minimal number of power sums needed is an exact enumeration for $a \le 3$, $b \le 20$

Proof: in this note · **Covers:** not formalized

NEW **Classical and new:** the paper: «we claim no novelty for it, only for the explicit bound t_a through [I]».

↳ Used in: Proposition 3.8 (p. 17), 10.5 (p. 52)

Proof

> Let $M_c(z) = \sum_{i,j} z^{H_i(j)}$. By [I, Section 6, Step 3], $(1-z^{L_a})M_c$ is a sum of at most t_a monomials for every c, so $P = (1-z^{L_a})(M_c - M_d) = \sum_{k=1}^{s} \kappa_k z^{e_k}$ with distinct exponents $e_k \ge 0$ and $s \le 2t_a$. Let $\vartheta = z\frac{d}{dz}$. Then $\vartheta^j(M_c - M_d)(1) = I_j(c) - I_j(d) = 0$ for $j \le 2t_a - 2$. Since $z^i \frac{d^i}{dz^i} = \vartheta(\vartheta-1)\cdots(\vartheta-i+1)$, the derivatives of $M_c - M_d$ of order $\le 2t_a - 2$ vanish at $z=1$, so $M_c - M_d$ vanishes there to order $\ge 2t_a - 1$, and P, with the simple zero of $1-z^{L_a}$, to order $\ge 2t_a \ge s$. By the same relation, $\vartheta^j P(1) = \sum_k \kappa_k e_k^j = 0$ for $0 \le j \le s-1$. The matrix $(e_k^j)_{k,j}$ is a Vandermonde matrix on distinct nodes, hence invertible (this is Hajós's lemma [14 , 15] : a nonzero polynomial with s monomials has no root $z \ne 0$ of multiplicity $\ge s$), so every $\kappa_k = 0$, $P = 0$, and $M_c = M_d$ because $\mathbb{Z}[z]$ is a domain.

Evidence & scope

- **Program:** least J with (I_0, …, I_J) separating all words: J(1,b) = 3 (2 ≤ b ≤ 20), J(2,b) = 3 except J = 5 at b = 6, 12, 16, J(3,b) = 3 for b ≤ 14 and J(3,15) = 4. working script `V377_propuestas.py`, not published

Proposition 3.8. (written proof) p. 14

> *Let $d_a = \sum_{k=1}^{a+1} \varphi(k)$ and $s_a = 2(a+1)(a+2)(d_a+1)$. If c, d are realisable for the same b and $I_j(c) = I_j(d)$ for $0 \le j \le 2s_a - 2$, then $Z(c) = Z(d)$. So $2s_a - 1 = O(a^4)$ power sums determine the word.*

Trust base: ⚠️ not checked. Written proof (Hajós's lemma on the sparse form of $\mathcal{Q}M_c$), not machine-checked

Proof: in this note · **Covers:** not formalized

🆕 **Classical and new:** the paper: the Vandermonde step is Hajós's lemma (known); the power-sum bounds are problem-specific consequences of this known machinery (Section 1).

↳ Not used by another result of the paper.

Proof

> Let $\mathcal{Q} = \prod_{k \le a+1} \Phi_k$, of degree d_a, divisible by $1 - z^s$ for $1 \le s \le a+1$, with a simple zero at $z = 1$. Each term $f(c_h + j)$ of a row changes slope at most once for $j \in [0, b]$: if $c_h < 0$ only the lower clipping point can occur there, and if $c_h \ge 0$ only the upper one can. Thus H_i is affine on at most $a+2$ intervals, with slope $s \in \{0, \dots, a+1\}$. An interval of slope $s \ge 1$, first value u and N terms contributes $(\mathcal{Q}/(1-z^s))(z^u - z^{u+sN})$ to $\mathcal{Q}M_c$, at most $2(d_a - s + 1) \le 2(d_a + 1)$ monomials, and one of slope 0 contributes $Nz^u\mathcal{Q}$, at most $d_a + 1$. So $\mathcal{Q}M_c$ has at most $(a+1)(a+2) \cdot 2(d_a+1) = s_a$ monomials, and $P = \mathcal{Q}(M_c - M_d)$ at most $2s_a$. The proof of Theorem 3.7 applies with $\mathcal{Q}$ in place of $1 - z^{L_a}$: $M_c - M_d$ vanishes at 1 to order $\ge 2s_a - 1$, so P to order $\ge 2s_a$, and Hajós's lemma gives $P = 0$. Finally $d_a \le \binom{a+2}{2}$, so $s_a = O(a^4)$.

Evidence & scope

- **Program:** improved bound s_a = 120, 280, 660, 1092, 2128 for a = 2..6; a row has at most a+2 affine segments because each summand has at most one clipping breakpoint. working script `simplificar_formulas.py`, not published

Proposition 3.9. p. 15

> *Let $a = b$. The map $\sigma(\Lambda, \theta) = (\Lambda/\theta, 1/\theta)$ permutes the cells, preserves words and fixes exactly $2a+1$ cells. Hence*
>
> $$W(a,a) \le \frac{R(a,a) + 2a + 1}{2}.$$

Isabelle · ours · part **show**

Only a part is in Isabelle: the counting part: 2W ≤ R + f for an involution that preserves words and has f fixed cells; that σ permutes the cells and fixes exactly 2a+1 of them is the geometric part, taken as a hypothesis

`prop_burnside` · `Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy`, line 792

```
theorem prop_burnside:
  assumes fin: "finite X" and sig: "\<And>x. x \<in> X \<Longrightarrow> sg x \<in> X"
and inv: "\<And>x. x \<in> X \<Longrightarrow> sg (sg x) = x"
    and pres: "\<And>x. x \<in> X \<Longrightarrow> w (sg x) = w x" and nfix: "card {x
\<in> X. sg x = x} = f"
  shows "2 * card (w ` X) \<le> card X + f"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Isabelle: 1 result, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the counting part: 2W ≤ R + f for an involution that preserves words and has f fixed cells; that σ permutes the cells and fixes exactly 2a+1 of them is the geometric part, taken as a hypothesis

↳ Used in: 3.10 (p. 18), 10.9 (p. 53)

Proof

σ is an involution of the open quadrant. A point of $\ell_{h,\rho}$ goes to a point with $\Lambda' - \rho\theta' = (\Lambda - \rho)/\theta = -h$, so $\sigma(\ell_{h,\rho}) = \ell_{-\rho,-h}$, a line of the arrangement because $a = b$; hence σ permutes the cells. The point $(\log t/\log s, \log r/\log s)$ of $n = 2^{e-1}r^a s^a$ goes to $(\log t/\log r, \log s/\log r)$, the point of the same n with r and s exchanged; every cell contains points of this kind [I, Lemma 2.3] and the word is constant on cells, so σ preserves words. In homogeneous coordinates $(\Lambda, \theta) = (X/Z, Y/Z)$, σ is the linear map $(X, Y, Z) \mapsto (X, Z, Y)$; it maps the open octant $X, Y, Z{>}0$ to itself, and the cone $\{(Z\Lambda, Z\theta, Z) : (\Lambda, \theta) \in C,\ Z{>}0\}$ to the cone over $\sigma(C)$. If $\sigma(C) = C$, the cone over C is therefore convex and σ-invariant, so $x + \sigma(x)$ lies in it for any x in it, and C contains a fixed point, that is, a point with $\theta = 1$; conversely a cell meeting $\theta = 1$ is fixed. The lines meet the half-line $\theta = 1$, $\Lambda{>}0$ at $\Lambda = \rho - h$, that is, at $\Lambda = 1, \ldots, 2a$, and cut it into $2a + 1$ open intervals; each lies in one cell, and no two lie in the same cell, which is convex and avoids the lines. So $2a + 1$ cells are fixed, and σ has $(R(a, a) - (2a + 1))/2 + (2a + 1)$ orbits, each inside one fibre.

Evidence & scope

- **Program:** a = b ≤ 4: σ an involution on cells preserving the word, checked cell by cell; fixed cells 3, 5, 7, 9 = 2a+1; orbits 6, 29, 99, 262; W = 6, 28, 95, 255. working script `V376_propuestas.py`, not published

Remark 3.10. (exchanging the primes when $a \neq b$; written proof for the first part) p. 15

For $a \neq b$ the same map σ sends $\ell_{h,\rho}$ to $\ell_{-\rho,-h}$, that is, the arrangement of the box (a, b) onto that of (b, a). The point σP of $n = 2^{e-1}r^a s^b$ is its point when the primes are exchanged, the exponents then being (b, a) in the box; every cell is realised by primes [I, Lemma 2.3] and the word is constant on cells, so σ is a bijection from the cells of (a, b) onto those of (b, a) that preserves words. Hence

$$R(a, b) = R(b, a), \qquad W(a, b) = W(b, a).$$

(Exact values of $R(a, b)$ and $W(a, b)$ for $(a, b) = (1, 2)$: 21, 16; $(1, 5)$: 87, 69; $(2, 3)$: 100, 80; $(2, 4)$: 162, 136; $(2, 5)$: 238, 200; $(2, 6)$: 329, 288; $(3, 4)$: 313, 268; $(3, 5)$: 462, 410; each equals the value at (b, a).) But σ does not permute the cells of one arrangement unless $a = b$, so the integers $2^{e-1}r^a s^b$ and $2^{e-1}s^a r^b$ need not have the same word. They do exactly when σP lies in a cell of the fibre of P. Computed on cells for $a = 2$, $b = 12$, over the 149 736 triples (r, s, e) with $r \neq s$ odd primes ≤ 200 and $1 \leq e \leq \min(200, e_{\max})$, where $e_{\max} = \lfloor (b + a\max(\theta, 1/\theta) + 1)\log s/\log 2 \rfloor + 1$, excluding the triples for which P or σP lies in the trivial cell (all $c_h = b$) or within 10^{-9} of a line (floating point, the cell of each point then read exactly): the words coincide for 22.0% of the triples; in 21.2% the point σP is in the *same* cell as P (this is the effect of σ fixing $\theta = 1$: in this sample the same-cell proportion is 0 for $\min(\theta, 1/\theta){<}0.8$, and the coincidence of words is 0 below 0.5), and in 0.85% it is in another cell of the fibre, which happens for only three pairs of cells. The most frequent is the pair of Proposition 3.11 below; an instance is $11^2\,7^{12}$ and $7^2\,11^{12}$, whose words (78 symbols) coincide. So in this sample there is no symmetry $p^a q^b \leftrightarrow q^a p^b$ for $a \neq b$, only $\sigma P \in$ cell of P near $\theta = 1$ and three exceptional pairs of cells.

↳ Used in: Proposition 3.11 (p. 18)

Proposition 3.11. (written proof) p. 16

> *Let $a = 2$ and $b \geq 2$. The vectors $c = (-3, -1, 0, 1, 2)$ and $d = (-2, -1, 0, 1, 1)$ are realisable, distinct, and $Z(c) = Z(d)$.*

Trust base: ⚠️ not checked. Written proof (two moves of Lemma M that telescope), not machine-checked; checked for $b \leq 300$

Proof: in this note · **Covers:** not formalized

↳ Not used by another result of the paper.

Proof

The points $(\frac{3}{10}, \frac{6}{5})$ and $(\frac{3}{10}, \frac{4}{5})$ are witnesses: $\Lambda + h\theta$ is $-\frac{21}{10}, -\frac{9}{10}, \frac{3}{10}, \frac{3}{2}, \frac{27}{10}$ and $-\frac{13}{10}, -\frac{1}{2}, \frac{3}{10}, \frac{11}{10}, \frac{19}{10}$ for $h = -2, \ldots, 2$, with floors c and d, unchanged by clipping since $b \geq 2$. From c to d, c_{-2} rises by 1 and c_2 falls by 1. By Lemma 2.1, $c_{-2} = -3$ enters only row 0, on $J = [2, b]$, and $c_2 = 2$ only row 2, on $J = [0, b-2]$. On $[2, b]$ row 0 of c is $f(j) + f(j-1) + f(j-3) = (j+1) + j + (j-2) = 3j - 1$, and on $[0, b-2]$ row 2 is $f(j+2) + f(j+1) + f(j) = 3j + 6$. So

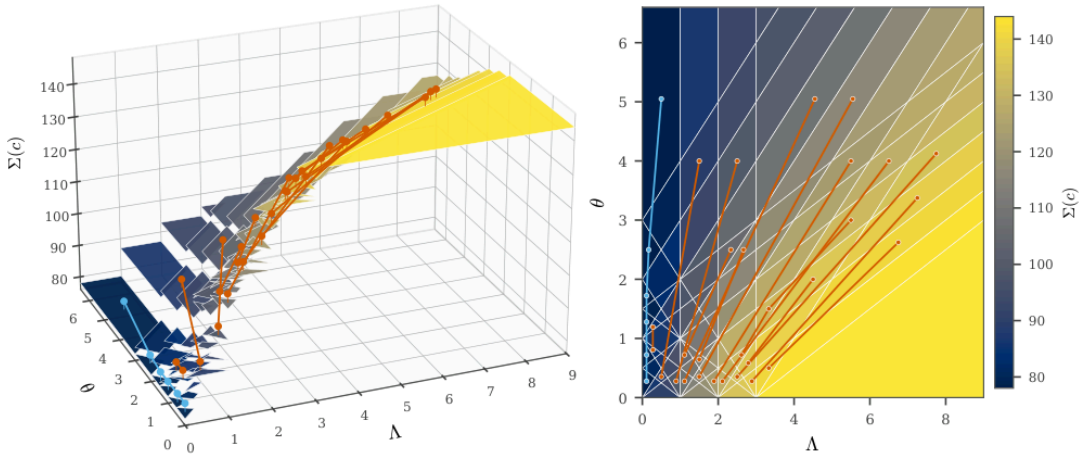
$$Z(d) - Z(c) = \sum_{j=2}^{b} (\delta_{3j} - \delta_{3j-1}) + \sum_{j=0}^{b-2} (\delta_{3j+5} - \delta_{3j+6}) = 0,$$

since the second sum is the first with the opposite sign ($j \mapsto j + 2$).

Evidence & scope

- **Program:** b = 2..300: witnesses and Z(c) = Z(d), 0 failures of 299; control (one change only) differs in 299/299. working script `V387_comprobaciones.py`, not published

Figure 2



Wall weights for $a = 2$, $b = 3$. Left: the cells of the arrangement in the window $0<\Lambda<9$, $0<\theta<6.6$ (99 of the $R(2, 3) = 100$ cells; the missing one shares its word with no other cell), each drawn as a horizontal terrace at height $\Sigma(c)$; right: the same cells seen from above. Crossing a wall upwards raises Σ by $\omega(h, \rho)$ (Proposition 3.1). The 15 non-trivial fibres ($E(2, 3) = 20$) are joined by bridges at their common height: the fibre of $O(0)$ (light blue, 6 cells near $\Lambda = 0$) and 14 others (vermilion). Every bridge is horizontal and joins cells that are neither adjacent nor comparable (Corollary 3.2). Equal height is necessary, not sufficient, for equal words. Cells computed exactly as intersections of half-planes; fibres by exact enumeration.

4 The dimension criterion

Proposition 4.1. (dimension criterion) p. 16

> *Fix a and a decomposition of B_a into linear sets. Then $E(a,b) = O(b)$ if and only if every component has rank at most 2. In particular the condition does not depend on the decomposition.*

Trust base: ⚠️ not checked. Written proof (a short consequence of the Ginsburg–Spanier theorem), re-derived by an independent AI agent, not checked by a human referee; not machine-checked

Proof: in this note · **Covers:** not formalized

🆕 **Classical and new:** the paper: «we did not find the criterion in this form, but it is a short consequence of the Ginsburg–Spanier theorem and we do not present it as a deep result».

↳ Used in: 4.2 (p. 20), 4.3 (p. 20), 10.4 (p. 52)

Proof

Zero generators can be deleted without changing L, so assume $v_i \neq 0$ for all i; a component of rank 0 is the single point x_0 and contributes at most one vector, to one value of b.

($\Leftarrow$) Let $L = x_0 + \sum \mathbb{N} v_i$ be a component. Each $v_i \neq 0$ has positive b-coordinate: otherwise the points $x_0 + t v_i$, $t \in \mathbb{N}$, would be infinitely many points of B_a with the same b, while $B_a(b) \subset [0, 2b+1]^{2a+1}$ is finite. So the span V of the v_i contains a vector with nonzero b-coordinate, and the section of L at $b = b_0$ lies in the affine space $(x_0 + V) \cap \{b = b_0\}$, of dimension $\operatorname{rank} L - 1 \leq 1$. A point contributes one vector. A line lies in $\{b = b_0\}$, so some c-coordinate varies along it, and it meets the cube $[0, 2b_0+1]^{2a+1}$ in at most $2b_0 + 2$ integer points. Summing over the finitely many components gives $E(a,b) = O(b)$.

($\Rightarrow$) Suppose a component has rank ≥ 3, and take independent v_1, v_2, v_3 among its generators, with b-coordinates $\beta_i > 0$. Put $\ell = \operatorname{lcm}(\beta_1, \beta_2, \beta_3)$ and $w_i = (\ell/\beta_i) v_i$. The points $x_0 + k_1 w_1 + k_2 w_2 + k_3 w_3$ with $k_i \in \mathbb{N}$, $k_1 + k_2 + k_3 = t$, lie in L, have $b = b_0 + t\ell$, and are pairwise distinct because the w_i are independent. So $E(a, b_0 + t\ell) \geq \binom{t+2}{2}$, which is not $O(b)$.

Remark 4.2. p. 17

$E(a,b) = R(a,b) - W(a,b)$ is eventually a quasipolynomial, since both terms are [I, Corollary 5.2 and Theorem 6.1]. With $E(a,b) = O(b)$ and the linear lower bound of [I, Lemmas 7.1 and 7.5] for $a \geq 2$, every constituent then has degree exactly one. Proposition 4.1 is not used below: the reduction bounds $E(a,b)$ directly. It explains what is being certified: the absence of two-parameter families of coincidences inside one value of b. Woods's theorem [8] gives that $E(a,b)$ is eventually a quasipolynomial, not its degree in terms of the ranks of the components; we did not find the criterion in this form, but it is a short consequence of the Ginsburg–Spanier theorem and we do not present it as a deep result.

↳ Not used by another result of the paper.

Remark 4.3. (effectivity) p. 17

A semilinear representation of a Presburger-definable set can be computed from its defining formula (quantifier elimination for Presburger arithmetic and the effective form of the Ginsburg–Spanier theorem [5]), and the ranks of the components are then linear algebra. So, for each fixed a, whether $E(a,b) = O(b)$ is decidable. Moreover the proof of ($\Leftarrow$) gives, if the decomposition has N components all of rank ≤ 2, $E(a,b) \leq N(2b+2)$ for *every* $b \geq 1$. This is a decision procedure in principle only: N is not known and the general algorithms are far too costly. The finite reduction of Section 5 is a practical sufficient condition, and a failure of its conditions for some a would not by itself disprove rigidity.

↳ Not used by another result of the paper.

5 The finite reduction

Theorem 5.1. (finite reduction) p. 19

> *Fix a. Suppose that*
>
> 1. *every piece of a chamber that contains vectors with at least two free coordinates has observation vector of (n, m)-rank 2; and*
> 2. *for every pair of pieces P, P' with the same signature, writing $V_{P'}(b, \phi(b, n, m)) = V_P(b, n, m)$ for the affine map ϕ so defined, and $\phi_{\hom}$ for the affine map of the normalised plane with $\phi(b, n, m) = \big(b,\ b\,\phi_{\hom}(n/b, m/b) + k\big)$, k constant: either $c_{P'} \circ \phi = c_P$ as affine maps, or no integer point is mapped by ϕ to an integer point, or the set $\{(\lambda, \mu) \in C_P : \phi_{\hom}(\lambda, \mu) \in C_{P'}\}$ has area 0 (area measured in the plane of C_P).*
>
> *Then $E(a, b) = O(b)$.*

Trust base: ⚠ not checked. Written proof, re-derived step by step by an AI agent other than the one that wrote it, NOT checked by a human referee; every intermediate statement compared with exact data for a = 1, 2, 3, b ≤ 200, each with a negative control

Proof: in this note · **Covers:** not formalized

🆕 **Classical and new:** the paper counts the finite reduction and its certificate among its own results, as far as its searches could see (Section 1).

↳ Used in: Proposition 3.8 (p. 17), 5.2 (p. 22), Theorem 6.1 (p. 22), Corollary 6.5 (p. 27), Proposition 7.1 (p. 31), Lemma 7.5 (p. 34), Proposition 7.8 (p. 36), Theorem 8.1 (p. 39), Lemma 8.3 (p. 40), Lemma 8.4 (p. 41), 10.1 (p. 52)

Proof

$E(a, b)$ is at most the number of realisable c that have a partner $d \neq c$ with $Z(d) = Z(c)$. We bound four kinds.

(i) c thin or exceptional: $O(b)$, by the box and (2).

(ii) c generic and d thin or exceptional. Here "generic" refers only to pieces of chambers whose vectors have at least two free coordinates, so c lies in such a piece P and (C1) applies to it; d need not have an observation vector. By

(C1) and (3), for a fixed d and a fixed piece there is at most one generic (n, m) with $V_P(b, n, m)$ equal to the sparse representation of K_d. So the number of such c is at most the number of pieces times the number of thin or exceptional d, which is $O(b)$ by the box (thin vectors are $O(b)$) and (2).

(iii) $c \in P$, $d \in P'$ generic, with different signatures. Then $S = \{(b, x, y) : V_P(b, x) = V_{P'}(b, y)\}$ is isomorphic to $A_P \cap A_{P'}$, a proper affine subspace of A_P, so $\dim S \leq 2$. If b is constant on S, it concerns one value of b. Otherwise each section $S \cap \{b = b_0\}$ has dimension ≤ 1, its x-projection is injective (by

(C1)) and lies on a line or a point, and contains $O(b)$ integer points of the box.

(iv) $c \in P$, $d \in P'$ generic, with the same signature. Then $A_P = A_{P'}$, so $y = \phi(b, x)$ for the affine rational map ϕ of (C2), which preserves b. In the first alternative of

(C2), $d = c$. In the second, there is no pair. In the third, a pair gives a point $(\lambda, \mu) = (n/b, m/b)$ of C_P with $\phi_{\hom}(\lambda, \mu) + k/b \in C_{P'}$. So (λ, μ) satisfies, up to an error $O(1/b)$, the finitely many linear inequalities that define the closed polygon $X = \overline{C_P} \cap \phi_{\hom}^{-1}(\overline{C_{P'}})$ (open or closed chambers give the same area). By Hoffman's error bound for systems of linear inequalities [9] , such a point lies within $O(1/b)$ of X if $X \neq \emptyset$.

> If $X = \emptyset$ there is no such point for b large: otherwise points of the compact polygon $\overline{C_P}$ with errors tending to 0 would have a limit point in X. A convex polygon of area 0 is a segment or a point, and its $O(1/b)$-neighbourhood scaled by b contains $O(b)$ integer points.
>
> There are finitely many pieces and pairs of pieces, so the four bounds add up to $O(b)$.

💡 **In plain words (commentary, not part of the paper)**

A cell is described by integers (n, m) and a finite "template" ε; cut the normalised plane $(n/b, m/b)$ into finitely many chambers where every relevant sign is fixed. On each piece (chamber plus template) the word, multiplied by a fixed cyclotomic product, becomes a sparse polynomial whose exponents and coefficients are affine in (b, n, m): the *observation vector*. Two generic cells have the same word exactly when their observation vectors agree.

So $E(a, b) = O(b)$ reduces to two checks on finitely many affine objects: (C1) on each piece the observation vector determines (n, m), and (C2) two pieces with the same signature never produce a two-parameter family of integer coincidences. Everything else (thin cells, cells near chamber edges, pairs with different signatures) costs only $O(b)$. The reduction is a written proof, re-derived by an AI agent other than the one that wrote it and not checked by a human referee; the paper says so in its table of status.

🛡️ **Evidence & scope**

- **Audit:** step-by-step verdicts of the independent audit (no step changed; four points made explicit). working script `?`, not published
- **Program:** templates found by a complete sweep: 4, 14, 36, 76, 136, 224, 346 for a = 1..7 (Mignosi's count, OEIS A005598); the vectors g(n,m,ε) = all realisable vectors for a = 1, 2, 3, b = 7, 20, 40. working script `certificado_rigidez.py`, not published

Remark 5.2. p. 20

> A positive area in (C2) does not by itself produce coincidences. A pair of pieces gives a genuine two-parameter family of coincidences inside each b, and then $E(a, b) \geq cb^2$ on an arithmetic progression of values of b, when in addition (1) the map ϕ sends some integer point to an integer point (integer compatibility), (2) the vectors $c_P(b, x)$ and $c_{P'}(b, \phi(b, x))$ are distinct at these points, not merely given by different affine formulas, and (3) a positive proportion of these integer points stays generic for both pieces as b grows (persistence of generic points under the strips and the conditions of the two pieces). Integrality of a rational affine map is a congruence condition, so the integer points mapped to integer points form a union of cosets of a full-rank sublattice of $\mathbb{Z}^3$; they may meet only some residue classes of b, which is why only a progression of values of b is obtained. Since the strips have bounded width, (3) should follow from (1) and a positive area, removing $O(b)$ points; we have not written this out. For $a = 1$ all three hold (Section 6); for $a = 2, \ldots, 5$ every candidate already fails (1).

↳ Not used by another result of the paper.

6 The certificate for $a \leq 5$

Theorem 6.1. (rigidity for $a \leq 5$; computer-assisted) p. 22

> *Assume Theorem 5.1 (written proof) and the correctness of the program* `certificado_rigidez.py` *(not checked by a human referee; its output re-checked by independent scripts, §6.6; run unchanged for $a = 5$). For $a \in \{2, 3, 4, 5\}$ there are $K_a, b_0(a)$ such that $E(a, b) \leq K_a b$ for $b \geq b_0(a)$; the vectors with at most one free coordinate are included in this count (they are $O(b)$). With the lower bound, $E(a, b) = \Theta(b)$. Consequently*

$$W(2,b) = \tfrac{22}{3}b^2 + O(b), \qquad W(3,b) = \tfrac{73}{5}b^2 + O(b),$$
$$W(4,b) = \tfrac{1709}{70}b^2 + O(b), \qquad W(5,b) = \tfrac{7657}{210}b^2 + O(b),$$

and for these a every constituent of the quasipolynomial $W(a,b)$ ($b \geq N_a$) has leading coefficient C_a, and every constituent of $E(a,b)$ has degree one. By Proposition 2.3, $W(a,b) = C_a b^2 + (D_a - \eta_a(b))b + O(1)$, where $\eta_a(b)$ is the slope of the constituent of $E(a,b)$.

Trust base: computation (not a formal proof). Computer-assisted: the written reduction (Theorem 5.1, not checked by a human referee) and the program certificado_rigidez.py (not checked by a human referee; exact rational arithmetic; output re-checked by independent scripts; run unchanged for a = 5): no piece of rank < 2, and every candidate of (C2) not integer-compatible (6, 15, 42, 45 for a = 2..5); a = 1 is the positive control (8 integer-compatible pairs, the exchange)

Proof: in this note · **Covers:** not formalized

Classical and new: the paper counts the certificate among its own results; it answers for a ≤ 5 Question 11.1 of [I].

↳ Used in: Corollary 6.2 (p. 23), Corollary 6.5 (p. 27), Corollary 7.2 (p. 32), 9.8 (p. 50), Proposition 10.6 (p. 53)

Proof

The upper bound is Theorem 5.1 with the output above. The lower bound $E(a,b) \geq \sum_{h \leq a} \#\{j \leq b : \gcd(j,h) = 1\} + \sum_{2 \leq j \leq a} \lfloor b/j \rfloor$ is [I, Remark 7.6]. Then $W = R - E$ and $R(a,b) = C_a b^2 + O(b)$ in every residue class [I, Corollary 5.2].

In plain words (commentary, not part of the paper)

For the integers $2^{e-1} r^a s^b$ with $a = 2, 3, 4, 5$, only linearly many cells (in b) lose their word to another cell, so the number of distinct words grows like the number of cells: $W(a,b) = C_a b^2 + O(b)$, e.g. $W(5,b) \sim \frac{7657}{210} b^2$. For $a = 1$ it is different, and known from paper I: a quadratic family (the exchange of progressions) loses about $b^2/4$ cells.

How it is obtained: the program builds every chamber, template and piece (12 920 generic pieces for $a = 5$) in exact rational arithmetic and checks the two conditions of the finite reduction. Every candidate for a two-parameter family shifts n by a non-integral multiple of $1/(a+1)$, so none is realised by integers. For $a = 1$ the same program finds the exchange: a positive control. Status: *computer-assisted*; neither the reduction nor the program has been checked by a human referee, and the output was re-checked by independent scripts. The lab shows the cascade $E(a,b)/b$ from the exact enumerations of the paper.

Evidence & scope

- **Program:** the certificate, Table 2: templates 4, 14, 36, 76, 136; chambers 10, 25, 47, 76, 112; generic pieces 20, 238, 1296, 4712, 12920; candidates 9, 6, 15, 42, 45; integer-compatible 8, 0, 0, 0, 0 (a = 1, …, 5); 57 s for a = 4, 167 s for a = 5. working script `certificado_rigidez.py`, not published
- **Program:** multi-point regression: 360, 4284, 23328, 84816, 232543 points for a = 1..5, no failure, rank 4 in every piece; control with a corrupted coefficient of $Q/(1-z)$: all 2856 points of a = 2 fail. working script `regresion_multi_a5.py`, not published
- **Program:** contrast with exact enumeration: for a = 1 the exchange appears; for a = 2..5 every real coincidence is exceptional on both sides (weak for a = 5). working script `contraste_rigidez.py`, not published
- **Program:** the lab «Rigidity live»: every enumerated (R, W) of the paper's logs for a ≤ 5, R recomputed by Alekseyev's formula, W against Theorems A, B of [I] and Conjecture 9.8, and the lower bound of the proof. `labjs/labs.js`, `lab/hacer_datos.py`

Corollary 6.2. (asymptotic faithfulness for $a \leq 5$) p. 23

Under the hypotheses of Theorem 6.1, for $a \in \{2,3,4,5\}$ and $b \geq b_0(a)$, $\mathrm{Amb}(a,b) \leq 2K_a b$ and

$$\frac{\mathrm{Amb}(a,b)}{R(a,b)} = \Theta_a\Big(\frac{1}{b}\Big),$$

so the proportion of cells whose word is shared with another cell tends to 0*, at the exact rate* $1/b$*.*

Trust base: computation (not a formal proof). A consequence of Theorem 6.1 (computer-assisted) and Lemma 3.6; the values of Table 3 are exact enumerations

Proof: in this note · **Covers:** not formalized

↳ Not used by another result of the paper.

Proof

By Lemma 3.6, $E \le \mathrm{Amb} \le 2E \le 2K_a b$, and $R(a,b) = C_a b^2 + O(b)$ with $C_a > 0$ [I, Corollary 5.2]. For the lower bound use $\mathrm{Amb} \ge E \ge c\,b$ (the lower bound of Theorem 6.1).

Evidence & scope

- **Program:** Table 3: Amb/R from exact enumeration in 240 cases (all b in 2 ≤ b ≤ 100 for a = 2, 3 ≤ b ≤ 80 for a = 3, 4 ≤ b ≤ 40 for a = 4, 5 ≤ b ≤ 30 for a = 5). working scripts `V385_multiplicidades.py`, `V385_resumen.py`, not published

Proposition 6.3. (two numerations of one progression; written proof) p. 24

Let $a \ge 2$, $b \ge 1$, $K = (a+1)(b+1)$, $q = (a+1)b + a = K - 1$, *and for* $0 \le k \le K-2$ *let* $P_1(k) = \left(\frac{k+1/2}{a+1}, \frac{1}{a+1}\right)$ *and* $P_2(k) = \left(k + \frac{1}{2}, b+1\right)$.

1. $P_1(k)$ *and* $P_2(k)$ *are off the lines and* $Z(P_1(k)) = Z(P_2(k)) = \{\{\min(K, N+k+1) : 0 \le N \le K-1\}\}$.
2. *If* $k \le K-3$, *then* $P_1(k)$ *and* $P_2(k)$ *lie in distinct cells, so these cells have the same word and form a fibre of at least two cells.*
3. *Let* $0 \le d < \frac{1}{a(a+1)}$, $u = a(a+1)d$ *and* $\theta = \frac{1}{a+1} + d$. *For each* $0 \le k \le K-3$ *the cell of* $P_1(k)$ *contains a segment of the line* $\{\theta\} \times (0, b + a\theta)$ *of length* $\frac{1-u}{a+1}$*; these segments are pairwise disjoint. Hence the proportion* $A_{a,b}(\theta)$ *of that line lying in cells whose word is shared with another cell satisfies*

$$A_{a,b}(\theta) \;\ge\; \frac{(q-1)(1-u)}{q+u} \;\ge\; F_a(d) - \frac{1}{q}, \qquad F_a(d) = 1 - a(a+1)d,$$

and the last bound holds trivially also at $d = \frac{1}{a(a+1)}$.

4. *Let* $0 \le d < \frac{1}{2a(a+1)}$, $u = a(a+1)d$ *and* $\theta = \frac{1}{a+1} - d$. *For each* $0 \le k \le K-3$ *the cell of* $P_1(k)$ *contains the segment* $\left|\Lambda - \frac{k+1/2}{a+1}\right| < \frac{1}{2(a+1)} - ad$ *of the line* $\{\theta\} \times (0, b + a\theta)$*; these segments are pairwise disjoint, and*

$$A_{a,b}(\theta) \;\ge\; \frac{(q-1)(1-2u)}{q-u}.$$

Isabelle · ours · part **show**

Only a part is in Isabelle: (i) for a ≥ 2, and (ii) as a line of the arrangement that separates the two points (an integer ρ ∈ [−b, b], so the coordinates c_{−a} differ); (iii) and (iv) (segments, lengths and the bounds on each side) and the identification of a cell with a connected region are not formalized

`prop_twonum_i` · `Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Twonum.thy`, line 280

