How Many Dyck Words Do the Divisors of an Integer Make? Exact Counts and Quasipolynomial Growth

Combinatorics on words · divisors of integers · enumeration of hyperplane arrangements

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At a glance

What the paper contributes

The language L of the words $\langle\langle n \rangle\rangle_2$ and the question of describing it are Rodríguez Caballero's. Starting from the reduction to the odd part, the paper counts the words of each semilength:

- **Part I (structure).** The words of a pattern are the words of the cells of a hyperplane arrangement in the logarithms of the primes, and every cell is realised by primes (Lemma 2.3). L_k is produced by the single finest pattern $p_1^{q_1-1} \cdots p_r^{q_r-1}$, $k = q_1 \cdots q_r$ (Theorem 3.1), so membership in L is decidable (Corollary 4.2). Written proofs.
- **Part II (general enumeration).** The number $R(a,b)$ of cells of the pattern $p^a p'^b$ is a known count: $R(a,b) = (N(2b,2a) + 4a + 4b + 6)/8$, with N Alekseyev's number of two-dimensional threshold functions on a grid (Proposition 5.3; Acketa–Žunić, Alekseyev); the paper gives a direct proof, with leading coefficient $C_a \sim 12a^2/\pi^2$ (Theorem 5.1, Corollary 5.2); $W(a,b)$ is eventually a quasipolynomial in b of degree exactly two (Theorem 6.1, an application of Woods' theorem, existence only; Theorem 6.3, Corollary 6.4); three mechanisms by which distinct cells give the same word (Section 7). Written proofs.
- **Part III (explicit counts).** $|L_{2q}| = (9q^2 - 10q + 13)/4$ for odd primes q (Theorem A, written proof) and $|L_{3q}| = (44q^2 - 69q + 58 - 3(\frac{q}{3}))/6$ for primes $q \geq 5$ (Theorem B: one lemma is decided by exact SMT queries with Z3, without a checked proof object); a table of $|L_k|$ for small k by exact enumeration.

The paper claims no novelty for the Kassel–Reutenauer reformulation of the appendix, which is implicit in Rodríguez Caballero's work (JNT 2018). The arrangements and their link with semiorders go back to Stanley; counting cells through vertices is the planar case of Zaslavsky's face-count formula; the quasipolynomiality rests on a theorem of Woods.

🚀 **Start here: Lemma 2.1**: reduction to the odd part: $D_n \backslash 2D_n = D_m$, $2D_n \backslash D_n = tD_m$, semilength $\tau(m)$.

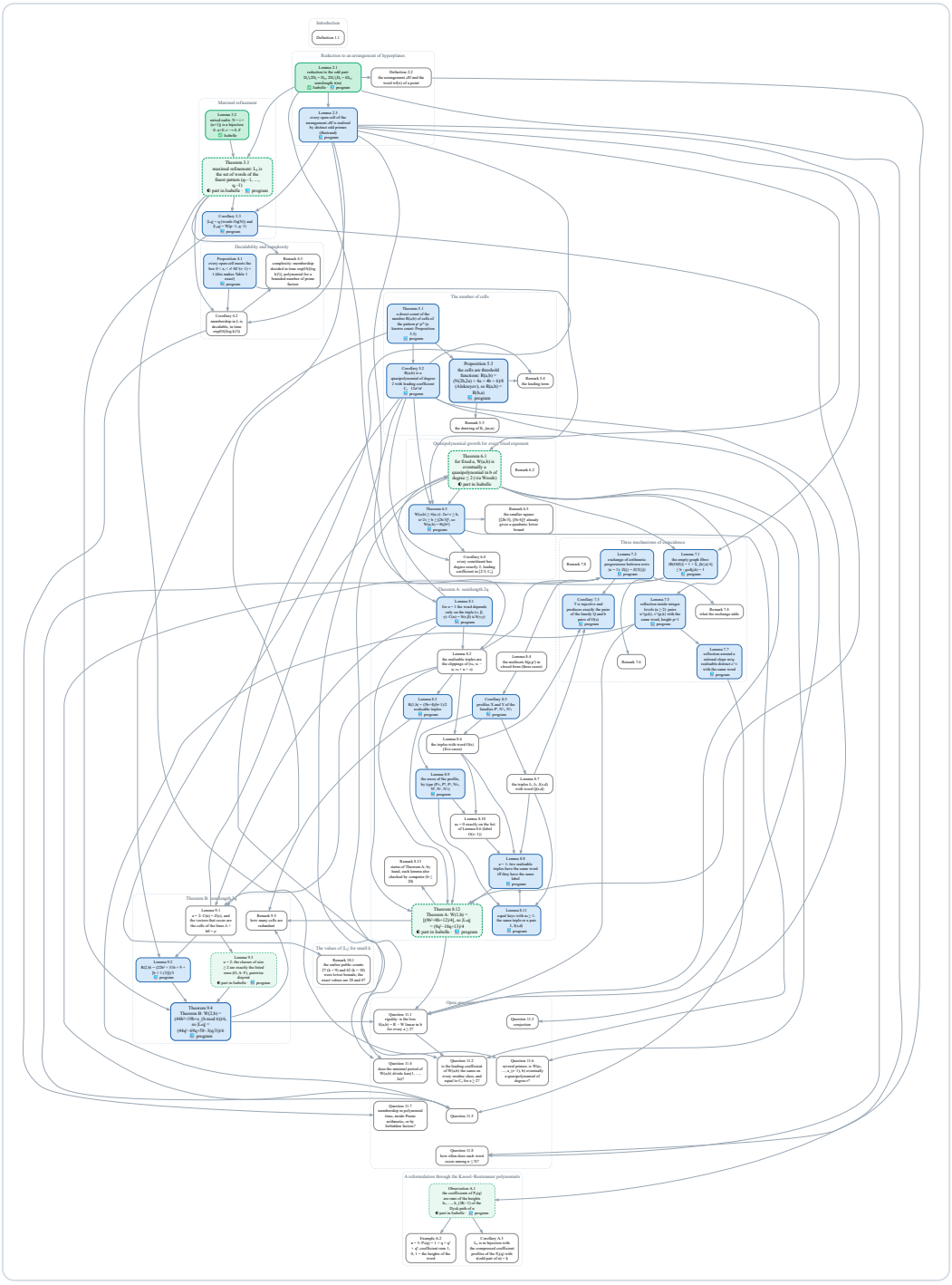
Main results:

- **Theorem 3.1**: maximal refinement: L_k is the set of words of the finest pattern $(q_1-1, \ldots, q_r-1)$
- **Proposition 5.3**: the cells are threshold functions: R(a,b) = (N(2b,2a) + 4a + 4b + 6)/8 (Alekseyev), so R(a,b) = R(b,a)
- **Theorem 6.1**: for fixed a, W(a,b) is eventually a quasipolynomial in b of degree ≤ 2 (via Woods)
- **Theorem 8.12**: Theorem A: $W(1,b) = \lfloor (9b^2+8b+12)/4 \rfloor$, so $|L_{2q}| = (9q^2-10q+13)/4$
- **Theorem 9.4**: Theorem B: W(2,b) = (44b²+19b+e_(b mod 6))/6, so $|L_{3q}| = (44q^2-69q+58-3(q/3))/6$

58 results | ✅ **0** entirely in Lean | ◐ **5** partly in Lean | ✅ **2** in Isabelle | 💻 **25** checked by a program | written proof only | (a result can carry several tags)

🗺️ Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



📐 Setting (copied from the paper: §1, §2, the introduction of Part II, §5 and §6)

Let D_n be the set of divisors of n and $2D_n = \{2d : d \in D_n\}$. Write the elements of the symmetric difference $D_n \triangle 2D_n$ in increasing order and replace each element by (if it lies in $D_n \setminus 2D_n$ and by) if it lies in $2D_n \setminus D_n$; the result is $\langle\langle n \rangle\rangle_2$. (§1)

For $k \geq 1$ let L_k be the set of words of L of semilength k (length $2k$), where $L = \{\langle\langle n \rangle\rangle_2 : n \geq 1\}$. (§1, Definition 1.1)

Let $m = p_1^{f_1} \cdots p_r^{f_r}$, where the p_i are distinct odd primes and the exponents $f_i \geq 1$. The multiset $\{f_1, \ldots, f_r\}$ is the *pattern* (signature) of m, and $\tau(m) = \prod(f_i + 1)$. Put

$$x_i = \frac{\log p_i}{\log t} > 0, \qquad I_f = \prod_{i=1}^{r} \{0, \ldots, f_i\}, \qquad u_a(x) = \sum_i a_i x_i \quad (a \in I_f).$$

Then $u_a(x) = \log d / \log t$ for $d = \prod p_i^{a_i}$, and $d<td'$ if and only if $u_a(x)<1+u_{a'}(x)$. Let $\mathcal{A}_f$ be the finite set of hyperplanes $\{x : \alpha \cdot x = 1\}$ with $\alpha \in \prod_i \{-f_i, \dots, f_i\}$, $\alpha \neq 0$. (§2, with $n = 2^{e-1}m$, m odd, and $t = 2^e$ of Lemma 2.1)

