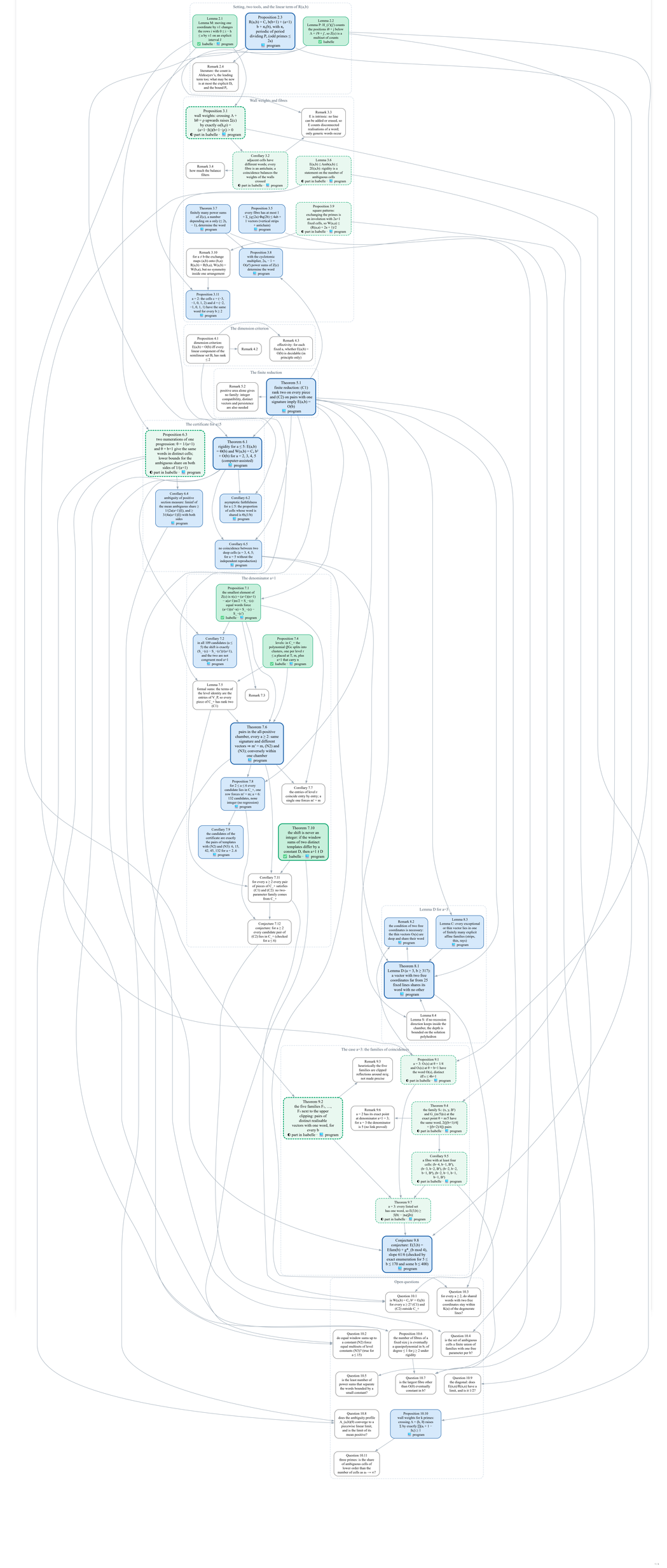
Map of *How Many Dyck Words Do the Divisors of an Integer Make? II. Generic Rigidity and Persistent Ambiguity*

Combinatorics on words · divisors of integers · line arrangements · Presburger arithmetic · computer-assisted proof

60 results · **0** entirely in Lean · **10** partly in Lean · **5** in Isabelle · **33** checked by a program · written proof only · (a result can carry several tags)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



What the paper contributes

The language of the words $\langle\!\langle n\rangle\!\rangle_2$ and its questions are Rodríguez Caballero's; paper I (doi:10.5281/zenodo.24541288) counted the cells $R(a,b)$ of the pattern r^as^b and asked whether the excess $E(a,b)=R(a,b)-W(a,b)$ is linear in b for $a\ge2$ (rigidity). This draft answers it for $a\le5$ and studies where the coincidences live, at three levels of status:

- **Level 1, general theory (every a; written proofs, re-derived by an independent AI agent, not checked by a human referee).** A dimension criterion (Proposition 4.1) and a finite reduction to affine observation vectors on pieces of the parameter space (Theorem 5.1). Crossing the wall $\Lambda+h\theta=\rho$ raises $\Sigma(c)$ by exactly $\omega(h,\rho)=(a+1-|h|)(b+1-|\rho|)$ (Proposition 3.1), so adjacent cells have different words, every fibre is an antichain of at most $4ab+1$ cells, $E\le\mathrm{Amb}\le2E$, and boundedly many power sums of $Z(c)$ determine the word. From Alekseyev's count, $R(a,b)=C_ab(b+1)+(a+1)b+\pi_a(b)$ with π_a of period dividing P_a (Proposition 2.3; the count itself is classical).
- **Level 2, computer-assisted rigidity (program and reduction not checked by a human referee).** $E(a,b)=\Theta(b)$ and $W(a,b)=C_ab^2+O(b)$ for $a=2,3,4,5$ (Theorem 6.1); the only affine candidates shift n by a non-integral multiple of $1/(a+1)$, explained by the smallest element $\tau(c)$ of $Z(c)$ (Proposition 7.1). For every $a\ge2$ the all-positive chamber has no two-parameter family (Theorems 7.6 and 7.10, Corollary 7.11, written proofs; Theorem 7.10 and Proposition 7.4 are checked in Isabelle/HOL). Measured by length, ambiguity does not vanish: lower bounds on both sides of $\theta=1/(a+1)$ give a mean ambiguous share with $\liminf\ge3/(4a(a+1)|I|)$ (Proposition 6.3, Corollary 6.4). Lemma D for $a=3$, $b\ge317$ (Theorem 8.1).
- **Level 3, a conjectural classification for $a=3$.** Nine one-parameter families of coincidences, each proved by hand (Theorems 9.2, 9.4, Proposition 9.1, Corollary 9.5), give $E(3,b)$ up to a periodic constant fitted to the data (Conjecture 9.8): it agrees with exact enumeration for $5\le b\le170$ and some $b\le400$; completeness is not proved.