```
theorem prop_twonum_i:
  assumes "2 \<le> a" "1 \<le> b" "0 \<le> k" "k \<le> (a + 1) * (b + 1) - 2"
  shows "0 < L1 a k \<and> 0 < T1 a \<and> 0 < L2 k \<and> 0 < T2 b"
    and "\<And>h. -a \<le> h \<Longrightarrow> h \<le> a \<Longrightarrow> L1 a k +
real_of_int h * T1 a \<notin> \<int>"
    and "\<And>h. -a \<le> h \<Longrightarrow> h \<le> a \<Longrightarrow> L2 k +
real_of_int h * T2 b \<notin> \<int>"
    and "Zms a b (cpt b (L1 a k) (T1 a)) = image_mset (\<lambda>N. min ((a + 1) * (b +
1)) (N + k + 1)) (mset_set {0..(a + 1) * (b + 1) - 1})"
```

```
    and "Zms a b (cpt b (L2 k) (T2 b)) = image_mset (\<lambda>N. min ((a + 1) * (b +
1)) (N + k + 1)) (mset_set {0..(a + 1) * (b + 1) - 1})"
```

prop_twonum_ii · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Twonum.thy, line 303

```
theorem prop_twonum_ii:
  assumes "2 \<le> a" "1 \<le> b" "0 \<le> k" "k \<le> (a + 1) * (b + 1) - 3"
  shows "\<exists>\<rho>. -b \<le> \<rho> \<and> \<rho> \<le> b \<and> L2 k +
real_of_int (-a) * T2 b < real_of_int \<rho>
              \<and> real_of_int \<rho> < L1 a k + real_of_int (-a) * T1 a"
    and "cpt b (L1 a k) (T1 a) (-a) \<noteq> cpt b (L2 k) (T2 b) (-a)"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Isabelle: 2 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: (i) for a ≥ 2, and (ii) as a line of the arrangement that separates the two points (an integer ρ ∈ [−b, b], so the coordinates c_{−a} differ); (iii) and (iv) (segments, lengths and the bounds on each side) and the identification of a cell with a connected region are not formalized

NEW **Classical and new:** the paper: the case a = 3 of (i), (ii) is Proposition 9.1 (and [I, Supplement, Lemma 2.3(a)]); every a ≥ 2 with the separating line, and (iii), are new here (comment in the source).

↳ Used in: Corollary 6.4 (p. 26), Proposition 9.1 (p. 42), 10.8 (p. 53)

Proof

(i) At P_2, $\Lambda + h\theta = k + \frac{1}{2} + h(b+1)$ is a half-integer; at P_1 it is $\frac{2(k+h)+1}{2(a+1)}$, not an integer because its numerator is odd. By Lemma 2.2, Z is the multiset of the counts $\#\{p \in \mathcal{P} : p<p_0 + \Lambda\}$. At P_1 the positions $i\theta + j = \frac{i+(a+1)j}{a+1}$ $(0 \le i \le a,\ 0 \le j \le b)$ are $N/(a+1)$ with $N = i + (a+1)j$ running once through $0, \dots, K-1$ (division with remainder), and $p<p_0 + \Lambda$ reads $N'<N + k + \frac{1}{2}$, that is $N' \le N + k$: the count is $\min(K, N + k + 1)$. At P_2 the positions $i(b+1) + j$ are again $N = 0, \dots, K-1$ each once, and $\Lambda = k + \frac{1}{2}$ gives the same counts.

(ii) Put $x_1 = \Lambda_1 - a\theta_1 = \frac{2k+1-2a}{2(a+1)}$ and $x_2 = \Lambda_2 - a\theta_2 = k + \frac{1}{2} - a(b+1)$ (values of $\Lambda + h\theta$ at $h = -a$). Then $x_1 - x_2 = a\,\frac{K-k-3/2}{a+1} \ge \frac{3a}{2(a+1)} \ge 1$ for $a \ge 2$, $k \le K-3$. Neither x_i is an integer, so the open interval (x_2, x_1) contains an integer ρ. Moreover $x_1 > -1$ and $x_1 \le b + 1 - \frac{a+5/2}{a+1} < b$, so ρ can be taken in $[-b, b]$ (take $\rho = -b$ if $x_2 < -b$): the line $\ell_{-a,\rho}$ of the arrangement separates the two points.

(iii) Write $\Lambda = \frac{k+1/2}{a+1} + \lambda$. Then $\Lambda + h\theta = \frac{k+h+1/2}{a+1} + \lambda + hd$, and its floor stays $\lfloor (k+h)/(a+1) \rfloor$ (so c_h is that of $P_1(k)$, clipped or not) iff $-\frac{r_h+1/2}{a+1} < \lambda + hd < \frac{a+1/2-r_h}{a+1}$, where $r_h = (k+h) \bmod (a+1)$. Let $h_0 \in [-a, 0]$ be the h with $r_{h_0} = 0$; then $r_h = a$ for $h = h_0 + a \in [0, a]$. We claim that for $0 \le d \le \frac{1}{a(a+1)}$ the lower constraint of h_0 and the upper one of $h_0 + a$ imply all the others. For the lower constraint of h, $r_h/(a+1) + (h - h_0)d \ge 0$ is clear if $h \ge h_0$; if $h = h_0 - j$, $1 \le j \le a$, then $r_h = a + 1 - j \ge 1$ and $jd \le ad \le \frac{1}{a+1} \le \frac{r_h}{a+1}$. The upper constraints are symmetric (for $h = h_0 + a + j$, $r_h = j - 1$, and $\frac{a+1-j}{a+1} \ge jd$ in the same way). So the cell contains all λ in the interval from $-\frac{1}{2(a+1)} - h_0 d$ to $\frac{1}{2(a+1)} - (h_0 + a)d$, of length $\frac{1}{a+1} - ad$. For consecutive k the upper end of the interval of k is $\frac{k+1}{a+1} - (h_0(k) + a)d$ and the lower end of that of $k+1$ is $\frac{k+1}{a+1} - h_0(k+1)d$, with $h_0(k+1) \le h_0(k) + a$ (it is $h_0(k) - 1$, or $0 = h_0(k) + a$ if $h_0(k) = -a$); so they do not overlap. All lie in $(0, b + a\theta)$ since $h_0 \le 0$. The sum of the lengths is $(q-1)\frac{1-u}{a+1}$ with $u = a(a+1)d$, while the line has length $\frac{q+u}{a+1}$. Finally $\frac{(q-1)(1-u)}{q+u} \ge (1-u) - \frac{1}{q}$, since $\frac{(1-u)(1+u)}{q+u} \le \frac{1}{q}$. By (ii) every one of these segments lies in a cell of a fibre of at least two cells, and they are disjoint, so their lengths add. For $d = \frac{1}{a(a+1)}$ the right-hand side $F_a(d) - \frac{1}{q}$ is negative.

(iv) Now $\Lambda + h\theta = \frac{k+h+1/2}{a+1} + \lambda - hd$, and its floor stays $\lfloor (k+h)/(a+1) \rfloor$ iff $-\frac{r_h+1/2}{a+1} < \lambda - hd < \frac{a+1/2-r_h}{a+1}$. Since $0 \le r_h \le a$, both margins are at least $\frac{1}{2(a+1)}$, and $|hd| \le ad$; so every λ with $|\lambda| < \frac{1}{2(a+1)} - ad$ keeps every floor, and the point has the vector

of $P_1(k)$. The segments have length $\frac{1-2u}{a+1} < \frac{1}{a+1}$ and centres $\frac{k+1/2}{a+1}$ spaced by $\frac{1}{a+1}$, so they are disjoint; they lie in $(0, b+a\theta)$, because $\frac{k}{a+1} + ad > 0$ and $\frac{k+1}{a+1} - ad \le \frac{K-2}{a+1} < b + a\theta = \frac{q-u}{a+1}$. There are $q-1$ of them, each in a cell of a fibre of at least two cells by (ii), and the line has length $\frac{q-u}{a+1}$.

In plain words (commentary, not part of the paper)

Take $\theta = \frac{1}{a+1}$: the positions $i\theta + j$ are $N/(a+1)$ with $N = i + (a+1)j$ running once through $0, \dots, K-1$. Take $\theta = b+1$: the positions $i(b+1)+j$ are again $0, \dots, K-1$, each once. Two ways of numbering the same progression give the same multiset Z, hence the same word, at points that lie in different cells (a line of the arrangement separates them). These pairs are few among the cells, but the cells at $\theta \approx \frac{1}{a+1}$ are long in Λ: just to the right of $\frac{1}{a+1}$ a proportion at least $1 - a(a+1)d - \frac{1}{q}$ of the vertical line is ambiguous, and part (iv) gives a weaker bound on the left, $(q-1)(1-2u)/(q-u)$ with $u = a(a+1)|d|$, about $1-2u$. These are lower bounds for the profile of Figure 4, whose ridge rises with b; the exact profile can be higher.

So “almost every cell has its own word” (Corollary 6.2) and “a positive share of the parameter length is ambiguous” (Corollary 6.4) are both true: one counts cells, the other measures length. Parts (i) and (ii) are checked in Isabelle/HOL; (iii) and (iv) are written proofs. The lab recomputes (i)–(ii) for every k in your browser.

Evidence & scope

- **Program:** (i), (ii): 1540 points a = 2..5, b = 2..12, every k, exact Fractions, words and Z equal, cells equal iff k = K−2; control θ₂ = b+1+1/7 differs in 70 of 483; (iii): exact widths equal to 1/(a+1) − ad, failing formula (1.1×) detected. working script `n26_auditoria.py`, not published
- **Program:** (iii) and (iv): every segment inside the exact section of the cell of P₁(k), a = 2..6, b = 1..60, 8 values of d on each side, every k ≤ K−3: 373200 + 373200 segments, 0 failures; control (a wider left segment) fails in 194670 cases. working script `V387_comprobaciones.py`, not published
- **Program:** the lab «Tent live» recomputes (i)–(ii) for every k and the exact section A_a,b(θ) near 1/(a+1) in your browser, against the bounds of (iii) on the right and (iv) on the left. `labjs/nucleo.js`

Corollary 6.4. (ambiguity of positive measure; written proof) p. 26

Let $a \ge 2$, $q = (a+1)b + a$, let $I \subset (0, \infty)$ be a compact interval, and let $P_{a,b}(I)$ be the mean over $\theta \in I$ of $A_{a,b}(\theta)$, the proportion of the line $\{\theta\} \times (0, b + a\theta)$ lying in cells whose word is shared with another cell.

1. *If $I \supseteq [\frac{1}{a+1}, \frac{1}{a}]$, then for every $b \ge 1$*

$$P_{a,b}(I) \;\ge\; \frac{q-1}{a(a+1)|I|}\Big[(q+1)\log\Big(1+\frac{1}{q}\Big) - 1\Big],$$

so

$$\liminf_{b\to\infty} P_{a,b}(I) \;\ge\; \frac{1}{2a(a+1)\,|I|}.$$

2. *If $I \supseteq [\frac{2a-1}{2a(a+1)}, \frac{1}{a}]$, then for every $b \ge 1$*

$$P_{a,b}(I) \;\ge\; \frac{q-1}{a(a+1)|I|}\Big[(q+1)\log\Big(1+\frac{1}{q}\Big) - (2q-1)\log\Big(1+\frac{1}{2q-1}\Big)\Big],$$

so

$$\liminf_{b\to\infty} P_{a,b}(I) \;\ge\; \frac{3}{4a(a+1)\,|I|} > 0.$$

Trust base: ⚠ not checked. Written proof (integrals of Proposition 6.3(iii) and (iv), with explicit bounds for every b), not machine-checked; the limit of the mean is open

Proof: in this note · **Covers:** not formalized

↳ Used in: 10.8 (p. 53)

Proof

Write $u = a(a+1)d$, so $d\theta = du/(a(a+1))$. By Proposition 6.3(iii), the right band $\theta \in [\frac{1}{a+1}, \frac{1}{a}]$ contributes at least $\frac{1}{a(a+1)} \int_0^1 \frac{(q-1)(1-u)}{q+u}\, du = \frac{q-1}{a(a+1)} \int_0^1 \left(\frac{q+1}{q+u} - 1\right) du = \frac{q-1}{a(a+1)} \left[(q+1)\log\frac{q+1}{q} - 1\right]$ to $|I|\, P_{a,b}(I)$, and by (iv) the left band $\theta \in [\frac{2a-1}{2a(a+1)}, \frac{1}{a+1}]$ contributes at least $\frac{1}{a(a+1)} \int_0^{1/2} \frac{(q-1)(1-2u)}{q-u}\, du = \frac{q-1}{a(a+1)} \left[1 - (2q-1)\log\frac{2q}{2q-1}\right]$; the rest of I contributes ≥ 0. As $q \to \infty$, $(q-1)[(q+1)\log(1+\frac{1}{q}) - 1] \to \frac{1}{2}$ and $(q-1)[1 - (2q-1)\log(1+\frac{1}{2q-1})] \to \frac{1}{4}$.