For integers $a, b \geq 1$, $W(a,b)$ is the number of distinct words $\langle\langle n \rangle\rangle_2$ with $n = 2^{e-1} r^a s^b$, where $r \neq s$ are odd primes and $e \geq 1$. We measure in units of $\log s$, with $\Lambda = \log t / \log s$ and $\theta = \log r / \log s$, and write $\mathrm{clip}(v) = \min(b, \max(-b-1, v))$ and $f(v) = \min(b+1, \max(0, v+1))$. (Introduction of Part II)

Let $\mathcal{L}_{a,b}$ be the set of lines $\ell_{h,\rho} : \Lambda + h\theta = \rho$ with $|h| \leq a$ and $\rho \in \mathbb{Z} \cap [-b, b]$, and let $R(a,b)$ be the number of cells (connected components) of the open quadrant $\mathcal{Q} = \{\Lambda>0, \theta>0\}$ minus these lines. (§5)

Put

$$c_h = \mathrm{clip}\lfloor \Lambda + h\theta \rfloor \quad (-a \leq h \leq a), \qquad H_{i'}(j') = \sum_{i=0}^{a} f(c_{i'-i} + j'), \qquad Z(c) = \biguplus_{i'=0}^{a} \{H_{i'}(j$$

(§6, Step 1)

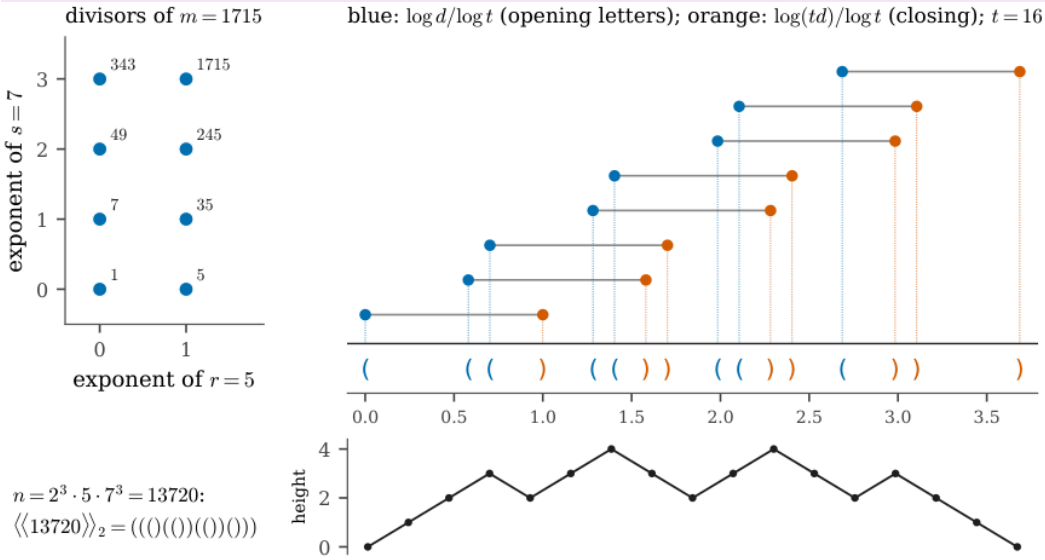
1 Introduction

Definition 1.1. p. 3

For $k \geq 1$ let L_k be the set of words of L of semilength k (length $2k$).

↳ Not used by another result of the paper.

Figure 1



The word of $n = 2^3 \cdot 5 \cdot 7^3$. Left: the divisors of the odd part $m = 5 \cdot 7^3$ as the grid of exponents $\{0,1\} \times \{0, \dots, 3\}$. Right: each divisor d projected to $\log d / \log t$ (blue) and joined to its partner $\log(td)/\log t$ (orange), one unit to the right ($t = 16$). Reading the line from left to right gives $\langle\langle n \rangle\rangle_2$; its Dyck path is drawn below.

2 Reduction to an arrangement of hyperplanes

Lemma 2.1. Isabelle p. 4

> *Write $n = 2^{e-1}m$ with m odd, $e \geq 1$, and put $t = 2^e$. Then $D_n \setminus 2D_n = D_m$ and $2D_n \setminus D_n = tD_m$. Hence $\langle\langle n \rangle\rangle_2$ is obtained by merging D_m (letters `(`) with tD_m (letters `)`), and its semilength is $\tau(m)$.*

Isabelle · ours **show**

`reduction_lemma` · Isabelle/CaballeroDyckWords/Caballero_Divisor_Words.thy, line 183

```
theorem reduction_lemma:
  assumes "odd m" "e \<ge> 1" "n = 2 ^ (e - 1) * m"
  shows "divs n - (*) 2 ` divs n = divs m"
    and "(*) 2 ` divs n - divs n = (*) (2 ^ e) ` divs m"
```