Fifteen written results are checked, wholly or in part, in Isabelle/HOL (public: karlesmarin/formalizations, Isabelle/CaballeroDyckWords). What the paper claims as its own, as far as its searches could see: the finite reduction and its certificate, Lemma D, the all-positive-chamber analysis and the families of Section 9; the rest is known, standard or elementary and is labelled so in the paper. **Draft of 26 November 2026, not refereed, not published.**

🚀 **Start here:** **Proposition 3.1**: wall weights: crossing $\Lambda + h\theta = \rho$ upwards raises $\Sigma(c)$ by exactly $\omega(h,\rho) = (a+1-|h|)(b+1-|\rho|) > 0$.

Main results:

- **Proposition 2.3**: $R(a,b) = C_a\, b(b+1) + (a+1)\, b + \pi_a(b)$, with π_a periodic of period dividing P_a (odd primes $\le 2a$)

- **Proposition 3.1**: wall weights: crossing $\Lambda + h\theta = \rho$ upwards raises $\Sigma(c)$ by exactly $\omega(h,\rho) = (a+1-|h|)(b+1-|\rho|) > 0$
- **Theorem 5.1**: finite reduction: (C1) rank two on every piece and (C2) on pairs with one signature imply $E(a,b) = O(b)$
- **Theorem 6.1**: rigidity for $a \leq 5$: $E(a,b) = \Theta(b)$ and $W(a,b) = C_a\, b^2 + O(b)$ for $a = 2, 3, 4, 5$ (computer-assisted)
- **Proposition 6.3**: two numerations of one progression: $\theta = 1/(a+1)$ and $\theta = b+1$ give the same words in distinct cells; lower bounds for the ambiguous share on both sides of $1/(a+1)$
- **Theorem 7.6**: pairs in the all-positive chamber, every $a \geq 2$: same signature and different vectors $\Rightarrow m' = m$, (N2) and (N3); conversely within one chamber
- **Theorem 7.10**: the shift is never an integer: if the window sums of two distinct templates differ by a constant D, then $a+1 \nmid D$
- **Theorem 8.1**: Lemma D ($a = 3$, $b \geq 317$): a vector with two free coordinates far from 25 fixed lines shares its word with no other
- **Theorem 9.2**: the five families $F_1, \ldots, F_5$ next to the upper clipping: pairs of distinct realisable vectors with one word, for every b
- **Conjecture 9.8**: conjecture: $E(3,b) = Efam(b) + g^*_(b \bmod 4)$, slope $61/6$ (checked by exact enumeration for $5 \leq b \leq 170$ and some $b \leq 400$)

Fields: where the paper lives and what it connects

Tools taken from other branches

- **Paper I of the series (Rodríguez Caballero's language L)** → the cells of the lines $\Lambda + h\theta = \rho$, the multiset $Z(c)$ that encodes the word, realisation by primes, the count $R(a,b)$, Woods' quasipolynomiality, $W(1,b)$ and $W(2,b)$, the two tools of its Supplement (Lemma M and Lemma P) — Lemma 2.1, Lemma 2.2, Proposition 2.3, Theorem 6.1
- **Threshold functions (Acketa–Žunić 1991; Alekseyev 2010)** → $R(a,b) = (N(2b,2a) + 4a + 4b + 6)/8$ through Alekseyev's exact count; the linear term $D_a = C_a + a + 1$ and the period bound P_a are read off it by Möbius inversion — Proposition 2.3
- **Semilinear sets (Ginsburg–Spanier) and polyhedra (Hoffman; Minkowski–Weyl)** → the dimension criterion for $E(a,b) = O(b)$; the error bound in case (iv) of the finite reduction; the recession argument of Lemma S — Proposition 4.1, Theorem 5.1, Lemma 8.4
- **Sturmian words (Mignosi; Berstel–Pocchiola; Lothaire)** → the templates ε are the factors of length $2a$ of mechanical words, counted by Mignosi's formula; the non-integrality theorem is a statement on equal window sums of two such factors — Theorem 5.1, Theorem 7.10
- **Sparse polynomials (Hajós's lemma)** → finitely many power sums of $Z(c)$, a number depending on a only, determine the word — Theorem 3.7

What the results serve

- **Rigidity of the words of divisors (Question 11.1 of paper I)** ← $E(a,b) = \Theta(b)$ and $W(a,b) = C_a\, b^2 + O(b)$ for $a \leq 5$, the all-positive chamber for every a, where the coincidences live (Lemma D), the families for $a = 3$ and open questions on every a,

> fibres, periods and the ambiguity profile — Theorem 6.1, Theorem 7.6, Theorem 8.1, 9.8