Evidence & scope

- **Program:** exact means over θ ∈ [0.14, 6.9] (600 values of θ): a = 2: 0.307 … 0.053 (b = 8 … 320), a = 3: 0.257 … 0.034, all above the bounds 0.0123 and 0.0062. working script `n23_media_theta.py`, not published

Corollary 6.5. (no coincidence between deep cells; computer-assisted) p. 30

Let $a = 3$. There is a constant C such that, for b large, two distinct realisable vectors with at least two free coordinates whose cells both contain a point at distance $>C$ from $\mathcal{L}_{\deg}$ have different words. The same holds for $a = 4$ and $a = 5$ with the corresponding sets of lines.

Trust base: computation (not a formal proof). Computer-assisted: pares_rigidez.py solves exactly every pair of pieces with different signatures (a = 3: 20353 pairs; a = 4: 128151; a = 5: 656246); for a ≤ 4 reproduced pair by pair by an independent solver, integer contrast for a = 2, 3 up to b = 601; for a = 5 not reproduced

Proof: in this note · **Covers:** not formalized

↳ Used in: Theorem 8.1 (p. 39), 10.3 (p. 52)

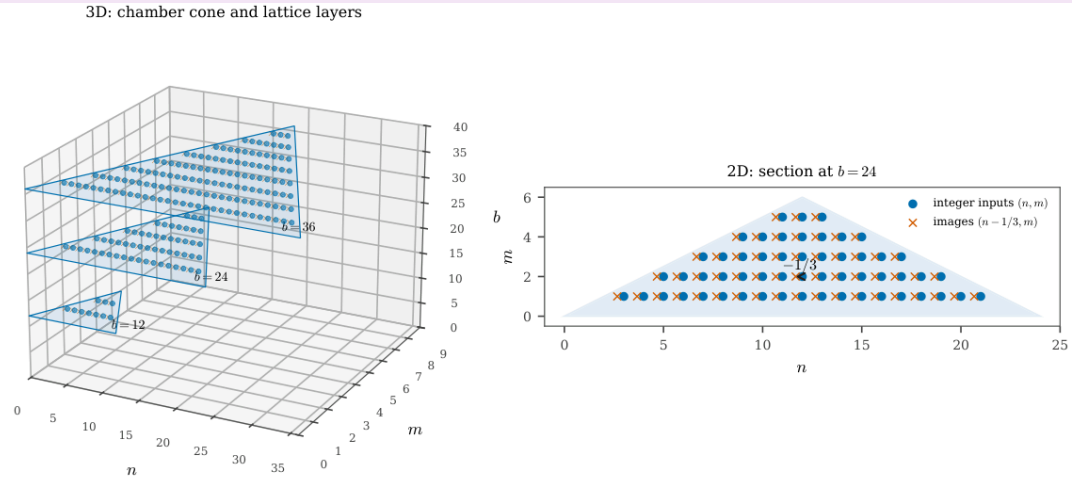
Proof

Sketch. A cell with a point at distance $>C$ from $\mathcal{L}_{\deg}$ has a generic representative, for C large. Two generic vectors with the same word fall into case (iii) or (iv) of the proof of Theorem 5.1. Case (iv) has no integer solutions (Table 6). In case (iii), by the output of `pares_rigidez.py`, at least one of the two cells lies within $O(1)$ of a chamber boundary, hence of $\mathcal{L}_{\deg}$; this is enough, since both were assumed deep. That output was reproduced pair by pair by an independent solver ($a \leq 5$), with an integer contrast for $a = 3$ up to $b = 601$ finding no generic coincidence.

Evidence & scope

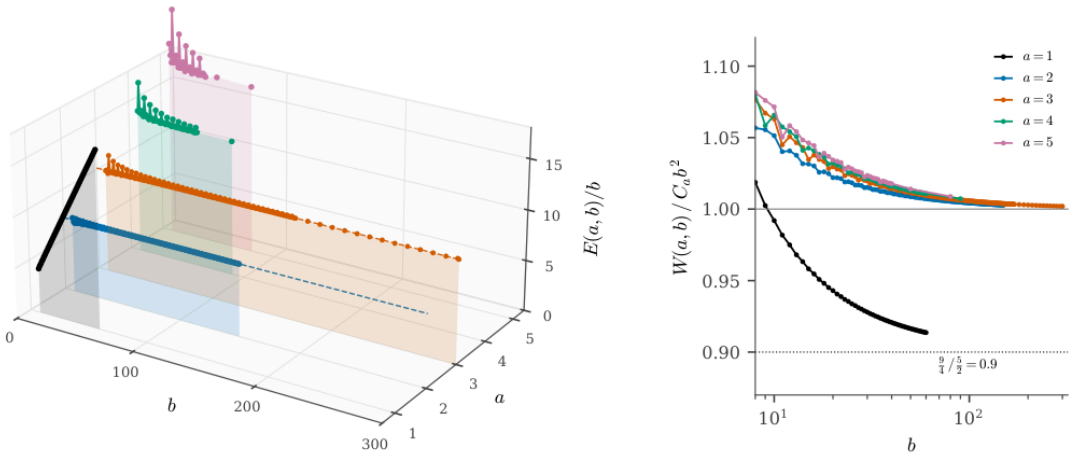
- **Program:** pairs of pieces of equal length, same signature, incompatible, dimension 2, dimension 1 for a = 3, 4, 5. working script `pares_rigidez.py`, not published
- **Audit:** pair-by-pair reproduction (a ≤ 4) and integer contrast. working script ?, not published

Figure 3



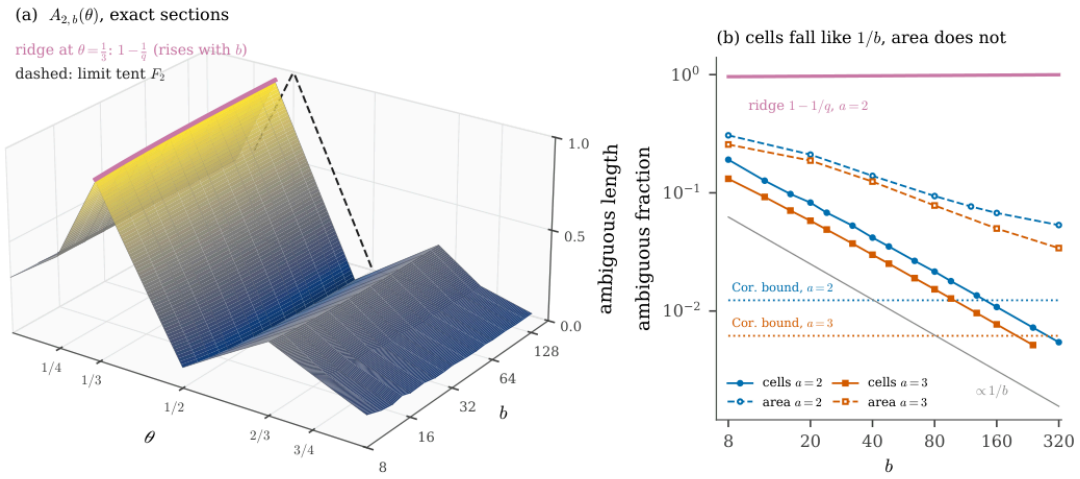
An equal-signature candidate with no integer solutions, from the portable $a = 2$ certificate. Two pieces share the all-positive triangular chamber and their observation vectors agree under $(b, n, m) \mapsto (b, n - \frac{1}{3}, m)$. Left (3D): the chamber cone at three values of b; blue dots are chamber lattice points and orange crosses their shifted images. Right (2D): the section $b = 24$ shows the two disjoint translates of the integer lattice. The drawing illustrates the integrality obstruction; points are selected by chamber inequalities only, without asserting piece-specific genericity.

Figure 4



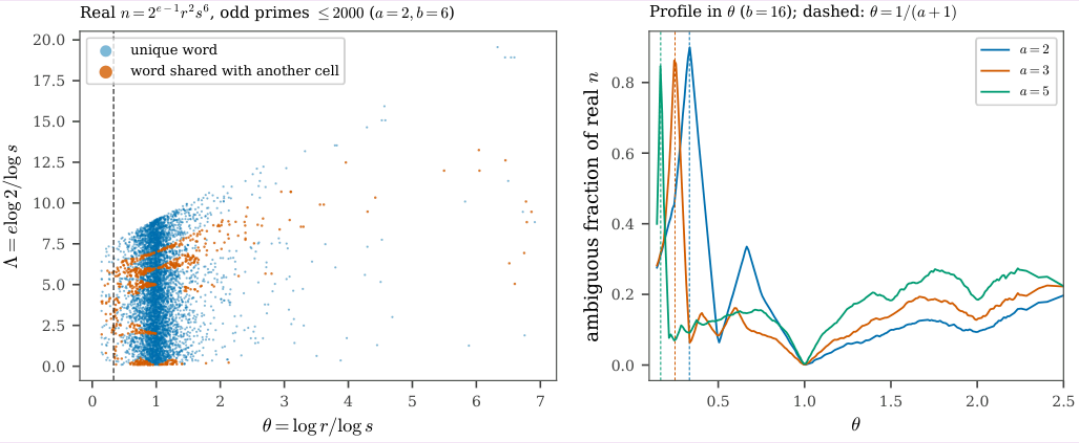
Rigidity in the exact data. Left: $E(a, b)/b$ over the (b, a)-plane, $8 \le b \le 300$, from exact enumeration of cells and words ($a = 1$: $b \le 60$; $a = 2$: $b \le 150$; $a = 3$: $b \le 170$ and $b = 180, 190, \ldots, 290, 299, 300$; $a = 4$: $b \le 60$ and $b = 90$; $a = 5$: $b \le 40$ and $b = 50, 80$; isolated values are drawn as dots). For $a = 1$ it grows like $b/4$, the quadratic exchange family of [I]; for $a = 2, \ldots, 5$ it stays bounded, as Theorem 6.1 asserts. The theorem gives boundedness only, not a single limit slope (each residue class of b may have its own slope η_a); the data suggest a single slope. Dashed: the slopes $\frac{43}{6}$ ($a = 2$, [I, Theorem 9.4]) and $\frac{61}{6}$ ($a = 3$, Conjecture 9.8). Right: $W(a, b)/(C_a b^2)$, which tends to 1 for $a = 2, \ldots, 5$ (Theorem 6.1) and to $\frac{9}{4} / \frac{5}{2} = 0.9$ for $a = 1$.

Figure 5



A non-shrinking tent-shaped lower bound for the ambiguity profile. (a) $A_{2,b}(\theta)$, the proportion of the line $\{\theta\} \times (0, b+2\theta)$ lying in cells whose word is shared with another cell, for $b = 8, \dots, 128$ (exact sections, $\theta = k/720$). The ridge at $\theta = \frac{1}{3}$ (reddish purple) has height $1 - \frac{1}{q}$, $q = 3b+2$, and rises with b; the dashed curve on the back wall is the limiting lower bound F_2 on the right (Proposition 6.3(iii)) and the computed profile on the left (where (iv) proves the weaker $1 - 12d$). The profile itself can be higher. Away from the ridge the surface does not vanish either: at $b = 128$ it is about 0.40 at $\theta = 0.2$ and 0.34 at $\theta = \frac{2}{3}$ (against 0.40 and 0.39 at $b = 8$); among the observed values it is near 0 only around $\theta = \frac{1}{2}$ (0.111, 0.030, 0.008 at $b = 8, 32, 128$). (b) The same arrangement counted two ways, $a = 2$ (circles) and $a = 3$ (squares): the proportion of ambiguous cells $\mathrm{Amb}\,/R$ (solid) falls like $1/b$; the mean $P_{a,b}(I)$ of the proportion of the line over $\theta \in I = [0.14, 6.9]$ (dashed) falls more slowly and stays above the one-sided lower bound $1/(2a(a+1)|I|)$ of Corollary 6.4(a) (dotted). Numerical values exact up to the sampling of θ; the limit of $P_{a,b}(I)$ is not known.

Figure 6



Integers and ambiguity. Left: 6000 integers $2^{e-1}r^2s^6$ with odd primes ≤ 2000 in the plane (θ, Λ); orange: the word is shared with another cell. Right: proportion of ambiguous integers as a function of θ ($b = 16$, $a = 2, 3, 5$); the dashed lines are $\theta = 1/(a+1)$. The sampled peaks at $1/(a+1)$ align with the cell-level ridge of Proposition 6.3 (their height is lower than $1 - 1/q$; the sample includes the trivial strip and a discrete set of e, which we did not analyse further); the minimum at $\theta = 1$ (cells meeting $\theta = 1$ have no partner for $a \le 5$, $b \le 16$, checked) and the second peak at $\theta \approx \frac{2}{3}$ for $a = 2$ are observed, not proved.

a	templ.	chambers	generic pieces	rank <2	signatures	same sign., c differs	integer-compatible	area >0
1	4	10	20	0	7	9	8	8
2	14	25	238	0	83	6	0	0
3	36	47	1296	0	401	15	0	0

4	76	76	4712	0	1320	42	0	0
5	136	112	12 920	0	3452	45	0	0

Colour: blue shade = the number's size within its column.

Table 6: Output of the certificate. The last three columns concern (C2): pairs of pieces with the same signature and different vector formulas; those with an integer point mapped to an integer point; those with positive area. Running times: 57 s for $a = 4$, 167 s for $a = 5$. $a = 1$, the positive control (a quadratic family is found); $a = 2, \dots, 5$, the computer-assisted cases.

a	b	$R(a,b)$	$\mathrm{Amb}(a,b)$	Amb/R	$b\,\mathrm{Amb}/R$
2	20	3143	260	0.0827	1.65
2	30	6913	381	0.0551	1.65
2	100	74370	1278	0.0172	1.72
3	20	6216	361	0.0581	1.16
3	30	13702	540	0.0394	1.18
3	80	94932	1451	0.0153	1.22
4	20	10359	496	0.0479	0.96
4	30	22860	734	0.0321	0.96
4	40	40246	975	0.0242	0.97
5	20	15440	631	0.0409	0.82
5	30	34095	903	0.0265	0.80

Table 3: Cells whose word is shared by another cell, from exact enumeration (all values of b in $2 \le b \le 100$ for $a = 2$, $3 \le b \le 80$ for $a = 3$, $4 \le b \le 40$ for $a = 4$, $5 \le b \le 30$ for $a = 5$; $E \le \mathrm{Amb} \le 2E$ holds in every one of the 240 cases). For $b \ge 2a$ the product $b\,\mathrm{Amb}/R$ stays between 1.14 and 1.73 ($a = 2$), 0.97 and 1.29 ($a = 3$), 0.90 and 1.12 ($a = 4$), 0.75 and 1.02 ($a = 5$). exact enumeration; the two shades separate the values of a.

Colour: the paper's own colours (see the caption).

a	b	$R(a,b)$	cells, Amb/R	integer sample	section measure
2	20	3143	0.083	0.082	0.203
2	40	12150	0.042	0.071	0.136
3	20	6216	0.058	0.052	0.182
3	40	24109	0.030	0.032	0.121
5	20	15440	0.041	0.080	0.168
5	30	34095	0.026	0.059	0.141

Table 4: Proportion of ambiguous cells (the word is shared with another cell), counted three ways, rounded to three decimals (so 0.026 here is the 0.0265 of Table 3). *Integer sample*: the integers $2^{e-1} r^a s^b$ with odd primes $r \ne s \le 2000$ (about 10^7 per row), computed with exact integer words; a word is ambiguous if its cell has a fibre of at least two cells. *Section measure*: the measure $\mu_{a,b}$ of the text; the choice of θ-range is a design choice, not a canonical measure. Numerical, not proved. The shading only separates the values of a.

Colour: the paper's own colours (see the caption).

K	pairs	$\lvert L(a,b) \cap L(a',b') \rvert$	share of $L(K-1)$ (%)	one fine cell	no common cell found
12	1	28	43	15	1
16	1	33	48	17	0
18	1	47	38	27	2

20	1	56	36	32	4
24	3	68–92	26–35	133	30
28	1	94	30	60	6
30	3	91–113	27–33	152	54
32	1	101	32	68	1
36	6	83–151	24–43	336	133
40	3	130–189	21–31	255	84
42	3	139–169	25–30	230	95
44	1	169	26	110	15
45	1	188	24	90	53
48	6	166–212	23–29	618	231
total	32	3979	1127 words	2143	709

Table 5: Words shared by two distinct two-prime shapes with K divisors (all pairs, $12 \le K \le 48$; ranges over the pairs with the same K). *Share of* $L(K-1)$: $K/|L(a,b) \cap L(a',b')|$, the part explained by the common coarsening. Of the remaining words, *one fine cell*: one cell of the arrangement of the finest shape with K divisors meets both embedded shapes; *no common cell found*: the word is not explained by a common fine cell under the embeddings tested, so the two shapes reach it, as far as tested, through different fine cells. Exact computation in rational arithmetic; a numerical observation, not a theorem. The shading only alternates the values of K.

Colour: the paper's own colours (see the caption).

7 The denominator $a+1$

Proposition 7.1. Isabelle p. 32

For every realisable c, the smallest element of $Z(c)$ is $\tau(c) = H_0(0) = \sum_{h=-a}^{0} f(c_h)$. If $c = g(n,m,\varepsilon)$ and $-1 \le n + hm + \varepsilon_h \le b$ for $-a \le h \le 0$, then

$$\tau(c) = (a+1)(n+1) - \frac{a(a+1)}{2}\, m + S_-(\varepsilon).$$

Consequently, if $c = g(n,m,\varepsilon)$ and $d = g(n',m,\varepsilon')$ both satisfy this condition and $Z(c) = Z(d)$, then

$$(a+1)(n'-n) = S_-(\varepsilon) - S_-(\varepsilon'),$$

and in particular $S_-(\varepsilon) \equiv S_-(\varepsilon') \pmod{a+1}$.

Isabelle · ours **show**

prop_tau_min · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy, line 435

```
theorem prop_tau_min:
  assumes mono: "\<And>h h'. -a \<le> h \<Longrightarrow> h \<le> h' \<Longrightarrow>
                 h' \<le> a \<Longrightarrow> c h \<le> c h'"
                and "0 \<le> a" "0 \<le> b"
  shows "Hval a b c 0 0 \<in># Zms a b c" and "\<forall>x \<in># Zms a b c. Hval a b c
                0 0 \<le> x"
              and "Hval a b c 0 0 = (\<Sum>h \<in> {-a..0}. fcl b (c h))"
```

prop_tau_formula · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy, line 470

```
theorem prop_tau_formula:
  assumes "0 \<le> a" "0 \<le> b" and cond: "\<And>h. -a \<le> h \<Longrightarrow> h \
  <le> 0 \<Longrightarrow> -1 \<le> n + h * m + e h \<and> n + h * m + e h \<le> b"
```

```
shows "2 * (\<Sum>h \<in> {-a..0}. fcl b (gvec b n m e h)) = 2 * (a + 1) * (n + 1) -
                                a * (a + 1) * m + 2 * Sminus a e"
```

prop_tau_formula_real · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy, line 491

```
                        corollary prop_tau_formula_real:
  assumes "0 \<le> a" "0 \<le> b" and "\<And>h. -a \<le> h \<Longrightarrow> h \<le> 0
      \<Longrightarrow> -1 \<le> n + h * m + e h \<and> n + h * m + e h \<le> b"
           shows "real_of_int (\<Sum>h \<in> {-a..0}. fcl b (gvec b n m e h))
        = (real_of_int a + 1) * (real_of_int n + 1) - real_of_int a * (real_of_int a
                              + 1) / 2 * real_of_int m
                                  + real_of_int (Sminus a e)"
```

prop_tau_cons · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy, line 511

```
                                theorem prop_tau_cons:
                              assumes "0 \<le> a" "0 \<le> b"
    and mc: "\<And>h h'. -a \<le> h \<Longrightarrow> h \<le> h' \<Longrightarrow> h'
           \<le> a \<Longrightarrow> gvec b n m e h \<le> gvec b n m e h'"
    and md: "\<And>h h'. -a \<le> h \<Longrightarrow> h \<le> h' \<Longrightarrow> h'
         \<le> a \<Longrightarrow> gvec b n' m e' h \<le> gvec b n' m e' h'"
     and cc: "\<And>h. -a \<le> h \<Longrightarrow> h \<le> 0 \<Longrightarrow> -1 \
               <le> n + h * m + e h \<and> n + h * m + e h \<le> b"
     and cd: "\<And>h. -a \<le> h \<Longrightarrow> h \<le> 0 \<Longrightarrow> -1 \
              <le> n' + h * m + e' h \<and> n' + h * m + e' h \<le> b"
               and Z: "Zms a b (gvec b n m e) = Zms a b (gvec b n' m e')"
                 shows "(a + 1) * (n' - n) = Sminus a e - Sminus a e'"
               and "(Sminus a e) mod (a + 1) = (Sminus a e') mod (a + 1)"
```

Trust base: 🔒 kernel only. Isabelle: 4 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** the whole statement

↳ Used in: Corollary 7.2 (p. 32), 7.3 (p. 33), Theorem 7.6 (p. 34), Corollary 7.7 (p. 36)

Proof

c is non-decreasing in h (floors of the increasing function $h \mapsto \Lambda + h\theta$, then clipped) and f is non-decreasing, so $H_i(j) = \sum_{l=0}^{a} f(c_{i-l} + j) \geq \sum_{l=0}^{a} f(c_{-l}) = H_0(0)$ for all $i, j \geq 0$. Under the condition, clipping leaves $c_h = n + hm + \varepsilon_h$ unchanged and $f(c_h) = c_h + 1$ for $-a \leq h \leq 0$; summing, and using $\sum_{h=-a}^{0} h = -a(a+1)/2$, gives the formula. Equal words have equal multisets, hence equal minima.

Evidence & scope

- **Program:** τ = H_0(0) = min Z(c) at 2000 random points of the all-positive chamber, a = 1..5: 2000/2000. working script `revision_mayor.py`, not published

Corollary 7.2. (computer-assisted) p. 32

In each of the 109 *candidates of Table 6 (*$a \leq 5$*) the shift found by the certificate is exactly* $\kappa = \big(S_-(\varepsilon) - S_-(\varepsilon')\big)/(a+1)$*, and* $S_-(\varepsilon) \not\equiv S_-(\varepsilon') \pmod{a+1}$*. So in every case the obstruction to integrality is this congruence. The residues of* $S_-(\varepsilon) - S_-(\varepsilon')$ *modulo* $a+1$ *that occur are all the nonzero ones, for each* a*.*

Trust base: computation (not a formal proof). Computer-assisted: the 109 [no_entera] lines of the certificate logs read and checked one by one (form n′ = n + κ, m′ = m; same chamber; κ formula; non-congruence: 109/109); control with denominator a: 0/109

Proof: in this note · **Covers:** not formalized

↳ Not used by another result of the paper.

Evidence & scope

- **Program:** block C: 109/109 on every item; residues 1, 1,2, 1,2,3, 1..4, 1..5 for a = 1..5. working script `revision_mayor.py`, not published

Remark 7.3. p. 33

Proposition 7.1 explains the denominator, not the phenomenon. Three facts would remove case (iv) of the reduction for every a (Question 10.1):

1. every candidate of (C2) lies in the all-positive chamber;

   every candidate keeps m;

   $S_-(\varepsilon) \not\equiv S_-(\varepsilon')$ whenever two templates there have the same signature and different vector formulas.

Proposition 7.1 proves none of them. Theorem 7.6 below proves the second for every $a \geq 2$, for the candidates inside the all-positive chamber, and Theorem 7.10 proves the third for every a. The first is checked for $a \leq 6$ (Conjecture 7.12).

↳ Not used by another result of the paper.

Proposition 7.4. Isabelle **(levels; written proof)** p. 33

Let c be non-decreasing with $0 \leq c_h \leq b-1$ for all h. Put $S = \sum_{h=-a}^{0} c_h$ and $w_{i,r} = \sum_{t=0}^{r-1}(t+1)(c_{i-a+t+1} - c_{i-a+t})$. Then

$$\mathcal{Q}G_c = (S+a+1)\mathcal{Q} + (a+1)z\mathcal{R}_1 + \sum_{r=1}^{a}\Big(\sum_{i=0}^{a} z^{w_{i,r}}\Big)B_r - \sum_{i=0}^{a} z^{\,T-\sum_{h\in W_i} c_h}\mathcal{R}_{a+1}.$$

If $c = g(n,m,\varepsilon)$ with no coordinate clipped, then

$$w_{i,r} = T_r m + \kappa_{i,r}(\varepsilon), \qquad T - \sum_{h\in W_i} c_h = (a+1)(b+1-n) - (a+1)\big(i-\tfrac{a}{2}\big)m - \sum_{h\in W_i}\varepsilon_h$$

Isabelle · ours **show**

prop_levels · Isabelle/CaballeroDyckWords/Caballero_Levels.thy, line 473

```
theorem prop_levels:
  "Q * Gc a b c z
= of_int ((\<Sum>h = - int a..0. c h) + int (a + 1)) * Q + of_nat (a + 1) * (zi z
    1 * R 1)
    + (\<Sum>r = 1..a. (\<Sum>i = 0..int a. zi z (wlev a c i r)) * Bcl z R r)
  - (\<Sum>i = 0..int a. zi z (int (a + 1) * (b + 1) - (\<Sum>h = i - int a..i. c
    h))) * R (a + 1)"
```

Trust base: 🔒 kernel only. Isabelle: 1 result, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** the whole statement

↳ Used in: Lemma 7.5 (p. 34), Theorem 7.6 (p. 34)

Proof

For $c_h \geq 0$ and $0 \leq j \leq b$, $f(c_h + j) = b + 1 - (e_h - j)_+$ with $e_h = b - c_h$. So the deficits of row i are $T - H_i(j) = \sum_{h\in W_i}(e_h - j)_+$.

Put $y = e_{i-a} - j$ and $q_r = c_{i-a+r} - c_{i-a}$ $(0 \leq r \leq a)$. Then $e_h - j = y - q_r$ for $h = i - a + r$, and as j runs over $[0,b]$, y runs over $[-c_{i-a}, Y]$ with $Y = b - c_{i-a}$. The deficit $\Psi(y) = \sum_{r=0}^{a}(y - q_r)_+$:

- vanishes for $y \leq 0$, which gives $c_{i-a} + 1$ values;
- has slope r on $[q_{r-1}, q_r]$;
- has slope $a+1$ on $[q_a, Y]$, where $q_a \leq Y$ because $c_i \leq b$.

On each segment the sum is a geometric progression: $\sum_{y=q_{r-1}+1}^{q_r} z^{\Psi(y)} = (z^{\Psi(q_{r-1})+r} - z^{\Psi(q_r)+r})/(1-z^r)$, also when $q_{r-1} = q_r$; the same holds on $[q_a, Y]$ with $r = a+1$.

Multiply by $\mathcal{Q}$ and collect the terms at each kink:

- $\Psi(0) = 0$ gives $z\mathcal{R}_1$;
- $\Psi(q_r) = \sum_{r'<r}(q_r - q_{r'}) = w_{i,r}$ gives $z^{w_{i,r}} B_r$;
- $\Psi(Y) + a + 1 = (a+1)(Y+1) - \sum_r q_r = T - \sum_{h \in W_i} c_h$ gives the last term.

The zero deficits of all rows give $(S + a + 1)\mathcal{Q}$. For $c = g(n, m, \varepsilon)$ the gaps are $c_{h+1} - c_h = m + \eta_h$ and $\sum_{h \in W_i} h = (a+1)(i - \frac{a}{2})$.

Evidence & scope

- **Program:** block K1: the identity against Z(c) computed directly, 300 random non-decreasing vectors per a for a ≤ 4 and 120 for a = 5, 6 (b ≤ 60), 0 failures; control with B_r replaced by −z^r R_r: differs in every case. working script `aud5_kappa.py`, not published

Lemma 7.5. (formal sums and rank two; written proof) p. 34

Let $a \geq 1$, let P be a generic piece of C_+ with template ε, and let F_ε be the right-hand side of Proposition 7.4 for $c = g(n, m, \varepsilon)$, read as a formal sum of terms κz^e whose exponents and coefficients are affine forms in $x = (b, n, m)$, with identical exponent forms merged and zero coefficients deleted.

1. *After the reflection $e \mapsto T + \deg \mathcal{Q} - e$ of the exponents (and the change of sign), the terms of F_ε and the entries of V_P correspond one to one as affine forms.*
2. *V_P has (n, m)-rank 2. So (C1) holds for every piece of C_+, for every $a \geq 1$.*

Trust base: ⚠ not checked. Written proof, not checked by a human referee, not machine-checked (it needs the framework of pieces and observation vectors, which is not formalized)

Proof: in this note · **Covers:** not formalized

↳ Used in: Theorem 7.6 (p. 34), Corollary 7.11 (p. 39)

Proof

(a) At an integer point of P that avoids finitely many planes (where two distinct exponent forms of F_ε agree or a coefficient form vanishes), the value of F_ε is the sparse form of $\mathcal{Q}G_c$, and reflected it is the list $V_P(x)$. So at each such point some bijection between the terms of F_ε and the entries of V_P matches exponents and coefficients in value. If a bijection pairs two different affine forms, it works only on the plane where they agree. There are finitely many bijections, and the points in question are not contained in finitely many planes (for b large they include a positive proportion of the integer points of $b\,C_P$). So some bijection pairs only identical forms.

(b) The cluster of F_ε at level $r = 1$ is $z^m P_1 B_1$ with $P_1 = \sum_i z^{\kappa_{i,1}} \neq 0$ and $B_1 \neq 0$, so F_ε has a term whose exponent has linear part m. The upper clusters are monomials times $-\mathcal{R}_{a+1} \neq 0$, so it has a term whose exponent has linear part $(a+1)(b-n) - (a+1)(i - \frac{a}{2})m$. By (a), V_P has two exponent entries with (n, m)-parts $(0, -1)$ and $\big(a+1, (a+1)(i - \frac{a}{2})\big)$, whose determinant is $a + 1 \neq 0$.

Theorem 7.6. (pairs in the all-positive chamber; written proof) p. 34

Let $a \geq 2$. Let P, P' be generic pieces of C_+ with templates $\varepsilon, \varepsilon'$ and the same signature, and let ϕ be the map of (C2). If $c_{P'} \circ \phi \neq c_P$, then

$$m' = m, \qquad n' = n + \nu, \qquad (a+1)\,\nu = S_-(\varepsilon) - S_-(\varepsilon'),$$

and the templates satisfy

1. $\sum_{h\in W_i}(\varepsilon_h-\varepsilon'_h)=S_-(\varepsilon)-S_-(\varepsilon')$ *for* $0\le i\le a$;
2. *for* $1\le r\le a$, *the multisets* $\{\kappa_{i,r}(\varepsilon)\}_{0\le i\le a}$ *and* $\{\kappa_{i,r}(\varepsilon')\}_{0\le i\le a}$ *are equal.*

Conversely, two templates $\varepsilon\neq\varepsilon'$ *that satisfy (N2) and (N3) give pieces* $P=(C,\varepsilon)$, $P'=(C,\varepsilon')$ *of any chamber* $C\subseteq C_+$ *(for* $a\le 6$, $C=C_+$*) with the same signature, related by this* ϕ*, and with* $c_{P'}\circ\phi\neq c_P$.

Trust base: ⚠ not checked. Written proof, not checked by a human referee, not machine-checked; the converse was repaired after the referee report V387 (canonical forms commute with φ, no integer points used); checked: the pairs with (N2) and (N3) are exactly the certificate's candidates for a ≤ 6, and all templates for a ≤ 15

Proof: in this note · **Covers:** not formalized

🆕 **Classical and new:** the paper counts the all-positive-chamber analysis among its own results, as far as its searches could see (Section 1).

↳ Used in: Corollary 7.7 (p. 36), Proposition 7.8 (p. 36), Corollary 7.9 (p. 37), Corollary 7.11 (p. 39), 10.2 (p. 52)

Proof

Formal sums. With F_ε as in Lemma 7.5, part (a) of the lemma shows that $V_P=V_{P'}\circ\phi$ means $F_\varepsilon=F_{\varepsilon'}\circ\phi$ as formal sums.

Linear parts. Group the terms by the linear part of the exponent (the exponent form minus its constant), and call the sum of the terms with a given linear part its *cluster*. F_ε has exactly $2a+2$ nonzero clusters:

- linear part 0, with cluster $(S+a+1)\mathcal{Q}+(a+1)z\mathcal{R}_1$, where $S=(a+1)n-T_am+S_-(\varepsilon)$;
- linear part T_rm for $1\le r\le a$, with cluster $z^{T_rm}P_rB_r$, where $P_r=\sum_i z^{\kappa_{i,r}(\varepsilon)}$ has positive coefficients;
- linear part $\ell_i=(a+1)(b-n)-(a+1)(i-\frac{a}{2})m$ for $0\le i\le a$, with cluster a monomial times $-\mathcal{R}_{a+1}$.

Each cluster is nonzero because $\mathbb{Z}[z]$ is a domain. As points of the space of linear forms in (b,n,m), the first $a+1$ linear parts lie on the line $\mathbb{R}m$ through 0. The other $a+1$ lie on the parallel line $(a+1)(b-n)+\mathbb{R}m$, which misses 0.

Let $\mu_{\text{lin}},\nu_{\text{lin}}$ be the linear parts of $m'\circ\phi$ and $n'\circ\phi$ (not to be confused with the constant ν below). Then $\mu_{\text{lin}}\neq 0$: for fixed b, ϕ is a bijection of the (n,m)-plane, since both observation vectors are injective by Lemma 7.5(b) and have the same image. The linear parts of $F_{\varepsilon'}\circ\phi$ are 0, $T_r\mu_{\text{lin}}$ and $(a+1)(b-\nu_{\text{lin}})-(a+1)(i-\frac{a}{2})\mu_{\text{lin}}$. If two of them coincided, $F_{\varepsilon'}\circ\phi$ would have at most $2a+1$ nonzero clusters. So the two sets of linear parts are equal.

Each set lies on two parallel lines, each carrying $a+1\ge 3$ points, and no other line carries three of them. So the line through 0 is the same: $\mu_{\text{lin}}=tm$, and $\{T_1,\dots,T_a\}=\{tT_1,\dots,tT_a\}$ forces $t=1$. On the other line, $\nu_{\text{lin}}=n+sm$, and matching its $a+1$ points forces $s=0$. Hence $m'\circ\phi=m+\delta$ and $n'\circ\phi=n+\nu$ with constants δ,ν. (For $a=1$ each line carries only two points, and indeed the exchange has $m'=2b-2n-m$.)

Clusters. Equal linear parts carry equal clusters. At level r, cancelling $z^{T_rm}B_r$ in the domain $\mathbb{Z}[z^{1/N}]$ (N a common denominator) gives $P_r=z^{T_r\delta}P'_r$, with $P'_r=\sum_i z^{\kappa_{i,r}(\varepsilon')}$. For $r=1$ the exponents $\kappa_{i,1}=\eta_{i-a}$ lie in $\{0,1\}$ on both sides, so $\delta\in\{-1,0,1\}$.

Suppose $\delta=1$. Then $\{\kappa_{i,r}(\varepsilon)\}=\{\kappa_{i,r}(\varepsilon')+T_r\}$, and since $0\le\kappa\le T_r$, every $\kappa_{i,r}(\varepsilon)=T_r$ and every $\kappa_{i,r}(\varepsilon')=0$. At $r=a$ this means $\eta\equiv 1$ and $\eta'\equiv 0$ on $[-a,a-1]$, that is, $\varepsilon_h=h$ and $\varepsilon'_h=0$. The clusters at ℓ_i give $\sum_{W_i}c_h=\sum_{W_i}c'_h\circ\phi$, where $c_h=n+h(m+1)$ and $c'_h\circ\phi=n+\nu+h(m+1)$. So $\nu=0$ and $c'\circ\phi=c$, which is excluded. The case $\delta=-1$ is the same.

Hence $\delta=0$ and $P_r=P'_r$, which is (N3). The clusters at ℓ_i give $\sum_{W_i}(c_h-c'_h\circ\phi)=0$, that is, $\sum_{W_i}(\varepsilon_h-\varepsilon'_h)=(a+1)\nu$ for every i. The case $i=0$ is the value of ν, and all of them together are

(N2). The cluster at 0 then agrees, because its S is the case $i = 0$.

Converse. Under (N2) and (N3), with $\phi(b, n, m) = (b, n + \nu, m)$, the clusters of F_ε and $F_{\varepsilon'} \circ \phi$ agree, so $F_\varepsilon = F_{\varepsilon'} \circ \phi$ as formal sums. We compare the observation vectors as lists of affine forms, without evaluating them at points: since $\nu \notin \mathbb{Z}$ in general (Theorem 7.10), ϕ maps no integer point to an integer point, and no argument through integer points generic for both pieces is available. By Lemma 7.5(a), V_P is the list of the terms of F_ε (reflected), and $V_{P'}$ that of $F_{\varepsilon'}$, each put in canonical form by steps (a)–(c) of Section 5 with respect to the common chamber C. These steps commute with composition by ϕ: ϕ is an affine bijection that fixes b, so it preserves identity and distinctness of exponent forms (step (a)) and the identical vanishing of coefficient forms (step (b)), and the reflection $e \mapsto T + \deg \mathcal{Q} - e$ depends on b only. For step (c), let e_1, e_2 be two exponent forms of $F_{\varepsilon'}$. The linear part of ϕ is the identity, so $(e_1 - e_2) \circ \phi$ has the same homogeneous part as $e_1 - e_2$. If that part is 0, then $e_1 - e_2$ is a constant whose n-coefficient is 0, and $(e_1 - e_2) \circ \phi = e_1 - e_2$: the same constant decides the order. Otherwise the order is decided by the sign of the same homogeneous part on the interior of the same chamber C. So the canonical form of $F_{\varepsilon'} \circ \phi$ is the canonical form of $F_{\varepsilon'}$ composed with ϕ, that is, $V_{P'} \circ \phi$; and it is the canonical form of F_ε, that is, V_P. Hence $V_P = V_{P'} \circ \phi$ as affine maps, and $A_P = A_{P'}$: the signatures are equal. Moreover $c'_h \circ \phi - c_h = \nu + \varepsilon'_h - \varepsilon_h$; at $h = 0$ this is ν, so it vanishes for every h only if $\nu = 0$ and $\varepsilon' = \varepsilon$.

In plain words (commentary, not part of the paper)

In the chamber where every coordinate is free and positive, the level identity (Proposition 7.4) splits the observation into clusters: one at 0, one at $T_r m$ for each level $r \le a$, and $a + 1$ more that carry n. Two pieces with the same signature must match cluster by cluster. The levels lie on one line through the origin with $a + 1 \ge 3$ points, the upper clusters on a parallel line, and matching these configurations forces $m' = m$ and a constant shift of n equal to $(S_-(\varepsilon) - S_-(\varepsilon'))/(a + 1)$. What remains is a purely combinatorial condition on the two templates: equal window sums up to a constant (N2) and equal multisets of level constants (N3).

This is a written proof for *every* $a \ge 2$, not checked by a human referee and not machine-checked; it reproduces exactly the candidates of the certificate for $a \le 6$ (6, 15, 42, 45, 132). With Theorem 7.10 it rules out every two-parameter family inside this chamber (Corollary 7.11). Outside it, the question stays open.

Evidence & scope

- **Program:** blocks K2–K4: the sets of pairs with (N2) and (N3) equal those read from the certificate logs for a = 2..6 (6, 15, 42, 45, 132); controls: (N3) only for r = 1 or windows of length a give other counts. working script `aud5_kappa.py`, not published

Corollary 7.7. p. 36

In the situation of Theorem 7.6, the entries of V_P and $V_{P'}$ that come from level r coincide entry by entry. Each exponent is $(a + 1)b - T_r m + \kappa$, with the same constant κ in both, because the cluster $P_r B_r$ is the same. A single such entry already forces $m' = m$. The shift of n is that of Proposition 7.1.

↳ Not used by another result of the paper.

Proposition 7.8. (computer-assisted) p. 36

Let $2 \le a \le 6$.

1. *Every pair of generic pieces with the following four properties lies in C_+ on both sides:*
   - *observation vectors of the same length;*
   - *the same directions (the system $V_P(b, n, m) = V_{P'}(b, n', m')$ has rank 2 in (n, m, n', m'));*
   - *a consistent system (as affine equations in the five variables (b, n, m, n', m'): the augmented matrix has the same rank as the matrix);*

- *different vector formulas.*

Hence every candidate of (C2) lies in C_+.

In every candidate pair, a single row of the system implies $m' = m$. It is an exponent entry $(a+1)b - T_k m + \kappa$ with the same constant κ in both pieces.

For $a = 6$ the program finds:

- 224 *templates,* 155 *chambers and* 30 240 *generic pieces, none of rank* <2*;*
- 7707 *signatures and* 132 *candidate pairs;*
- *every candidate in C_+, with $m' = m$ and $n' = n - k/7$, $1 \le k \le 6$; none is integer.*

Trust base: computation (not a formal proof). Computer-assisted (filas_rigidez.py), a ≤ 6; for a = 6 a mode without the conditions of genericity, hence with no regression of the 30240 pieces: E(6,b) = O(b) is NOT claimed

Proof: in this note · **Covers:** not formalized

↳ Used in: Corollary 7.9 (p. 37), 7.12 (p. 39)

Evidence & scope

- **Program:** P7 and P8 OK for a = 2..6; a = 6: 224 templates, 155 chambers, 30240 generic pieces (rank < 2: 0), 7707 signatures, 132 candidates n′ = n − k/7; control: the row n − n′ is never certified. working script `filas_rigidez.py`, not published

Corollary 7.9. (computer-assisted) p. 37

For $a = 2, \dots, 6$ the candidate pairs of the certificate are exactly the unordered pairs of distinct templates that satisfy (N2) and (N3): 6, 15, 42, 45 *and* 132 *pairs. In each of them the linear part of ϕ is the identity on (n, m), $m' = m$, and $n' - n = (S_-(\varepsilon) - S_-(\varepsilon'))/(a+1)$.*

Trust base: computation (not a formal proof). Computer-assisted: an enumeration of templates that does not use the certificate, compared with the certificate's pairs for a = 2..6; without a certificate for a = 7..15

Proof: in this note · **Covers:** not formalized

↳ Not used by another result of the paper.

Evidence & scope

- **Program:** blocks K2–K4: the two sets of pairs agree; a = 7..15: 139, 238, 245, 572, 351, 970, 725, 942, 1149 pairs in C_+, none congruent. working script `aud5_kappa.py`, not published

Theorem 7.10. Isabelle **(written proof)** p. 37

Let $a \ge 1$, $L = a + 1$, and let $\varepsilon \ne \varepsilon'$ be templates whose window sums differ by a constant: $\sum_{h \in W_i} (\varepsilon_h - \varepsilon'_h) = D$ for $0 \le i \le a$. Then $L \nmid D$. In particular two distinct templates that satisfy (N2) have $S_-(\varepsilon) \not\equiv S_-(\varepsilon') \pmod{a+1}$, and a class of templates whose window sums agree up to a constant has at most $a + 1$ elements.

Isabelle · ours **show**

nonint · Isabelle/CaballeroDyckWords/Caballero_Nonint/Caballero_Nonint.thy, line 272

```
theorem nonint:
  assumes a: "1 \<le> a"
and e: "e \<in> tvec a" and e': "e' \<in> tvec a" and ne: "e \<noteq> e'"
                and ws: "const_wsum a e e' D"
  shows "\<not> (int a + 1) dvd D"
```

nonint_N2 · Isabelle/CaballeroDyckWords/Caballero_Nonint/Caballero_Nonint.thy, line 291

```
corollary nonint_N2:
assumes "1 \<le> a" "e \<in> tvec a" "e' \<in> tvec a" "e \<noteq> e'" "N2 a e e'"
    shows "Sminus a e mod (int a + 1) \<noteq> Sminus a e' mod (int a + 1)"
```

nonint_class_bound · Isabelle/CaballeroDyckWords/Caballero_Nonint/Caballero_Nonint.thy, line 301

```
corollary nonint_class_bound:
  assumes a: "1 \<le> a" and K: "K \<subseteq> tvec a"
  and cls: "\<And>e e'. e \<in> K \<Longrightarrow> e' \<in> K \<Longrightarrow> \
      <exists>D. const_wsum a e e' D"
  shows "finite K" and "card K \<le> a + 1"
```

Trust base: 🔒 kernel only. Isabelle: 3 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** the whole statement

🆕 **Classical and new:** the paper: «we did not find this statement in the literature on balanced and mechanical words that we searched (Lothaire, abelian complexity and balance of Sturmian words), and its two ingredients … are elementary».

↳ Used in: Corollary 7.11 (p. 39), 10.2 (p. 52)

Proof

Write $\varepsilon_h = \lfloor u + hv \rfloor$ and $\varepsilon'_h = \lfloor u' + hv' \rfloor$ with $u, v, u', v' \in (0,1)$, and suppose $D = kL$ with $k \in \mathbb{Z}$. Put $d_h = \varepsilon_h - \varepsilon'_h - k$ for $|h| \le a$. Every window of L consecutive indices in $[-a, a]$ has $\sum d_h = 0$, and the difference of two consecutive windows gives $d_{h+L} = d_h$ for $-a \le h \le -1$.

Suppose some $d_h \neq 0$. By periodicity some d_h with $-a \le h \le 0$ is nonzero, and since $\sum_{h=-a}^{0} d_h = 0$ the window $[-a, 0]$ contains an index p with $d_p > 0$ and an index q with $d_q < 0$. Every index of $[-a, -1]$ occurs again L places later; only 0 does not. So one of p, q, say p, is in $[-a, -1]$, and the $L - 1$ indices strictly between p and $p + L$ meet every residue class modulo L except that of p; one of them, q', has $q' \equiv q$, so $d_{q'} = d_q < 0$. (If $q \in [-a, -1]$ instead, exchange the signs.) This gives $h_1 < h_2 < h_3$ with d_{h_1}, d_{h_3} of one sign and d_{h_2} of the other.

Now $d_h = \lfloor u + hv \rfloor - \lfloor u' + k + hv' \rfloor$, and $\lfloor x \rfloor > \lfloor y \rfloor$ implies $x > y$. So the sign of d_h, when it is not 0, is the sign of $f(h) = (u - u' - k) + h(v - v')$. An affine function of h cannot take signs $+, -, +$ or $-, +, -$ at $h_1 < h_2 < h_3$. Hence $d \equiv 0$, that is, $\varepsilon_h - \varepsilon'_h = k$ for every h; at $h = 0$ both templates are $\lfloor u \rfloor = \lfloor u' \rfloor = 0$, so $k = 0$ and $\varepsilon = \varepsilon'$, a contradiction.

For (N2), $D = \sum_{h=-a}^{0} (\varepsilon_h - \varepsilon'_h) = S_-(\varepsilon) - S_-(\varepsilon')$. Having window sums that agree up to a constant is an equivalence relation, and inside one class the residues of S_- modulo $a + 1$ are pairwise distinct, so a class has at most $a + 1$ elements.

💡 **In plain words (commentary, not part of the paper)**

A template is the list $\varepsilon_h = \lfloor u + hv \rfloor$, $|h| \le a$: the floors of an affine function, a finite Sturmian word. If two different templates have sums over every window of $a + 1$ consecutive indices that differ by the same constant D, then D is not a multiple of $a + 1$.

Why: if $D = k(a + 1)$, the differences $d_h = \varepsilon_h - \varepsilon'_h - k$ have zero sum on every window, hence are periodic of period $a + 1$; a nonzero periodic pattern with zero sum changes sign at least twice in a row ($+, -, +$), but the sign of a difference of floors of two affine functions follows the sign of an affine function, which changes at most once. So the shift of n in Theorem 7.6 is never an integer and no integer point is matched. The whole statement is machine-checked in Isabelle/HOL (session Caballero_Nonint). For arbitrary $0/1$-gap sequences it fails from $a = 3$ on, so the floors are essential.

🛡️ **Evidence & scope**

- **Program:** a = 1..15: every class with equal window sums up to a constant has pairwise distinct S_ − mod a+1 (0 congruent pairs); the largest class has exactly a+1 elements for every a ≤ 15; control with all 0/1-gap sequences with ε₀ = 0 (not mechanical): 0, 2, 16, 132 congruent pairs for a = 2..5. working script `V383_comprobaciones.py`, not published

Corollary 7.11. p. 38

> *Let $a \geq 2$. Every piece of C_+ satisfies (C1), and every pair of pieces of C_+ with the same signature satisfies (C2): either $c_{P'} \circ \phi = c_P$, or $\phi(b, n, m) = (b, n + \nu, m)$ with $\nu \notin \mathbb{Z}$, which maps no integer point to an integer point.*

Trust base: ⚠️ not checked. Written proof, from Lemma 7.5, Theorem 7.6 (written, not machine-checked) and Theorem 7.10 (machine-checked in Isabelle/HOL)

Proof: in this note · **Covers:** not formalized

↳ Used in: 7.12 (p. 39), 10.1 (p. 52)

Proof

> Lemma 7.5(b), Theorem 7.6 and Theorem 7.10 with $(a+1)\nu = S_-(\varepsilon) - S_-(\varepsilon')$.

Conjecture 7.12. p. 38

Let $a \geq 2$. Every candidate pair of (C2) lies in C_+ (checked for $a \leq 6$, Proposition 7.8).

↳ Used in: 10.1 (p. 52)

8 Lemma D for $a = 3$

Theorem 8.1. (Lemma D; computer-assisted) p. 38

> *Let $a = 3$. There is a constant C such that, for every $b \geq 317$, a realisable vector with at least two free coordinates and depth $> C$ shares its word with no other realisable vector.*

Trust base: computation (not a formal proof). Computer-assisted: Lemmas C and S written proofs re-derived by an independent AI agent, not checked by a human referee; the programs lema_d.py and aud4_pares_asim.py not read by a human: 3624618 pairs solved exactly, 0 alarms; the constant C is not computed (the data suggest C ≈ 3); threshold b_0 = 317

Proof: in this note · **Covers:** not formalized

NEW **Classical and new:** the paper counts Theorem 8.1 among its own results, as far as its searches could see (Section 1).

↳ Used in: 8.2 (p. 40), 10.3 (p. 52)

Proof

> The program `lema_d.py` builds the families of Lemma C for $a = 3$ (83 556 strip families on 333 planes, 28 thin families, and 967 920 ray families in two passes), and for every pair (generic piece, family) that passes two necessary filters (equal length of the observation vectors; equal constant coefficients) it solves the linear system exactly and decides the hypothesis of Lemma S by an exact feasibility test on the recession cone. Of the 3 624 618 pairs solved, 1 398 370 are incompatible, 359 673 have b constant on S (all with $b \leq 165$), 1 543 823 satisfy Lemma S (bounded depth), 322 752 violate it only because $c_P = c_{\mathcal{F}}$ identically on S (the same vector, described by a strip that crosses another chamber), none violates it without integer points, and *none* is an alarm (unbounded depth, integer points, $c \neq d$). So for $b \geq b_0$ every coincidence of a deep c with an exceptional or thin d has bounded depth, uniformly over the finitely many pairs, and with the ordered generic check above this proves the theorem with $C = \max(C_0, \max_{P,\mathcal{F}} C_{P\mathcal{F}}) + \sqrt{53}$, where $\sqrt{53}$ is a uniform bound on the diameter of a cell with at least two free coordinates: if $c_h, c_{h'}$ $(h \neq h')$ are free, two points of the cell have $|\Delta\Lambda + h\Delta\theta| < 1$ and $|\Delta\Lambda + h'\Delta\theta| < 1$, so $|\Delta\theta| < 2/|h - h'| \leq 2$ and $|\Delta\Lambda| < 1 + 2a = 7$, and the diameter is at most $\sqrt{(2a+1)^2 + 4} = \sqrt{53}$. The thresholds computed by the program give $b_0 = 317$. By

source: forms $Ab + D$, 1; the margin of σ on the strip planes, 82 ($= b_1$); sign failures off the rays on the strips, 66.7 (with the wide rays; 580 with the basic rays alone, which is why the wide rays are added); all signs of a ray equal to their asymptotic signs, 317 (basic rays) and 270 (wide rays); pairs with b constant on S, $b \leq 165$. The maximum, 317, comes from the basic rays.

In plain words (commentary, not part of the paper)

For $a = 3$ and $b \geq 317$: if a cell has at least two free coordinates and reaches far (more than some constant C) from a fixed set of 25 lines, its word belongs to it alone. So the coincidences of such cells all live in bounded neighbourhoods of these lines, as the map of Figure 6 shows at $b = 40$.

How: one half (deep against generic) is an ordered check of all pairs of pieces; the other half (deep against thin or exceptional) covers the thin and exceptional vectors by finitely many affine families (Lemma C), and decides for each pair (piece, family) whether the depth can grow without bound, by an exact test on a recession cone (Lemma S): about 3.6×10^6 exact linear systems, no alarm. Status: *computer-assisted*; the two lemmas are written proofs re-derived by an independent AI agent; neither they nor the programs have been checked by a human referee. The constant C is not computed (the data suggest $C \approx 3$). The condition on free coordinates cannot be dropped (Remark 8.2).

Evidence & scope

- **Program:** families: 83556 strip families on 333 planes, 28 thin families, 967920 ray families; pairs solved 3624618: incompatible 1398370, b constant 359673 ($b \leq 165$), bounded depth 1543823, same vector 322752, alarms 0. working script `lema_d.py`, not published
- **Program:** ordered generic check: 7652 compatible pairs, 0 alarms (also 0 for a = 1, 2); with the chamber of P closed 3960 alarms (420 for a = 2): the check is not vacuous. working script `aud4_pares_asim.py`, not published
- **Program:** above b_0: an independent enumerator at b = 331 and 400 finds no vector with two free coordinates and depth > 3 in a coincidence (largest depth 3.0). working script `aud4_enum.py`, not published

Remark 8.2. p. 38

The restriction to at least two free coordinates is necessary. For $0<x<b$ and $s = 3b + 3 + x$, the one-prime pair of Proposition 9.1 is $O_2(s) = (x, b, b, b, b, b, b)$, a thin vector whose cell contains the half-line $\Lambda - 3\theta = x + \frac{1}{2}$, $\theta>b$, at distance $\min(x + \frac{1}{2}, b - x - \frac{1}{2})/\sqrt{10}$ from the parallel lines $\Lambda - 3\theta \in \{0, b\}$ and far from the others; its partner $O_1(s)$ lies at $\theta = \frac{1}{4}$, within $\frac{1}{4}$ of the line $\theta = 0$. Equal words checked directly for $b = 100, 300, 1000, 3001$ (depths about $15.8, 47.4, 158.1, 474.5$), with the control $x \mapsto x + 1$ giving different words.

↳ Not used by another result of the paper.

Evidence & scope

- **Program:** equal words for b = 100, 300, 1000, 3001 (depths about 15.8, 47.4, 158.1, 474.5); control x ↦ x+1 gives different words. working scripts `lema_d_fina_profunda.py`, `aud4_qD.py`, not published

Lemma 8.3. (Lemma C: covering; written proof) p. 39

There is a finite explicit set of families $\mathcal{F}$, each with integer parameters, an affine form giving b and an affine observation vector $V_{\mathcal{F}}$, such that for $b \geq b_1$ every exceptional or thin realisable vector d is the vector $c_{\mathcal{F}}$ of some family at a valid point (all recorded signs correct, all listed coefficients nonzero), with K_d given by $V_{\mathcal{F}}$ there. The families are of three kinds: strips (a plane $\ell(b, n, m) = t$, ℓ primitive with $(B, C) \neq (0, 0)$, with an integral parametrisation by (β, s), a template and a closed interval of $\sigma = s/\beta$), thin families ($((-b-1)^{h_0+a}, s, b^{a-h_0})$ with parameters (b, s)) and rays (a line inside a strip or thin family, with one integer parameter).

Trust base: ⚠ not checked. Written proof (sketch), re-derived by an independent AI agent, not checked by a human referee; coverage contrasted with exact enumeration up to b = 90

Proof: in this note · **Covers:** not formalized

↳ Used in: Theorem 8.1 (p. 39)

Proof

Sketch. An exceptional d has a representative (n, m, ε) in the box; $(n/b, m/b)$ lies in a closed chamber. On an edge, it satisfies the equation of that edge. Inside, a recorded form $f = g\ell + D$ has the wrong sign or a coefficient vanishes; since the homogeneous part has a strict sign on the interior, $0 < |\ell(b, n, m)| \le |D|/g$, so d lies on one of finitely many planes $\ell = t$; forms $Ab + D$ with $(B, C) = (0, 0)$ only fail for $b \le |D|/|A|$. This includes the chambers whose pieces have fewer than two free coordinates: if d has two, a clipping sign recorded there is wrong, and the same bound applies. All recorded forms and coefficients of all pieces (generic or not), and all edges, are taken. On a plane the deduction of the certificate is repeated in one variable $\sigma = s/\beta$, cutting the interval at every sign change; a form whose homogeneous part vanishes on the plane is constant there, so it is either right on the whole family or wrong on all of it (and the family is empty). A point that is still not valid lies on one of finitely many lines $X\beta + Ys = t'$ (a ray) or has β bounded. On a ray every form is $X\gamma + D$, with the sign of X for γ large. Thin vectors are treated in the same way with (b, s) and $\sigma = s/b \in [-\frac{5}{4}, \frac{5}{4}]$; this range contains $s = -b - 1$ and $s = b$, so the vectors with no free coordinate are among them.

Evidence & scope

- **Program:** coverage: at b = 60 and 90 every exceptional or thin vector (43872 and 77191) is covered with K of the formula equal to the direct K. working script `lema_d_cobertura.py`, not published

Lemma 8.4. (Lemma S: recession; written proof) p. 40

Let P be a generic piece and $\mathcal{F}$ a family, let S be the rational affine solution set of $V_P(b, n, m) = V_{\mathcal{F}}(x)$, $b = b_{\mathcal{F}}(x)$, and let $\Pi \subset S$ be the closed polyhedron cut out by the interval of σ (for strips), by $(n/b, m/b) \in C_P$ and by $b \ge 1$ (so every recession direction of Π has $b(v) \ge 0$). Let $f_e(b, n, m) \ge 0$ be the inequalities of the edges e of C_P and $g = \min_e f_e$. If no direction v of S satisfies $b(v) > 0$, the interval condition on σ, and $f_{e,\hom}(v) > 0$ for every e, then g is bounded above on Π.

Trust base: ⚠ not checked. Written proof (Minkowski–Weyl decomposition), re-derived by an independent AI agent, not checked by a human referee; not machine-checked

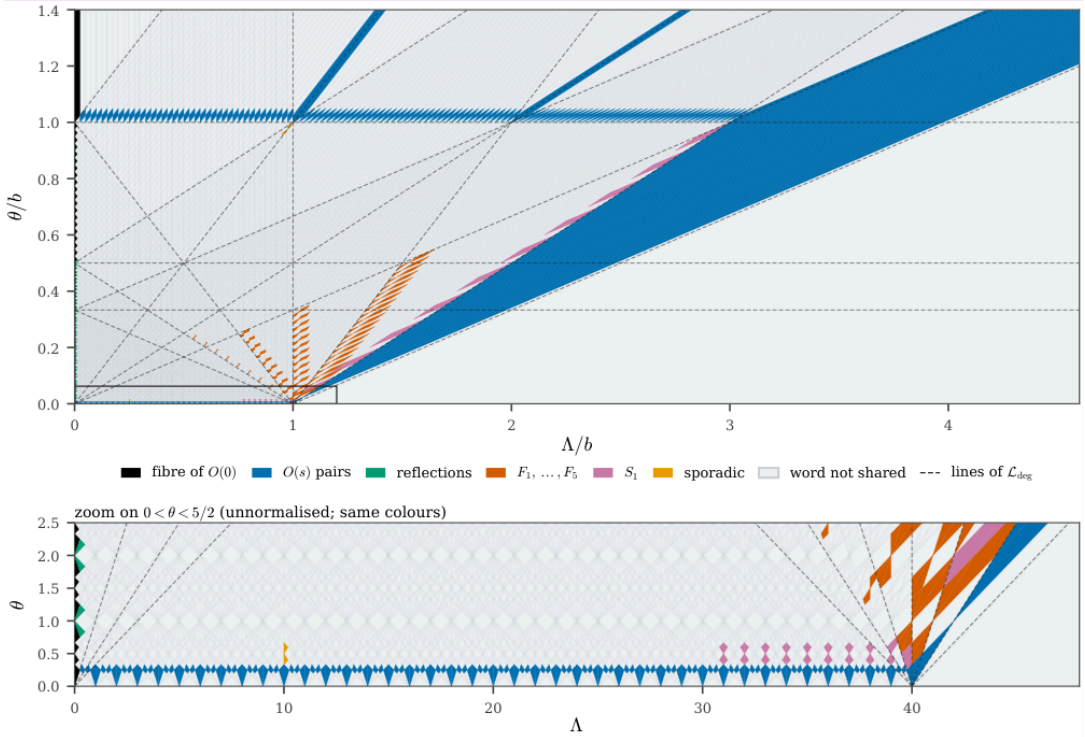
Proof: in this note · **Covers:** not formalized

↳ Used in: Theorem 8.1 (p. 39)

Proof

If $\Pi = \emptyset$ there is nothing to prove. Otherwise, by the decomposition theorem for polyhedra (Minkowski–Weyl in Motzkin's form, valid for every nonempty polyhedron, pointed or not; V need not consist of vertices; textbook material, e.g. [16 , Ch. 7]), $\Pi = \mathrm{conv}(V) + \mathrm{rec}(\Pi)$ with V finite. For $x \in \mathrm{conv}(V)$ and $r \in \mathrm{rec}(\Pi)$, $g(x + r) \le \max_e f_e(x) + \min_e f_{e,\hom}(r)$. Let $r \ne 0$ be a recession direction with $b(r) = 0$. For a strip, $b = q\beta + r_0$ gives $\beta(r) = 0$, and $lo \cdot \beta \le s \le hi \cdot \beta$ gives $s(r) = 0$; for a thin family the parameters are (b, s) with $lo \cdot b \le s \le hi \cdot b$, so again $s(r) = 0$; for a ray with b non-constant, $b(r) = 0$ forces $\gamma(r) = 0$ (rays with b constant are the pairs with b constant on S, which concern one value of b). So $V_{\mathcal{F}}$ is constant along r, and by the injectivity of $V_P(b, \cdot)$ (condition (C1)) $r = 0$, a contradiction. Since $b(r) \ge 0$ on $\mathrm{rec}(\Pi)$, $b(r) > 0$, and by hypothesis $\min_e f_{e,\hom}(r) \le 0$. Hence $g \le \max_{x \in \mathrm{conv}\, V} \max_e f_e(x)$. Equivalently, g is bounded on Π if and only if the linear program "maximise t subject to $t \le f_e$ for every e, on Π" is bounded; that program also gives the constant.

Figure 7



Where the coincidences live, $a = 3$, $b = 40$. Top: the exact cells in the window $0<\Lambda<4.6b$, $0<\theta<1.4b$ (23 134 of the 24 109 cells, in normalised coordinates). A cell is coloured when its word is shared (all 722 such cells meet the window), by the list of Section 9 that contains it, or as sporadic; dashed: the lines of $\mathcal{L}_{\mathrm{deg}}$. In this picture the coloured cells with at least two free coordinates lie along lines of $\mathcal{L}_{\mathrm{deg}}$ (Theorem 8.1 proves that they stay within a bounded distance of $\mathcal{L}_{\mathrm{deg}}$ for $b \geq 317$); the large blue wedges are cells of the thin vectors $O_2(s)$, which reach far from $\mathcal{L}_{\mathrm{deg}}$ (Remark 8.2). Bottom: the strip $0<\theta<\frac{5}{2}$, $0<\Lambda<48$ (the box in the top panel), not normalised: the partners $O_1(s)$ on $\theta = \frac{1}{4}$, the vectors $G_{m/5}(s)$ of S_1 on $\theta = \frac{2}{5}, \frac{3}{5}$, and the reflections and the fibre of $O(0)$ along $\Lambda = 0$. Fibres from the exact enumeration.

9 The case $a = 3$: the families of coincidences

Proposition 9.1. **($O(s)$ for $a = 3$)** p. 42

> *For $0 \leq s \leq 4b + 3$, $O_1(s)$ is the vector of $\left(\frac{2s+1}{8}, \frac{1}{4}\right)$, $O_2(s)$ that of $\left(s + \frac{1}{2}, b + 1\right)$, and $Z(O_1(s)) = Z(O_2(s)) = O(s)$. For $b \geq 2$, $O_1(s) \neq O_2(s)$ if and only if $s \leq 4b + 1$. For $s = 0$ both vectors lie in the fibre of $O(0)$.*

Isabelle · ours · part **show**

Only a part is in Isabelle: all but the last sentence (for s = 0 both vectors lie in the fibre of O(0))

prop_Os · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Twonum.thy, line 432

```
theorem prop_Os:
  assumes "1 \<le> b" "0 \<le> s" "s \<le> 4 * b + 3"
  shows "L1 3 s = real_of_int (2 * s + 1) / 8 \<and> T1 3 = 1 / 4 \<and> L2 s =
    real_of_int s + 1 / 2 \<and> T2 b = real_of_int (b + 1)"
      and "\<And>h. cpt b (L1 3 s) (T1 3) h = O1 b s h"
      and "\<And>h. cpt b (L2 s) (T2 b) h = O2 b s h"
  and "Zms 3 b (O1 b s) = Os_ms b s" and "Zms 3 b (O2 b s) = Os_ms b s"
and "2 \<le> b \<Longrightarrow> (\<exists>h. -3 \<le> h \<and> h \<le> 3 \<and> O1
    b s h \<noteq> O2 b s h) \<longleftrightarrow> s \<le> 4 * b + 1"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Isabelle: 1 result, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: all but the last sentence (for s = 0 both vectors lie in the fibre of O(0))

↳ Used in: 8.2 (p. 40), Theorem 9.7 (p. 49)

Proof

At $\theta=\frac{1}{4}$ the positions are $N/4$, $N=i+4j$, which runs once over $0,\dots,4b+3$. With $4\Lambda=s+\frac{1}{2}$ the values $\Lambda+h/4=(s+h+\frac{1}{2})/4$ are not integers and have floor $\lfloor(s+h)/4\rfloor$. By Lemma 2.2 the count at $N/4$ is $\#\{N'\le 4b+3: N'<N+s+\frac{1}{2}\}=\min(T,N+s+1)$, which takes the values $s+1,\dots,4b+3$ once and the value T for the $s+1$ largest N. At $\theta=b+1$ the positions $i(b+1)+j$ are again the integers $0,\dots,4b+3$, and with $\Lambda=s+\frac{1}{2}$ the counts are the same; the vector is $\mathrm{clip}(s+h(b+1))$.

Distinctness. If $s\le b$: $O_2(s)$ has $c_1=b$, while $O_1(s)$ has $c_1=\lfloor(s+1)/4\rfloor<b$. If $b<s\le 2b+1$: $c_0=b$ against $\lfloor s/4\rfloor<b$. If $2b+2\le s\le 3b+2$: $c_{-1}=\mathrm{clip}(s-b-1)=b$ against $\lfloor(s-1)/4\rfloor\le\lfloor(3b+1)/4\rfloor<b$ ($b\ge 2$). If $3b+3\le s\le 4b+3$: $O_2(s)=(s-3b-3,B^6)$, and $O_1(s)$ has $c_{-2}=\lfloor(s-2)/4\rfloor$, which is b iff $s\ge 4b+2$; for $s\in\{4b+2,4b+3\}$ also $c_{-3}=\lfloor(s-3)/4\rfloor=s-3b-3$, so $O_1(s)=O_2(s)$. Finally $O(0)$ is the word of the empty graph.

Evidence & scope

- **Program:** witnesses, words, $O_1\neq O_2$ iff $s\le 4b+1$, $b=3..60$ (7540 cases); control $O(s+1)$. working script `pruebas_a3.py`, not published

Theorem 9.2. (the families $F_1,\dots,F_5$) p. 42

In each line below, P and Q are distinct realisable vectors with $Z(P)=Z(Q)$, for all integers $i\ge 0$ and $\varepsilon\in\{0,1\}$ in the stated range.

1. $P=(x,y,z,B^4)$, $Q=(x+1,y,z-1,B^4)$; $x=b-3-3i$, $y=b-2-2i+\varepsilon$, $z=b-i$; $x\ge -1$.
2. $P=(x,y,z,w,B^3)$, $Q=(x+1,y,z,w-1,B^3)$; $x=b-4-4i$, $y=b-3-3i+\varepsilon$, $z=b-2-2i+\varepsilon$, $w=b-i$; $x\ge -1$.
3. $P=(x,y,B^5)$, $Q=(x+1,y-1,B^5)$; $x=b-6-2i$, $y=b-2-i$; $x\ge -1$.
4. $P=(x,y,z,w,B^3)$, $Q=(x+1,y+1,z-1,w-1,B^3)$; $x=b-4-4i$, $y=b-3-3i$, $z=b-1-2i$, $w=b-i$; $x\ge -1$.
5. $P=(x,y,z,w,v,B^2)$, $Q=(x,y+1,z,w-1,v,B^2)$; $x=b-7-5i$, $y=b-6-4i$, $z=b-4-3i$, $w=b-2-2i$, $v=b-1-i$; $y\ge -1$.

The numbers of pairs are $2\lfloor(b+1)/3\rfloor$, $2\lfloor(b+1)/4\rfloor$, $\lfloor(b-3)/2\rfloor$, $\lfloor(b+1)/4\rfloor$ and $\lfloor(b-1)/4\rfloor$.

Isabelle · ours · part **show**

Only a part is in Isabelle: the five families for every b, i and ε in the stated ranges (affine witnesses, distinct vectors, Z(P) = Z(Q) through an involution of the grid); the numbers of pairs are not formalized

thm_F1 · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Families.thy, line 173

```
theorem thm_F1:
  assumes i: "0 \<le> i" and x: "-1 \<le> b - 3 - 3 * i" and e: "e = 0 \<or> e = 1"
  shows "realis b (vF1 b i e 0)" "realis b (vF1 b i e 1)" "vF1 b i e 0 (-3) \<noteq>
    vF1 b i e 1 (-3)"
    "Zms 3 b (vF1 b i e 0) = Zms 3 b (vF1 b i e 1)"
```

thm_F2 · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Families.thy, line 280

```
theorem thm_F2:
  assumes i: "0 \<le> i" and x: "-1 \<le> b - 4 - 4 * i" and e: "e = 0 \<or> e = 1"
  shows "realis b (vF2 b i e 0)" "realis b (vF2 b i e 1)" "vF2 b i e 0 (-3) \<noteq>
```

```
vF2 b i e 1 (-3)"
"Zms 3 b (vF2 b i e 0) = Zms 3 b (vF2 b i e 1)"
```

thm_F3 · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Families.thy, line 383

```
theorem thm_F3:
  assumes i: "0 \<le> i" and x: "-1 \<le> b - 6 - 2 * i"
shows "realis b (vF3 b i 0)" "realis b (vF3 b i 1)" "vF3 b i 0 (-3) \<noteq> vF3 b i
1 (-3)"
"Zms 3 b (vF3 b i 0) = Zms 3 b (vF3 b i 1)"
```

thm_F4 · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Families.thy, line 465

```
theorem thm_F4:
  assumes i: "0 \<le> i" and x: "-1 \<le> b - 4 - 4 * i"
shows "realis b (vF4 b i 0)" "realis b (vF4 b i 1)" "vF4 b i 0 (-3) \<noteq> vF4 b i
1 (-3)"
"Zms 3 b (vF4 b i 0) = Zms 3 b (vF4 b i 1)"
```

thm_F5 · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Families.thy, line 543

```
theorem thm_F5:
  assumes i: "0 \<le> i" and x: "-1 \<le> b - 6 - 4 * i"
shows "realis b (vF5 b i 0)" "realis b (vF5 b i 1)" "vF5 b i 0 (-2) \<noteq> vF5 b i
1 (-2)"
"Zms 3 b (vF5 b i 0) = Zms 3 b (vF5 b i 1)"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Isabelle: 5 results, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the five families for every b, i and ε in the stated ranges (affine witnesses, distinct vectors, Z(P) = Z(Q) through an involution of the grid); the numbers of pairs are not formalized

🆕 **Classical and new:** the paper: «we know of no earlier list of coincidences of these words» (Section 1).

↳ Used in: 9.3 (p. 46), Theorem 9.7 (p. 49)

Proof

Realisability and distinctness. Table 6 gives a witness for each vector. In $F_1, \dots, F_4$ the vectors differ in c_{-3}, in F_5 in c_{-2}.

In $F_1, \dots, F_4$ all coordinates are ≥ -1 in the stated ranges. In F_5 all coordinates but x are ≥ -1; $x = y - 1 - i \geq (-b-3)/4 > -b-1$ is free, it enters only row 0, and on the interval $[3+2i, 5+4i]$ where row 0 changes, $x + j + 1 \geq b - 3 - 3i \geq i + 2 > 0$, so there $f(x+j) = x + j + 1$. For $v \geq -1$ and $j \geq 0$, $f(v+j) = \min(b+1, v+j+1)$, which equals $v + j + 1$ up to the *cut* $j = b - v$ and $b + 1$ after. In each case we move from P to Q by Lemma 2.1: a coordinate $c_h \geq -1$ raised by 1 changes the rows $0 \leq i - h \leq 3$ on $J = [0, b - c_h - 1]$, lowered by 1 on $J = [0, b - c_h]$. Below, "row r is $\alpha + \sigma j$ on $[j_1, j_2]$" refers to P, and $\pm$ is the net change of the row there.

F_3. Moves $c_{-3}\colon x \mapsto x + 1$ (row 0, $J = [0, 5 + 2i]$) and $c_{-2}\colon y \mapsto y - 1$ (rows 0, 1, $J = [0, 2 + i]$). Net: row 0 is $+1$ on $[3 + i, 5 + 2i]$, row 1 is -1 on $[0, 2 + i]$. Cuts: $b - y = 2 + i$, $b - x = 6 + 2i$. Row $0 = 2(b+1) + f(y+j) + f(x+j)$ is $4b - 2 - 2i + j$ on $[3 + i, 5 + 2i]$; row $1 = 3(b+1) + f(y+j)$ is $4b + 2 - i + j$ on $[0, 2 + i]$. The two contributions telescope to $\delta_T - \delta_{4b+1-i}$ and $\delta_{4b+1-i} - \delta_T$. This is the family E of [I, Supplement, Proposition 4.7] with one more row.

F_1. Moves $c_{-3}\colon x \mapsto x + 1$ (row 0, $J = [0, 2 + 3i]$) and $c_{-1}\colon z \mapsto z - 1$ (rows 0, 1, 2, $J = [0, i]$). Net: row 0 is $+1$ on $[i + 1, 2 + 3i]$; rows 1, 2 are -1 on $[0, i]$. Cuts: $b - z = i$, $b - y = 2 + 2i - \varepsilon$, $b - x = 3 + 3i$. On P:

$$\text{row } 0 = \begin{cases} 4b - 1 - 5i + \varepsilon + 2j, & [i + 1,\ 2 + 2i - \varepsilon], \\ 4b + 1 - 3i + j, & [3 + 2i - \varepsilon,\ 2 + 3i], \end{cases}$$

$$\text{row } 1 = 4b + 2 - 3i + \varepsilon + 2j, \qquad \text{row } 2 = 4b + 4 - i + j \qquad \text{on } [0, i].$$

The slope-1 pieces give $(\delta_T - \delta_{4b+4-i-\varepsilon}) + (\delta_{4b+3-i} - \delta_T)$. With $j = i + 1 + k$ the slope-2 piece of row 0 gives $\sum_{k=0}^{1+i-\varepsilon}(\delta_{4b+2-3i+\varepsilon+2k} - \delta_{4b+1-3i+\varepsilon+2k})$, and row 1 gives $\sum_{k=0}^{i}(\delta_{4b+1-3i+\varepsilon+2k} - \delta_{4b+2-3i+\varepsilon+2k})$. For $\varepsilon = 1$ all four cancel. For $\varepsilon = 0$ the slope-2 sums leave the term $k = i + 1$, $\delta_{4b+4-i} - \delta_{4b+3-i}$, which cancels the remainder $\delta_{4b+3-i} - \delta_{4b+4-i}$ of the slope-1 pieces.

F_2. Moves $c_{-3}\colon x \mapsto x + 1$ (row 0, $J = [0, 3 + 4i]$) and $c_0\colon w \mapsto w - 1$ (rows $0, \dots, 3$, $J = [0, i]$). Net: row 0 is $+1$ on $[i + 1, 3 + 4i]$; rows $1, 2, 3$ are -1 on $[0, i]$. Cuts: i, $2 + 2i - \varepsilon$, $3 + 3i - \varepsilon$, $4 + 4i$. On P:

$$\text{row } 0 = \begin{cases} 4b - 5 - 9i + 2\varepsilon + 3j, & [i + 1,\ 2 + 2i - \varepsilon], \\ 4b - 3 - 7i + \varepsilon + 2j, & [3 + 2i - \varepsilon,\ 3 + 3i - \varepsilon], \\ 4b - 4i + j, & [4 + 3i - \varepsilon,\ 3 + 4i], \end{cases}$$

and on $[0, i]$: row $1 = 4b - 1 - 6i + 2\varepsilon + 3j$, row $2 = 4b + 2 - 3i + \varepsilon + 2j$, row $3 = 4b + 4 - i + j$. Slope 1: row 0 gives $\delta_T - \delta_{4b+4-i-\varepsilon}$ and row 3 gives $\delta_{4b+3-i} - \delta_T$. Slope 3: with $j = i + 1 + k$, row 0 gives $\sum_{k=0}^{1+i-\varepsilon}(\delta_{4b-1-6i+2\varepsilon+3k} - \delta_{4b-2-6i+2\varepsilon+3k})$, and row 1 cancels its terms $k \le i$. Slope 2: with $j = 3 + 2i - \varepsilon + k$ ($0 \le k \le i$), row 0 gives $\sum_k(\delta_{4b+4-3i-\varepsilon+2k} - \delta_{4b+3-3i-\varepsilon+2k})$, and row 2 gives $\sum_{k=0}^{i}(\delta_{4b+1-3i+\varepsilon+2k} - \delta_{4b+2-3i+\varepsilon+2k})$. For $\varepsilon = 1$ everything cancels. For $\varepsilon = 0$ the two slope-2 sums telescope against each other to $(\delta_{4b+4-i} - \delta_{4b+3-i}) + (\delta_{4b+1-3i} - \delta_{4b+2-3i})$; the slope-3 remainder ($k = i + 1$) is $\delta_{4b+2-3i} - \delta_{4b+1-3i}$, and the slope-1 remainder is $\delta_{4b+3-i} - \delta_{4b+4-i}$. The total is 0.

F_4. Four moves: $x \mapsto x + 1$ (row 0, $J = [0, 3 + 4i]$), $y \mapsto y + 1$ (rows $0, 1$, $J = [0, 2 + 3i]$), $z \mapsto z - 1$ (rows $0, 1, 2$, $J = [0, 1 + 2i]$), $w \mapsto w - 1$ (rows $0, \dots, 3$, $J = [0, i]$). Net:

	$[0, i]$	$[i + 1, 1 + 2i]$	$[2 + 2i, 2 + 3i]$	$[3 + 3i, 3 + 4i]$
row 0	0	$+1$	$+2$	$+1$
row 1	-1	0	$+1$	0
row 2	-2	-1	0	0
row 3	-1	0	0	0

Colour: blue shade = the number's size within its column.

Cuts: i, $1 + 2i$, $3 + 3i$, $4 + 4i$. On the intervals with nonzero net change, P has: row $0 = 4b - 4 - 9i + 3j$, $4b - 3 - 7i + 2j$, $4b - 4i + j$ on the three intervals; row $1 = 4b - 6i + 3j$ on $[0, i]$ and $4b + 1 - 3i + j$ on $[2 + 2i, 2 + 3i]$; row $2 = 4b + 3 - 3i + 2j$ on $[0, i]$ and $4b + 3 - 2i + j$ on $[i + 1, 1 + 2i]$; row $3 = 4b + 4 - i + j$ on $[0, i]$. Slope 3: row 0 on $[i + 1, 1 + 2i]$ gives $\sum_{k=0}^{i}(\delta_{4b-6i+3k} - \delta_{4b-1-6i+3k})$, cancelled by row 1 on $[0, i]$. Slope 2, with the net change ± 2: row 0 gives $\sum_{k=0}^{i}(\delta_{4b+3-3i+2k} - \delta_{4b+1-3i+2k})$, cancelled by row 2 on $[0, i]$. Slope 1: row 0 on $[3 + 3i, 3 + 4i]$ and row 1 on $[2 + 2i, 2 + 3i]$ each give $\delta_T - \delta_{4b+3-i}$; row 2 on $[i + 1, 1 + 2i]$ and row 3 each give $\delta_{4b+3-i} - \delta_T$. The total is 0.

F_5. Moves $c_{-2}\colon y \mapsto y + 1$ (rows $0, 1$, $J = [0, 5 + 4i]$, as $y \ge -1$) and $c_0\colon w \mapsto w - 1$ (rows $0, \dots, 3$, $J = [0, 2 + 2i]$). Net: rows $0, 1$ are $+1$ on $[3 + 2i, 5 + 4i]$; rows $2, 3$ are -1 on $[0, 2 + 2i]$. Cuts: $b - v = 1 + i$, $b - w = 2 + 2i$, $b - z = 4 + 3i$, $b - y = 6 + 4i$. On P:

$$\begin{aligned} &\text{row } 0 = 4b - 13 - 12i + 3j\ [3 + 2i, 4 + 3i], && 4b - 9 - 9i + 2j\ [5 + 3i, 5 + 4i], \\ &\text{row } 1 = 4b - 6 - 7i + 2j\ [3 + 2i, 4 + 3i], && 4b - 2 - 4i + j\ [5 + 3i, 5 + 4i], \\ &\text{row } 2 = 4b - 3 - 6i + 3j\ [0, 1 + i], && 4b - 2 - 5i + 2j\ [2 + i, 2 + 2i], \\ &\text{row } 3 = 4b + 1 - 3i + 2j\ [0, 1 + i], && 4b + 2 - 2i + j\ [2 + i, 2 + 2i]. \end{aligned}$$

Reindex each piece from its first point ($k = 0, 1, \dots$). The slope-3 piece of row 0 takes the values $4b - 4 - 6i + 3k$ ($0 \le k \le 1 + i$) and raises each by 1; the slope-3 piece of row 2 takes $4b - 3 - 6i + 3k$ (same k) and lowers each by 1: the two contributions cancel term by

term. In the same way the slope-2 piece of row 0 on $[5+3i, 5+4i]$ (values $4b+1-3i+2k$, raised) cancels that of row 2 on $[2+i, 2+2i]$ (values $4b+2-3i+2k$, lowered), $0 \le k \le i$; and the slope-2 piece of row 1 on $[3+2i, 4+3i]$ (values $4b-3i+2k$, raised) cancels that of row 3 on $[0, 1+i]$ (values $4b+1-3i+2k$, lowered), $0 \le k \le 1+i$. The slope-1 pieces give $\delta_T - \delta_{4b+3-i}$ (row 1) and $\delta_{4b+3-i} - \delta_T$ (row 3).

Counts. The ranges are $0 \le i \le (b-2)/3$, $(b-3)/4$, $(b-5)/2$, $(b-3)/4$, $(b-5)/4$, which give the stated numbers (F_1 and F_2 have two values of ε).

In plain words (commentary, not part of the paper)

For $a = 3$ five explicit one-parameter families of pairs of cells $P \neq Q$ carry the same word, all next to the upper clipping (the last coordinates equal to b). In each family Q is obtained from P by moving two or four coordinates by ± 1; by Lemma M each move changes some rows by ± 1 on an interval, and in the multiset Z the changes telescope and cancel piece by piece (the proof is a written computation, family by family).

The numbers of pairs are $2\lfloor (b+1)/3 \rfloor$, $2\lfloor (b+1)/4 \rfloor$, $\lfloor (b-3)/2 \rfloor$, $\lfloor (b+1)/4 \rfloor$ and $\lfloor (b-1)/4 \rfloor$, so together they contribute to the slope $61/6$ of the conjectural $E(3, b)$. The paper knows of no earlier list of these coincidences. In Isabelle/HOL the five families are checked for every b, i and ε (witnesses, distinctness and $Z(P) = Z(Q)$); the counts are not.

Evidence & scope

- **Program:** b = 3..60: rows of P by pieces = formula (3800/3800), Q − P = net change of Lemma M (3800/3800), Z(P) = Z(Q) (3800/3800); controls: a piece shifted by 1 detected (19552/19552), one step outside the range gives different words (124/124). working script `pruebas_a3.py`, not published
- **Program:** the affine witnesses of Table 6: 7600/7600 vectors, b = 3..60; control θ + 1: 7600/7600 detected. working script `testigos_a3.py`, not published

Remark 9.3. p. 45

The five families live next to the upper clipping ($c_h = b$ for large h), at θ close to an integer plus $\frac{3}{2}$, $\frac{4}{3}$ or 4 (Table 6). Heuristically they are clipped variants of reflections around m/q: for $q \in \{2, 3\}$, [I, Lemma 7.7] needs $(p+1)q \le 3$, which fails for every $p \ge 1$, and for $q = 1$ the reflections exist ([I, Lemma 7.5], $p = 1, 2$) but not with these clipped vectors; the clipping truncates the levels and plays the role of the missing room. We have not made this precise. F_3 is the family E of $a = 2$ with the same formula, and F_1 is the analogue of the family D. At the first parameter i beyond the range, the first moving coordinate takes a value between -2 and -5 (depending on the family); there the two vectors are still realisable but the words differ, which recalls the condition $\beta' \ge -1$ of the exchange lemma [I, Lemma 7.2].

↳ Not used by another result of the paper.

Theorem 9.4. (the family S_1) p. 45

Let $m \in \{2, 3\}$, $\rho \in \{0, 1\}$ and $n \ge 1$ with $x = b - 4n + \rho \ge -1$, and put $s = 5b + 4 - 5n + \rho$ and $y = b + 1 - m$. Then $c = (x, y, B^5)$ and $G = G_{m/5}(s)$ are distinct realisable vectors and $Z(c) = Z(G)$; precisely, with $\sigma = 3b + 3 + x$ for $m = 2$ and $\sigma' = 3b + 2 + x$ for $m = 3$,

$$m = 2: \quad Z = \delta_\sigma + [\sigma + 2, 4b + 3] + \delta_{4b+3} + \sigma\, \delta_T,$$
$$m = 3: \quad Z = \delta_{\sigma'} + \delta_{\sigma'+2} + [\sigma' + 4, 4b + 3] + \delta_{4b+2} + \delta_{4b+3} + \sigma'\, \delta_T.$$

There are $2\big(\lfloor (b+1)/4 \rfloor + \lfloor (b+2)/4 \rfloor\big)$ such pairs. The pairs with $(n, \rho) = (1, 1)$ are the pairs of F_1 with $i = 0$ ($m = 2$ for $\varepsilon = 1$, $m = 3$ for $\varepsilon = 0$).

Isabelle · ours · part show

Only a part is in Isabelle: only the paragraph on witnesses: for m ∈ {2, 3}, ρ ∈ {0, 1} and n ≥ 1, c and G are realisable and differ in c_{−1}; the words Z(c), Z(G), their

equality and the count of pairs are not formalized

thm_S1_witnesses · Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Families.thy, line 578

```
theorem thm_S1_witnesses:
  assumes m: "m = 2 \<or> m = 3" and rho: "rho = 0 \<or> rho = 1" and n: "1 \<le> n"
                  and x: "-1 \<le> b - 4 * n + rho"
    shows "realis b (cS b n rho m)" "realis b (gS b (5 * b + 4 - 5 * n + rho) m)"
          "cS b n rho m (-1) \<noteq> gS b (5 * b + 4 - 5 * n + rho) m (-1)"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Isabelle: 1 result, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: only the paragraph on witnesses: for m ∈ {2, 3}, ρ ∈ {0, 1} and n ≥ 1, c and G are realisable and differ in c_{−1}; the words Z(c), Z(G), their equality and the count of pairs are not formalized

↳ Used in: Corollary 9.5 (p. 48), 9.6 (p. 49), Theorem 9.7 (p. 49)

Proof

Witnesses. For c take $\theta = y - x + \frac{1}{2}$ and $\Lambda = 3y - 2x + \frac{7}{4}$; then $\Lambda - 3\theta = x + \frac{1}{4}$, $\Lambda - 2\theta = y + \frac{3}{4}$ and $\Lambda - \theta = 2y - x + \frac{5}{4} \geq b + \frac{1}{4}$ (as $y \geq b - 2$ and $x \leq b - 3$), and every $\Lambda + h\theta$ has fractional part $\frac{1}{4}$ or $\frac{3}{4}$. For G take $(\Lambda, \theta) = (\frac{2s+1}{10}, \frac{m}{5})$: $\Lambda + h\theta = (s + mh + \frac{1}{2})/5$ is not an integer and has floor $\lfloor (s + mh)/5 \rfloor$. The vectors differ in c_{-1}: G has $c_{-1} = \lfloor (s - m)/5 \rfloor \leq b - n < b$.

The word of c. Rows $2, 3$ are constant, $= T$. For $m = 2$ ($y = b - 1$): $f(b - 1 + j)$ is b at $j = 0$ and $b + 1$ after, so row $1 = 3(b + 1) + f(b - 1 + j)$ is $4b + 3$ at $j = 0$ and T after, and row $0 = 2(b + 1) + f(b - 1 + j) + f(x + j)$ is $3b + 3 + x = \sigma$ at $j = 0$, $3b + 4 + x + j$ for $1 \leq j \leq b - x - 1$ (the values $\sigma + 2, \ldots, 4b + 3$), and T for the $x + 1$ values $j \geq b - x$. Hence $Z(c) = \delta_\sigma + [\sigma + 2, 4b + 3] + \delta_{4b+3} + (x + 1 + b + 2b + 2)\delta_T$, and $x + 3b + 3 = \sigma$. For $m = 3$ ($y = b - 2$): row 1 is $4b + 2$, $4b + 3$, then T; row 0 is $3b + 2 + x = \sigma'$, $3b + 4 + x$, then $3b + 4 + x + j$ for $2 \leq j \leq b - x - 1$ (the values $\sigma' + 4, \ldots, 4b + 3$), then T ($x + 1$ times); the number of T is $x + 1 + (b - 1) + 2(b + 1) = \sigma'$.

The word of G. At $\theta = m/5$ the positions are $N/5$ with $N = mi + 5j$; as mi runs over four distinct residues modulo 5, the multiset of positions is a set S_m:

$$S_2 = \{N \in [0, 5b + 6] : N \not\equiv 3\} \setminus \{1, 5b + 5\},$$
$$S_3 = \{N \in [0, 5b + 9] : N \not\equiv 2\} \setminus \{1, 4, 5b + 5, 5b + 8\}$$

(congruences modulo 5). With $5\Lambda = s + \frac{1}{2}$, Lemma 2.2 gives $Z(G) = \{\mathrm{cnt}(N + s) : N \in S_m\}$, where $\mathrm{cnt}(M) = \#\{N' \in S_m : N' \leq M\}$. Below the top, S_m meets every five consecutive integers in four elements; precisely

$$\mathrm{cnt}(M + 5) = \mathrm{cnt}(M) + 4 \qquad (\text{for } 1 \leq M\ (m = 2),\ 4 \leq M\ (m = 3),\ \text{and } M + 5 \leq 5b \cdot$$

while at the top $\mathrm{cnt}(5b + 5) = 4b + 3$ ($m = 2$), $\mathrm{cnt}(5b + 5) = 4b + 2$ and $\mathrm{cnt}(5b + 6) = \mathrm{cnt}(5b + 7) = \mathrm{cnt}(5b + 8) = 4b + 3$ ($m = 3$), and $\mathrm{cnt} = T$ from $5b + 6$ ($m = 2$), resp. $5b + 9$ ($m = 3$), on. For $M \leq 5b + 4$ one has $\mathrm{cnt}(M) = M - \lfloor (M + 2)/5 \rfloor$ ($m = 2$, $M \geq 1$) and $\mathrm{cnt}(M) = M - 1 - \lfloor (M + 3)/5 \rfloor$ ($m = 3$, $M \geq 4$). Note $s \geq 5b + 4 - \frac{5}{4}(b + 2) > 4$.

$m = 2$. Here $s \equiv 4 + \rho \pmod 5$ and $\sigma = 3b + 3 + x = 4b + 3 - 4n + \rho$; one checks $\mathrm{cnt}(s) = s - \lfloor (s + 2)/5 \rfloor = \sigma$ for $\rho = 0, 1$. Among $s + 1, \ldots, s + 4$ exactly one integer is $\equiv 3$ (namely $s + 4$ if $\rho = 0$, $s + 3$ if $\rho = 1$) and none is 1, so $\mathrm{cnt}(s + e) = \sigma + \kappa(e)$ for $e \in \{0, 1, 2, 4\}$, with $\kappa(0, 1, 2, 4) = (0, 1, 2, 3)$. Write $N \in S_2$ as $N = 5t + e$, $e \in \{0, 1, 2, 4\}$ (and $N \neq 1$). If $N + s \leq 5b + 4$, then $\mathrm{cnt}(N + s) = \sigma + 4t + \kappa(e)$. This condition reads $N \leq 5n - \rho$. Here $t = 0$ contributes $\sigma, \sigma + 2, \sigma + 3$ ($N = 0, 2, 4$) and each $t \geq 1$ the block $\sigma + 4t + \{0, 1, 2, 3\}$, so the elements with $t \leq n - 1$ give

$\{\sigma\} \cup [\sigma+2, \sigma+4n-1]$, each once. For $\rho=0$, $\sigma+4n-1 = 4b+2$, and the condition also admits $N = 5n$, which gives $\sigma+4n = 4b+3$. For $\rho=1$, $\sigma+4n-1 = 4b+3$ and nothing else is admitted. In both cases these N give $\{\sigma\} \cup [\sigma+2, 4b+3]$, each once. The next element of S_2, $N = 5n+1-\rho$, has $N+s = 5b+5$ and gives $4b+3$. All larger N give T, and there are $4b+4-\mathrm{cnt}(5n+1-\rho) = 4b+4-(4n+1-\rho) = \sigma$ of them. This is the stated word.

$m=3$. Here $\sigma' = 3b+2+x = 4b+2-4n+\rho$ and $\mathrm{cnt}(s) = s-1-\lfloor (s+3)/5 \rfloor = \sigma'$. Among $s+1, \ldots, s+4$ exactly one integer is $\equiv 2$ ($s+3$ if $\rho=0$, $s+2$ if $\rho=1$), so $\mathrm{cnt}(s+e) = \sigma' + \kappa(e)$ for $e \in \{0,1,3,4\}$, $\kappa(0,1,3,4) = (0,1,2,3)$. Write $N \in S_3$ as $N = 5t+e$ with $e \in \{0,1,3,4\}$; for $t=0$ only $N = 0, 3$ occur, with values σ', $\sigma'+2$. If $N+s \le 5b+4$, i.e. $N \le 5n-\rho$, then $\mathrm{cnt}(N+s) = \sigma' + 4t + \kappa(e)$, and the elements with $t \le n-1$ give $\{\sigma', \sigma'+2\} \cup [\sigma'+4, \sigma'+4n-1]$, each once. We separate the two values of ρ.

- [leftmargin=*]
- $\rho=0$: $\sigma' = 4b+2-4n$, so $\sigma'+4n-1 = 4b+1$; the condition also admits $N = 5n$, which gives $\sigma'+4n = 4b+2$. The part below $5b+5$ is $\{\sigma', \sigma'+2\} \cup [\sigma'+4, 4b+2]$. Above it, $N = 5n+1, 5n+3, 5n+4$ have $N+s = 5b+5, 5b+7, 5b+8$ and give $4b+2, 4b+3, 4b+3$.
- $\rho=1$: $\sigma' = 4b+3-4n$, so $\sigma'+4n-1 = 4b+2$, reached by $N = 5n-1$ (for $n=1$ the interval $[\sigma'+4, 4b+2]$ is empty and the part below $5b+5$ is $\{4b-1, 4b+1\}$). Nothing else is admitted, and the part below $5b+5$ is again $\{\sigma', \sigma'+2\} \cup [\sigma'+4, 4b+2]$. Above it, $N = 5n, 5n+1, 5n+3$ have $N+s = 5b+5, 5b+6, 5b+8$ and give $4b+2, 4b+3, 4b+3$.

In both cases all larger N give T, and there are $4b+4-(4n+2-\rho) = \sigma'$ of them. The values below T are therefore $\{\sigma', \sigma'+2\} \cup [\sigma'+4, 4b+2] \cup \{4b+2, 4b+3, 4b+3\} = \{\sigma', \sigma'+2\} \cup [\sigma'+4, 4b+3] \cup \{4b+2, 4b+3\}$, which is the stated word.

Count and overlap. For each m, $\rho=0$ gives $\lfloor (b+1)/4 \rfloor$ values of n and $\rho=1$ gives $\lfloor (b+2)/4 \rfloor$. For $(n,\rho) = (1,1)$, $c = (b-3, b+1-m, B^5)$ and $G_{m/5}(5b)$ is $(b-2, b-1, b-1, B^4)$ for $m=2$ and $(b-2, b-2, b-1, B^4)$ for $m=3$, which are the pairs of F_1 with $i=0$.

Evidence & scope

- **Program:** b = 3..60: Z(c) = formula (1828/1828), Z(G) = formula (1828/1828), equal words, witnesses (1828/1828), number of pairs (58/58); controls: ρ = 2 and G_(5−m)/5 give different words. working script `pruebas_a3.py`, not published
- **Program:** count = generator for b = 5..300 (296/296) and = pairs of S_1 in the exact fibres. working script `revision_mayor.py`, not published

Corollary 9.5. (a fibre containing at least four cells) p. 46

For $b \ge 5$ the four distinct vectors

$$(b-4, b-1, B^5), \quad (b-3, b-2, B^5), \quad (b-2, b-2, b-1, B^4), \quad (b-2, b-1, b-1, b-$$

have the same word. Among the pairs of Theorem 9.4, a pair with $m=2$ and a pair with $m=3$ have the same word only for $(n,\rho) = (1,0)$ and $(1,1)$ respectively, which are these four vectors.

Isabelle · ours · part **show**

Only a part is in Isabelle: the first sentence: the four vectors have the same multiset Z(c) (the equivalence of Z(c) with the word is taken from [I]); the second sentence is not formalized

`cor_fibre4` · `Isabelle/CaballeroDyckWords/Caballero_Fibre4/Caballero_Fibre4.thy`, line

192

```
theorem cor_fibre4:
  assumes b: "5 \<le> b"
shows "v1 b \<noteq> v2 b" "v1 b \<noteq> v3 b" "v1 b \<noteq> v4 b" "v2 b \<noteq>
    v3 b" "v2 b \<noteq> v4 b" "v3 b \<noteq> v4 b"
  and "Zms b (v1 b) = Zfibre b" "Zms b (v2 b) = Zfibre b" "Zms b (v3 b) = Zfibre b"
    "Zms b (v4 b) = Zfibre b"
```

Trust base: kernel + code evaluation — **only part of the statement is formal** (see the box). Isabelle, oracles: cor_fibre4 (Code_Generator.holds_by_evaluation)

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the first sentence: the four vectors have the same multiset Z(c) (the equivalence of Z(c) with the word is taken from [I]); the second sentence is not formalized

↳ Used in: Theorem 9.7 (p. 49), 10.7 (p. 53)

Proof

The first vector is c for $(n,\rho,m)=(1,0,2)$, with $\sigma=3b+3+x=4b-1$; its partner is $G_{2/5}(5b-1)=(b-2,b-1,b-1,b-1,B^3)$. The second is c for $(n,\rho,m)=(1,1,3)$, with $\sigma'=3b+2+x=4b-1$; its partner is $G_{3/5}(5b)=(b-2,b-2,b-1,B^4)$. With $\sigma=\sigma'=4b-1$ both formulas of Theorem 9.4 read $\delta_{4b-1}+\delta_{4b+1}+\delta_{4b+2}+2\delta_{4b+3}+(4b-1)\delta_T$. Conversely, the two formulas have the same total weight on T only if $\sigma=\sigma'$, and then they differ by $\delta_{\sigma+3}-\delta_{4b+2}$, which vanishes only for $\sigma=4b-1$, that is, $4n-\rho=4$ $(m=2)$ and $4n-\rho=3$ $(m=3)$.

Evidence & scope

- **Program:** block T4: b = 5..80, the four vectors distinct with equal Z; control: among the 43491 pairs (m = 2 pair, m = 3 pair) exactly 76 (one per b, the stated one) have the same word. working script `V383_comprobaciones.py`, not published
- **Program:** the lab «Figure 2 live» finds this fibre for a = 3 and b ≥ 5 among all the cells it enumerates. `labjs/nucleo.js`

Remark 9.6. p. 47

For $a=2$ the analogous family F of [I] pairs $(x,b-1,b,b,b)$ with the exact point $\theta=\frac{2}{3}$, of denominator $3=a+1$. For $a=3$ the denominator is 5, not $a+1$; this is consistent with the factor 5 in the period of $R(3,b)$, but we have not proved a link.

↳ Not used by another result of the paper.

Theorem 9.7. (the listed classes for $a=3$) p. 47

Let $b\geq 5$. Each of the following is a set of realisable vectors with a single word: the fibre of $O(0)$; the pairs $\{O_1(s),O_2(s)\}$, $1\leq s\leq 4b+1$; the reflection pairs with $p=1,2$; the pairs of $F_1,\dots,F_5$; the pairs of S_1. Let $\mathcal{G}_b$ be the graph whose vertices are the vectors of these sets, $\mathcal{U}_b$, with an edge between two vectors whenever they lie in a common listed set. Then

$$E(3,b)\ \geq\ |\mathcal{U}_b|-|\pi_0(\mathcal{G}_b)|,$$

where $\pi_0(\mathcal{G}_b)$ is the set of connected components of $\mathcal{G}_b$. The same holds for the graph $\mathcal{G}_b^{+}$ obtained by adding the set of Corollary 9.5.

Isabelle · ours · part **show**

Only a part is in Isabelle: the counting part: E ≥ |𝒰| − |π₀(𝒢_b)| once each listed set has one word; that they do is Proposition 9.1, Theorems 9.2 and 9.4 and Corollary 9.5

`thm_a3_families` · `Isabelle/CaballeroDyckWords/Caballero_Setting/Caballero_Setting.thy`, line 851

```
theorem thm_a3_families:
  assumes fin: "finite X" and sub: "\<And>S. S \<in> L \<Longrightarrow> S \<subseteq> X"
    and one: "\<And>S x y. S \<in> L \<Longrightarrow> x \<in> S \<Longrightarrow> y \<in> S \<Longrightarrow> w x = w y"
  shows "int (card (\<Union>L)) - int (card (Comps L)) \<le> Ecount X w"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Isabelle: 1 result, no oracles

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the counting part: $E \geq |\mathcal{U}| - |\pi_0(\mathcal{G}_b)|$ once each listed set has one word; that they do is Proposition 9.1, Theorems 9.2 and 9.4 and Corollary 9.5

↳ Used in: 9.8 (p. 50)

Proof

> By [I, Lemmas 7.1 and 7.5], Proposition 9.1, Theorems 9.2 and 9.4 and Corollary 9.5, each listed set has one word, so vectors in one connected component have one word, and $E(3,b)$ counts, for each word, its number of vectors minus one.

Evidence & scope

- **Program:** b = 5..300: sum of the lists = closed form, exactly two overlaps (S_1 with (n,ρ) = (1,1) = F_1 with i = 0), one word per component. working script `estructura_a3.py`, not published

Conjecture 9.8. (structural formula; checked) p. 47

For $b \geq 5$, $E(3,b) = E_{\text{floor}}(b) + g^*_{b \bmod 4}$ with $(g^*_0, g^*_1, g^*_2, g^*_3) = (3,5,3,18)$, that is,

$$E(3,b) = 9b + 2\left\lfloor \tfrac{b+1}{3} \right\rfloor + \left\lfloor \tfrac{b-1}{2} \right\rfloor + h_{b \bmod 4},$$

$(h_0, h_1, h_2, h_3) = (0, 1, -1, 17)$, and $W(3,b) = R(3,b) - E(3,b)$ with $R(3,b)$ of [I, Theorem 5.1]. The slope of $E(3,b)$ is $\frac{61}{6}$.

↳ Not used by another result of the paper.

In plain words (commentary, not part of the paper)

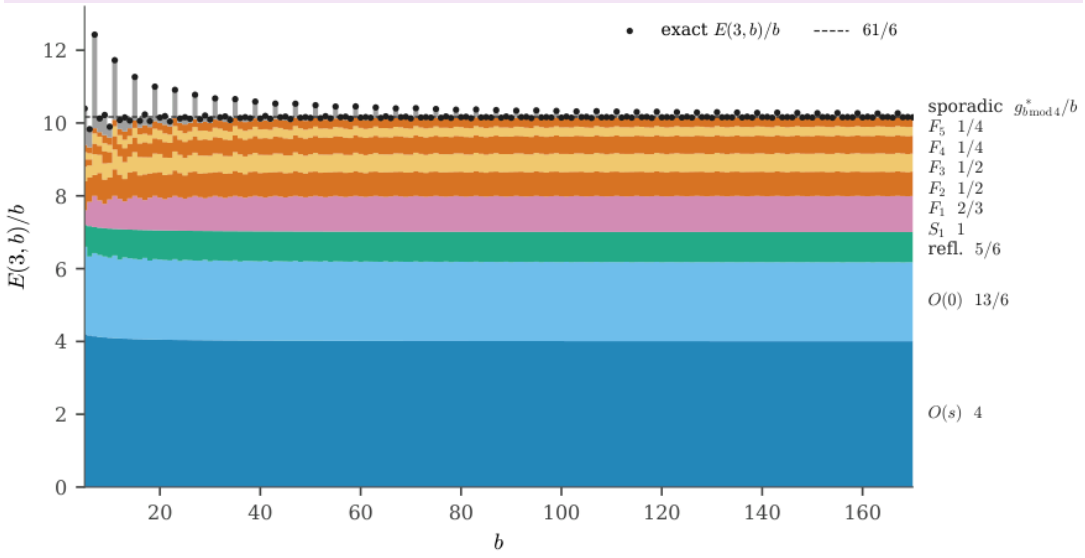
If the listed families (the fibre of $O(0)$, the one-prime pairs, two reflections, $F_1, \ldots, F_5$ and S_1) are all the coincidences up to a constant number of sporadic ones, then $E(3,b)$ has the explicit formula above, of slope $61/6$ and period 12, and $W(3,b) = R(3,b) - E(3,b)$ has period 60: the "12 plus 5" of the fit in paper I, 12 from the families and 5 from the cells.

Status: a *conjecture*. Each listed pair is proved to share its word; the constant g^* is fitted, not derived; and completeness is not proved. The formula agrees with exact enumeration for $5 \leq b \leq 170$, for $b = 180, 190, \ldots, 290, 299, 300$ and for $b = 331, 400$. The linear coefficient $\frac{253}{30}$ of $W(3,b)$ follows only if the conjecture holds; the quadratic one, $\frac{73}{5}$, is Theorem 6.1. The lab walks $b \mapsto b+1$ along the $(12,5)$ torus knot of Figure 8.

Evidence & scope

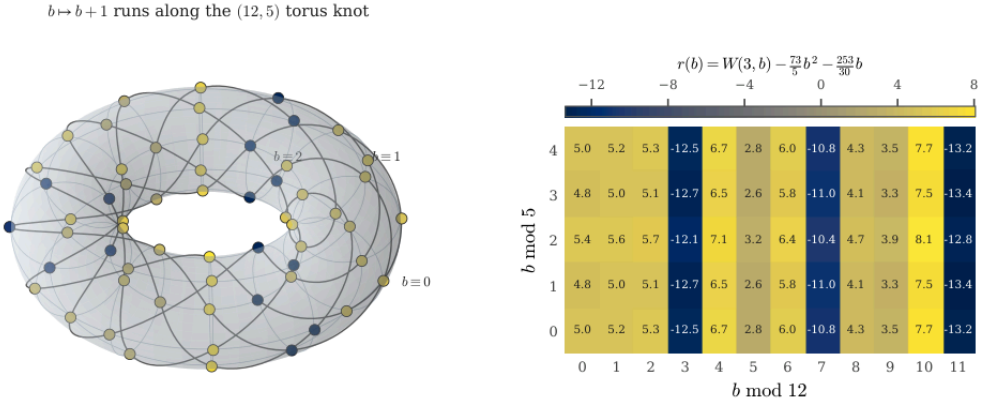
- **Program:** exact enumeration of all cells and words: equality for 5 ≤ b ≤ 170 and b ∈ 180, 190, …, 290, 299, 300; b = 331 and 400 (E = 3381, 4065) with the independent enumerator of the audit of Lemma D; fails for b ≤ 4. working scripts `enum_a3.py`, `aud4_enum.py`, not published
- **Program:** the lab «Torus live»: W(3,b) = R(3,b) − E(3,b) by the formula equals the exact data at all 180 enumerated values of b ≥ 5; the remainder r(b) is periodic mod 60 and not mod 12, 15, 20, 30. `labjs/labs.js`, `lab/hacer_datos.py`

Figure 8



The listed classes account for the conjectured slope of $E(3,b)$. For $5 \le b \le 170$, the contribution of each list of Theorem 9.7 to $E(3,b)$, divided by b and stacked (the two pairs listed both in S_1 and in F_1 are removed from S_1). Each band tends to the slope written at its right, and these slopes add up to $\frac{61}{6}$. The grey top band is the sporadic remainder $E(3,b) - E_{\text{floor}}(b)$, exact value minus listed classes: in the data it is $g^*_{b \bmod 4} = 3, 5, 3, 18$, hence the comb at $b \equiv 3 \pmod 4$ that fades like $18/b$. Dots: exact $E(3,b)/b$.

Figure 9



The residues of $W(3,b)$ modulo 60 on a torus: an observation on the exact values, and a property of the conjectural formula, not a theorem about $W(3,b)$. Since $\mathbb{Z}/60 = \mathbb{Z}/12 \times \mathbb{Z}/5$, the residue of b is a point of a torus with 12 steps around one circle and 5 around the other, and $b \mapsto b+1$ runs along the $(12,5)$ torus knot. Each of the 60 points is coloured by $r(b) = W(3,b) - \frac{73}{5}b^2 - \frac{253}{30}b$; right: the same values on the flat torus. In the 180 exact values of $W(3,b)$ for $5 \le b \le 300$, $r(b)$ depends only on b mod 60, and not only on b modulo 12, 15, 20 or 30; the formula of Conjecture 9.8 gives the same 60 values for all $5 \le b \le 1000$. The columns carry the period 12 of the proposed formula for $E(3,b)$ (the three dark columns are $b \equiv 3 \pmod 4$, where $h_{b \bmod 4}$ jumps to 17); the small changes along each column come from $R(3,b)$.

vector	Λ	θ	$\Lambda + h\theta - c_h$ (free h, from $h = -3$)	$\Lambda + h\theta - b$ (other h)
F_1: P, $\varepsilon = 0$	$b + \frac{7}{4}$	$i + \frac{3}{2}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{4}$	$\ge \frac{7}{4}$
F_1: Q, $\varepsilon = 0$	$b + \frac{1}{6}$	$i + \frac{2}{3}$	$\frac{1}{6}, \frac{5}{6}, \frac{1}{2}$	$\ge \frac{1}{6}$
F_1: P, $\varepsilon = 1$	$b + \frac{9}{4}$	$i + \frac{3}{2}$	$\frac{3}{4}, \frac{1}{4}, \frac{3}{4}$	$\ge \frac{9}{4}$
F_1: Q, $\varepsilon = 1$	$b + \frac{1}{4}$	$i + \frac{1}{2}$	$\frac{3}{4}, \frac{1}{4}, \frac{3}{4}$	$\ge \frac{1}{4}$
F_2: P, $\varepsilon = 0$	$b - i + \frac{1}{6}$	$i + \frac{4}{3}$	$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{1}{6}$	$\ge \frac{3}{2}$
F_2: Q, $\varepsilon = 0$	$b - i - \frac{5}{8}$	$i + \frac{3}{4}$	$\frac{1}{8}, \frac{7}{8}, \frac{5}{8}, \frac{3}{8}$	$\ge \frac{1}{8}$
F_2: P, $\varepsilon = 1$	$b - i + \frac{5}{6}$	$i + \frac{4}{3}$	$\frac{5}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}$	$\ge \frac{13}{6}$

F_2: Q, $\varepsilon = 1$	$b - i - \frac{1}{6}$	$i + \frac{2}{3}$	$\frac{5}{6}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}$	$\geq \frac{1}{2}$
F_3: P	$b + i + \frac{13}{2}$	$i + 4$	$\frac{1}{2}, \frac{1}{2}$	$\geq \frac{5}{2}$
F_3: Q	$b + i + \frac{11}{4}$	$i + \frac{5}{2}$	$\frac{1}{4}, \frac{3}{4}$	$\geq \frac{1}{4}$
F_4: P	$b - i + \frac{3}{4}$	$i + \frac{3}{2}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}$	$\geq \frac{9}{4}$
F_4: Q	$b - i - \frac{1}{2}$	$i + \frac{2}{3}$	$\frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{2}$	$\geq \frac{1}{6}$
F_5: P	$b - 2i - \frac{11}{6}$	$i + \frac{5}{3}$	$\frac{1}{6}, \frac{5}{6}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}$	$\geq \frac{3}{2}$
F_5: Q	$b - 2i - \frac{13}{6}$	$i + \frac{4}{3}$	$\frac{5}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{1}{6}$	$\geq \frac{1}{2}$

Table 6: Witnesses for $F_1, \dots, F_5$. The fourth column lists $\Lambda + h\theta - c_h \in (0, 1)$ for the coordinates that are written out in Theorem 9.2 (when one of them equals b, as $z = b$ for $i = 0$ in F_1, the value $b +$ offset still clips to b); the fifth gives a lower bound, valid for all $i \geq 0$, of $\Lambda + h\theta - b$ for the coordinates equal to b. All offsets are affine and do not depend on (b, i), the fractional parts of Λ and θ make every $\Lambda + h\theta$ a non-integer, and $\Lambda, \theta > 0$ in the stated ranges. The shading only separates the families.

Colour: the paper's own colours (see the caption).

10 Open questions

Question 10.1. p. 50

Is $W(a, b) = C_a b^2 + O_a(b)$ for every $a \geq 2$? More concretely:

1. Does (C1) hold outside C_+: does every full-dimensional generic piece with at least two free coordinates have an observation vector of (n, m)-rank 2?

Does every pair of pieces with the same signature, not both in C_+, satisfy one of the three alternatives of (C2): $c_{P'} \circ \phi = c_P$, no integer point mapped to an integer point, or an overlap of area 0?

↳ Not used by another result of the paper.

Question 10.2. p. 50

For every $a \geq 1$, if two templates have window sums that differ by a constant (condition (N2)), are their multisets of level constants equal (condition (N3)): $\{\kappa_{i,r}(\varepsilon)\}_i = \{\kappa_{i,r}(\varepsilon')\}_i$ for $1 \leq r \leq a$?

↳ Not used by another result of the paper.

Question 10.3. p. 50

Is there, for every $a \geq 2$, a constant $K(a)$ such that every realisable vector with at least two free coordinates that shares its word with another vector has its cell within distance $K(a)$ of $\mathcal{L}_{\deg}(a)$?

↳ Not used by another result of the paper.

Question 10.4. p. 51

For fixed $a \geq 2$, is $\mathcal{A}_a$ a finite union of affine families each of which, for fixed b, has at most one free integer parameter (plus finitely many sporadic points per b)? Can a decomposition of $\mathcal{A}_a$, separating its bounded pieces from its unbounded families, be computed for $a = 3$ without the full classification of Section 9?

↳ Not used by another result of the paper.

Question 10.5. p. 52

Is the least number $J(a,b)$ of Section 3 bounded by a small constant for fixed a, for instance $J(2,b) \le 5$ for all b? Which values of b need more than $I_0, \dots, I_3$, and why are they 6, 12 and 16 for $a = 2$ in the range computed?

↳ Not used by another result of the paper.

Proposition 10.6. (written proof) p. 52

> *Fix $a \ge 1$ and $j \ge 1$. Then $N_j(a,b)$ is, for b large, a quasipolynomial in b. If $E(a,b) = O_a(b)$ and $j \ge 2$, every constituent has degree at most one; so for $a = 2,3,4,5$ this holds by Theorem 6.1 (computer-assisted).*

Trust base: ⚠ not checked. Written proof (Woods' theorem on a Presburger formula), conditional on E(a,b) = O(b) for the degree bound, hence computer-assisted for a = 2..5; not checked by computation beyond the table of fibre sizes

Proof: in this note · **Covers:** not formalized

↳ Used in: 10.7 (p. 53)

Proof

Let $N'_j(a,b)$ count all fibres of size exactly j. A fibre is counted once through its lexicographically smallest vector c, and "c is realisable, no realisable $d <_{\text{lex}} c$ has $Z(d) = Z(c)$, there are realisable $d_1 <_{\text{lex}} \cdots <_{\text{lex}} d_{j-1}$, all $>_{\text{lex}} c$, with $Z(d_i) = Z(c)$, and every realisable d with $Z(d) = Z(c)$ is one of $c, d_1, \dots, d_{j-1}$" is a Presburger formula in (b,c) for fixed j, since realisability and $Z(c) = Z(d)$ are [I, Section 6, Steps 2 and 3]. Its sections are finite, so by Woods's theorem [8] , as in [I, Theorem 6.1], $N'_j(a,b)$ is eventually a quasipolynomial. The fibre of $O(0)$ has $1 + \sum_{h \le a} \Phi_h(b)$ vectors [I, Lemma 7.1], more than j for b large, so $N_j = N'_j$ eventually. A fibre of size $j \ge 2$ contributes $j - 1 \ge 1$ to E, so $(j-1)N_j(a,b) \le E(a,b)$; a quasipolynomial that is $O(b)$ has every constituent of degree ≤ 1.

Question 10.7. p. 52

For fixed $a \ge 2$, is the size of the largest fibre other than $O(0)$ eventually constant in b?

↳ Not used by another result of the paper.

Question 10.8. p. 53

(i) Does $A_{a,b}$ converge, as $b \to \infty$, to a piecewise linear limit profile with kinks at these fractions, uniformly on compact intervals avoiding them, and is the limit of the mean $P_{a,b}(I)$ positive? A sharper test than comparing values at $b = 80, 160, 320$: for $a = 2$, compute the exact profile in each residue class of b modulo the period and compare the coefficients of b in the ambiguous length; different leading coefficients in two classes on the same interval of slopes would refute a single limit. (The exact means decrease like $L + c/b$ with $L \approx 0.04$ ($a = 2$), 0.02 ($a = 3$) between $b = 80$ and $b = 320$, a fit to three values.) (ii) For $a = 2$ the profile has a second peak at $\theta = \frac{2}{3}$ whose height, computed exactly, is $0.409, 0.375, 0.355, 0.345$ for $b = 6, 12, 24, 48$: does it tend to $\frac{1}{3}$ with a defect of order $\frac{1}{2b}$? We have a pairing of cells that accounts for it for $b = 3, 5, 8, 12, 20$, by computation only, and no proof. (iii) The cells meeting $\theta = 1$ have a fibre of size one for $a \le 5$, $b \le 16$ (exact check): is this so for all a, b?

↳ Not used by another result of the paper.

Question 10.9. p. 54

Does $E(n,n)/R(n,n)$ have a limit, is it $\frac{1}{2}$, and what is the growth of $(R(n,n)+2n+1)/2 - W(n,n)$? For $a \neq b$ with both large, which ratios b/a keep the proportion of ambiguous cells away from 0?

↳ Not used by another result of the paper.

Proposition 10.10. (written proof) p. 54

> *Crossing the wall $\Lambda = \langle h, \vartheta \rangle$, $h \in B - B$, in the direction of increasing Λ, at a point on no other wall, raises Σ by exactly $\omega(h) = \prod_{i=1}^{k}(a_i + 1 - |h_i|) \geq 1$. Consequently $\Sigma = \sum_{h:\ \langle h,\vartheta\rangle<\Lambda} \omega(h)$, two cells separated by one wall have different multisets Z, and the cells with one multiset form an antichain for inclusion of the sets $\{h : \langle h, \vartheta\rangle < \Lambda\}$.*

Trust base: ⚠ not checked. Written proof, not machine-checked; checked on the shape (2,2,3); that Z encodes the word for k ≥ 3 is not checked

Proof: in this note · **Covers:** not formalized

↳ Used in: 10.11 (p. 54)

Proof

> Only the pairs with $y - x = h$ change status, and for each i there are $a_i + 1 - |h_i|$ pairs (x_i, y_i) in $\{0, \ldots, a_i\}^2$ with $y_i - x_i = h_i$; the choices are independent. Summing over the walls below Λ gives the formula for Σ, and all weights are positive.

Evidence & scope

- **Program:** shape (2,2,3), 6 random rational slopes, 1050 simple walls: jump of the pair count = ∏(a_i+1−|h_i|), 0 failures; control without the absolute value fails on 834. working script `V387_comprobaciones.py`, not published

Question 10.11. p. 54

For $k = 3$ and fixed a_1, a_2, is the number of cells whose word is shared by another cell of lower order than the number of cells, as $a_3 \to \infty$?

↳ Not used by another result of the paper.

Acknowledgements

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper (PDF).

A Reproducibility and independent checks

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper, p. 55.

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📊 Tables

Every table of the paper, in its order: click one to unfold it, with where the paper cites it, how its numbers were checked, and its data as CSV. The typeset table is one click away, on its page of the PDF.

Table 1. Status of the results, row by row (Levels 1 and 2; the categories are explained at the end of the table).

result	status
2@lLevel 1	
Proposition 2.3 (linear term D_a of $R(a,b)$; period divides P_a)	, a consequence of the published count (checked exactly, $a \le 6$, every class mod $\mathrm{lcm}(1,\dots,2a)$); minimal period $= P_a$: $a \le 6$
Proposition 3.1 (wall weights)	(a), (c) and $\omega \ge 1$; the crossing argument (b):
Corollary 3.2 (antichains)	(ii), (iii); (i):
Proposition 3.5 (fibres $\le 4ab+1$)	(finite checks, $a \le 3$, $b \le 15$)

Lemma 3.6 ($E \le \mathrm{Amb} \le 2E$)	(the counting lemma)
Theorem 3.7, Proposition 3.8 (finitely many power sums)	; the minimal number: $a \le 3$, $b \le 20$
Proposition 3.9 ($W(a,a) \le (R(a,a)+2a+1)/2$)	the counting step ($2W \le R+f$); the geometric step:
Proposition 3.11 (one pair of $a=2$, every $b \ge 2$)	(checked $b \le 300$)
Proposition 4.1 (dimension criterion)	
Theorem 5.1 (finite reduction)	; the encoding tested on exact data, $a \le 3$, $b \le 200$ (Appendix A)
2@lLevel 2	
Theorem 6.1: $E(a,b) = O(b)$, $W(a,b) = C_a b^2 + O(b)$ for $a = 2,3,4,5$	, conditional on the correctness of the program and of Theorem 5.1
Corollary 6.2 ($\mathrm{Amb}/R = \Theta_a(1/b)$ for $a = 2,\dots,5$)	: a written consequence of Theorem 6.1 and Lemma 3.6 (data in Table 3)
Proposition 6.3 (two numerations of one progression)	(i), and (ii) as a line of the arrangement that separates the two points; (iii), (iv):
Corollary 6.4 (ambiguity of positive section measure, every $a \ge 2$)	(segments checked in exact arithmetic, $a \le 6$, $b \le 60$)
Corollary 6.5 (no coincidence between two deep cells, $a = 3,4,5$)	
Proposition 7.1 (the identity for τ)	; Corollary 7.2 (the 109 candidates):
Proposition 7.4 (levels of C_+, every a)	
Lemma 7.5, Theorem 7.6 (all-positive chamber, every $a \ge 2$)	(checked: Corollary 7.9, $a \le 6$; all templates, $a \le 15$)

Colour: the paper's own colours (see the caption).

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Table 2. Status of the results (continued). *Written proof*: a mathematical argument given in the paper. *Isabelle/HOL*: the stated part is checked by the proof assistant (Section 1, Formalisation). *Computer-assisted*: a written reduction plus a finite exact computation in rational arithmetic that the result needs; no proof assistant was used for these, and the programs have not been checked by a human referee. *Exact enumeration*: exact data, with no conclusion for all b. *Conjecture*: not proved. The finite checks in parentheses, and the independent re-computations by an AI agent, are described in Appendix A. Row colours: catW catWb written proof, catI catIb written proof with a part checked in Isabelle/HOL, catC catCb computer-assisted, catQ conjecture (two shades alternate within a category).

result	status
Proposition 7.8 (candidates only in C_+; one row forces $m' = m$)	($a \le 6$; $a = 6$ without regression)
Theorem 7.10 (the shift is never an integer)	(the whole statement)
Corollary 7.11 ((C1), (C2) inside C_+)	, from written and formalised results
Conjecture 7.12 (candidates only in C_+)	(checked $a \le 6$)
Theorem 8.1 (Lemma D for $a = 3$, $b \ge 317$)	(Lemmas 8.3 and 8.4:)
2@lLevel 3	
Proposition 9.1 ($O(s)$ for $a = 3$)	
Theorem 9.2 (families $F_1,\dots,F_5$)	the identities and the witnesses; the counts:

Theorem 9.4 (family S_1)	the witnesses; the words, their equality and the count:
Corollary 9.5 (a fibre with at least four cells)	the equality of the four multisets; that the fibre has exactly four cells: observed
Theorem 9.7 (lower bound by the coincidence graph)	the counting implication; its premises are the results above
Conjecture 9.8: the list is complete ($\Rightarrow$), formula for $E(3, b)$	\ $b \le 170$, some $b \le 400$
2@lSection *10*	
Proposition 10.6 (fibre counts of fixed size are quasipolynomials of degree ≤ 1)	, conditional on rigidity (so for $a = 2, \dots, 5$)
Proposition 10.10 (wall weights for k primes)	(checked on the shape $(2, 2, 3)$)

Colour: the paper's own colours (see the caption).

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Table 2. Output of the certificate. The last three columns concern (C2): pairs of pieces with the same signature and different vector formulas; those with an integer point mapped to an integer point; those with positive area. Running times: 57 s for $a = 4$, 167 s for $a = 5$. catX $a = 1$, the positive control (a quadratic family is found); catC catCb $a = 2, \dots, 5$, the computer-assisted cases.

a	templ.	chambers	generic\	rank < 2	signatures	same sign.,\c differs	integer-\	area > 0
1	4	10	20	0	7	9	8	8
2	14	25	238	0	83	6	0	0
3	36	47	1296	0	401	15	0	0
4	76	76	4712	0	1320	42	0	0
5	136	112	12 920	0	3452	45	0	0

Colour: the paper's own colours (see the caption).

p. 21 in the PDF | ⬇ CSV | cited in §6.2 Results | cited in §6.5 Pairs with different signatures, and deep cells | cited in §7 The denominator a+1 | cited in Corollary 7.2

Table 3. Cells whose word is shared by another cell, from exact enumeration (all values of b in $2 \le b \le 100$ for $a = 2$, $3 \le b \le 80$ for $a = 3$, $4 \le b \le 40$ for $a = 4$, $5 \le b \le 30$ for $a = 5$; $E \le \mathrm{Amb} \le 2E$ holds in every one of the 240 cases). For $b \ge 2a$ the product $b\,\mathrm{Amb}\,/R$ stays between 1.14 and 1.73 ($a = 2$), 0.97 and 1.29 ($a = 3$), 0.90 and 1.12 ($a = 4$), 0.75 and 1.02 ($a = 5$). catE catEb exact enumeration; the two shades separate the values of a.

a	b	$R(a, b)$	$\mathrm{Amb}(a, b)$	$\mathrm{Amb}\,/R$	$b\,\mathrm{Amb}\,/R$
2	20	3143	260	0.0827	1.65
2	30	6913	381	0.0551	1.65
2	100	74370	1278	0.0172	1.72
3	20	6216	361	0.0581	1.16
3	30	13702	540	0.0394	1.18
3	80	94932	1451	0.0153	1.22
4	20	10359	496	0.0479	0.96
4	30	22860	734	0.0321	0.96
4	40	40246	975	0.0242	0.97

5	20	15440	631	0.0409	0.82
5	30	34095	903	0.0265	0.80

Colour: the paper's own colours (see the caption).

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Table 4. Proportion of ambiguous cells (the word is shared with another cell), counted three ways, rounded to three decimals (so 0.026 here is the 0.0265 of Table 3). *Integer sample*: the integers $2^{e-1}r^a s^b$ with odd primes $r \neq s \leq 2000$ (about 10^7 per row), computed with exact integer words; a word is ambiguous if its cell has a fibre of at least two cells. *Section measure*: the measure $\mu_{a,b}$ of the text; the choice of θ-range is a design choice, not a canonical measure. Numerical, not proved. The shading only separates the values of a.

a	b	$R(a,b)$	cells, Amb/R	integer sample	section measure
2	20	3143	0.083	0.082	0.203
2	40	12150	0.042	0.071	0.136
3	20	6216	0.058	0.052	0.182
3	40	24109	0.030	0.032	0.121
5	20	15440	0.041	0.080	0.168
5	30	34095	0.026	0.059	0.141

Colour: the paper's own colours (see the caption).

p. 25 in the PDF · ⬇ CSV · cited in §6.3 Faithfulness is a statement about cells, not about integers · cited in §A Reproducibility and independent checks

Table 5. Words shared by two distinct two-prime shapes with K divisors (all pairs, $12 \leq K \leq 48$; ranges over the pairs with the same K). *Share of* $L(K-1)$: $K/|L(a,b) \cap L(a',b')|$, the part explained by the common coarsening. Of the remaining words, *one fine cell*: one cell of the arrangement of the finest shape with K divisors meets both embedded shapes; *no common cell found*: the word is not explained by a common fine cell under the embeddings tested, so the two shapes reach it, as far as tested, through different fine cells. Exact computation in rational arithmetic; a numerical observation, not a theorem. The shading only alternates the values of K.

K	pairs	$\lvert L(a,b) \cap L(a',b')\rvert$	share of $L(K-1)$ (%)	one fine cell	no common cell found
12	1	28	43	15	1
16	1	33	48	17	0
18	1	47	38	27	2
20	1	56	36	32	4
24	3	68–92	26–35	133	30
28	1	94	30	60	6
30	3	91–113	27–33	152	54
32	1	101	32	68	1
36	6	83–151	24–43	336	133
40	3	130–189	21–31	255	84
42	3	139–169	25–30	230	95

44	1	169	26	110	15
45	1	188	24	90	53
48	6	166–212	23–29	618	231
total	32	3979	1127 words	2143	709

Colour: the paper's own colours (see the caption).

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cited in §6.4 Words shared by two shapes with the same number of divisors

Table 6. Witnesses for $F_1, \dots, F_5$. The fourth column lists $\Lambda + h\theta - c_h \in (0,1)$ for the coordinates that are written out in Theorem 9.2 (when one of them equals b, as $z = b$ for $i = 0$ in F_1, the value $b + \mathrm{offset}$ still clips to b); the fifth gives a lower bound, valid for all $i \geq 0$, of $\Lambda + h\theta - b$ for the coordinates equal to b. All offsets are affine and do not depend on (b, i), the fractional parts of Λ and θ make every $\Lambda + h\theta$ a non-integer, and $\Lambda, \theta > 0$ in the stated ranges. The shading only separates the families.

vector	Λ	θ	$\Lambda + h\theta - c_h$ (free h, from $h = -3$)	$\Lambda + h\theta - b$ (other h)
F_1: P, $\varepsilon = 0$	$b + \frac{7}{4}$	$i + \frac{3}{2}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{4}$	$\geq \frac{7}{4}$
F_1: Q, $\varepsilon = 0$	$b + \frac{1}{6}$	$i + \frac{2}{3}$	$\frac{1}{6}, \frac{5}{6}, \frac{1}{2}$	$\geq \frac{1}{6}$
F_1: P, $\varepsilon = 1$	$b + \frac{9}{4}$	$i + \frac{3}{2}$	$\frac{3}{4}, \frac{1}{4}, \frac{3}{4}$	$\geq \frac{9}{4}$
F_1: Q, $\varepsilon = 1$	$b + \frac{1}{4}$	$i + \frac{1}{2}$	$\frac{3}{4}, \frac{1}{4}, \frac{3}{4}$	$\geq \frac{1}{4}$
F_2: P, $\varepsilon = 0$	$b - i + \frac{1}{6}$	$i + \frac{4}{3}$	$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{1}{6}$	$\geq \frac{3}{2}$
F_2: Q, $\varepsilon = 0$	$b - i - \frac{5}{8}$	$i + \frac{3}{4}$	$\frac{1}{8}, \frac{7}{8}, \frac{5}{8}, \frac{3}{8}$	$\geq \frac{1}{8}$
F_2: P, $\varepsilon = 1$	$b - i + \frac{5}{6}$	$i + \frac{4}{3}$	$\frac{5}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}$	$\geq \frac{13}{6}$
F_2: Q, $\varepsilon = 1$	$b - i - \frac{1}{6}$	$i + \frac{2}{3}$	$\frac{5}{6}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}$	$\geq \frac{1}{2}$
F_3: P	$b + i + \frac{13}{2}$	$i + 4$	$\frac{1}{2}, \frac{1}{2}$	$\geq \frac{5}{2}$
F_3: Q	$b + i + \frac{11}{4}$	$i + \frac{5}{2}$	$\frac{1}{4}, \frac{3}{4}$	$\geq \frac{1}{4}$
F_4: P	$b - i + \frac{3}{4}$	$i + \frac{3}{2}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}$	$\geq \frac{9}{4}$
F_4: Q	$b - i - \frac{1}{2}$	$i + \frac{2}{3}$	$\frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{2}$	$\geq \frac{1}{6}$
F_5: P	$b - 2i - \frac{11}{6}$	$i + \frac{5}{3}$	$\frac{1}{6}, \frac{5}{6}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}$	$\geq \frac{3}{2}$
F_5: Q	$b - 2i - \frac{13}{6}$	$i + \frac{4}{3}$	$\frac{5}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{1}{6}$	$\geq \frac{1}{2}$

Colour: the paper's own colours (see the caption).

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cited in §9.2 Five families next to the upper clipping · cited in §A Reproducibility and independent checks

Table without a number, in the text of §9.2 Five families next to the upper clipping

	$[0, i]$	$[i+1, 1+2i]$	$[2+2i, 2+3i]$	$[3+3i, 3+4i]$
row 0	0	$+1$	$+2$	$+1$
row 1	-1	0	$+1$	0
row 2	-2	-1	0	0
row 3	-1	0	0	0

Colour: blue shade = the number's size within its column.

⬇ CSV