`dword_counts_and_dyck` · Isabelle/CaballeroDyckWords/Caballero_Divisor_Words.thy, line 194

```
theorem dword_counts_and_dyck:
  assumes "odd m" "e \<ge> 1" "n = 2 ^ (e - 1) * m"
  shows "length (filter id (dword n)) = card (divs m)"
    and "length (filter Not (dword n)) = card (divs m)"
    and "\<And>k. length (filter Not (take k (dword n))) \<le> length
(filter id (take k (dword n)))"
```

🆕 **Classical and new:** elementary and not new: implicit in Rodríguez Caballero's 2020 construction from the divisors of 2n (FOM, 17 March 2020); the identity in this form was noted in the public discussion that led to this note.

↳ Used in: 2.2 (p. 6), Lemma 2.3 (p. 6), Theorem 3.1 (p. 8), Lemma 8.1 (p. 28), A.1 (p. 45)

Proof

> $D_n = \{2^a d : 0 \leq a \leq e-1,\ d \mid m\}$ and $2D_n = \{2^a d : 1 \leq a \leq e,\ d \mid m\}$. The two sets share the elements with $1 \leq a \leq e-1$, so the difference $D_n \setminus 2D_n$ consists of the elements with $a = 0$ and $2D_n \setminus D_n$ of those with $a = e$.

💡 **In plain words (commentary, not part of the paper)**

Only the odd part of n and the power of two matter, and in a simple way. If $n = 2^{e-1}m$ with m odd, the letters `(` of the word are the divisors of m and the letters `)` are those divisors multiplied by $t = 2^e$; all the other divisors of n cancel against their doubles. So the word has $\tau(m)$ letters of each kind, and it is obtained by merging the two sorted lists D_m and tD_m. For $n = 13720 = 2^3 \cdot 5 \cdot 7^3$ (Figure 1), $t = 16$ and $m = 5 \cdot 7^3$ has 8 divisors, so the word has semilength 8.

The paper attributes the reduction to Rodríguez Caballero, who pointed it out in the discussion that led to the note; the paper includes the short proof.

🛡️ **Evidence & scope**

- **Audit:** machine-checked in Isabelle/HOL session Caballero_Base, theory Caballero_Divisor_Words (task 290): reduction_lemma proves D_n \ 2D_n = D_m and 2D_n \ D_n = 2^e D_m; dword_counts_and_dyck proves that the word has τ(m) letters

of each kind and is a Dyck word (every prefix). Compiles without sorry (Isabelle2026); negative controls (2^(e+1) in place of 2^e, a shifted count) fail. working script `Caballero_Divisor_Words.thy`, not published

- **Program:** n = 1 … 3000: the word rebuilt from the counts C(n) of (1) equals the word read from the definition; with t = 2^(e−1) (wrong reduction) the cell vector of n = 13720 fails. `lab/comprueba_lab.py`

Definition 2.2. p. 5

Let $\mathcal{A}_f$ be the finite set of hyperplanes $\{x : \alpha \cdot x = 1\}$ with $\alpha \in \prod_i \{-f_i, \dots, f_i\}$, $\alpha \neq 0$. For $x \in \mathbb{R}^r_{>0} \setminus \bigcup \mathcal{A}_f$ let $w_f(x)$ be the word whose letters) are preceded by

$$c_b(x) = \#\{a \in I_f : u_a(x) < 1 + u_b(x)\} \qquad (b \in I_f)$$

letters (, sorted increasingly. Explicitly, if $c_{(1)} \le \cdots \le c_{(K)}$ is the sorted list, then $w_f(x) = (^{c_{(1)}})\,(^{c_{(2)}-c_{(1)}}) \cdots (^{c_{(K)}-c_{(K-1)}})$.

↳ Used in: 11.8 (p. 44)

Lemma 2.3. (realisation by primes) p. 5

> *Let $x^0 \in \mathbb{R}^r_{>0} \setminus \bigcup \mathcal{A}_f$. There are $e \ge 1$ and distinct odd primes $p_1, \dots, p_r$ such that $n = 2^{e-1} p_1^{f_1} \cdots p_r^{f_r}$ satisfies $\langle\langle n \rangle\rangle_2 = w_f(x^0)$.*

Trust base: ⚠️ not checked. Written proof (audited); not machine-checked. The Isabelle session of task 291 leaves the realisation by Bertrand's postulate out (its phase 2)

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** the whole statement

🆕 **Classical and new:** the paper: arrangements of this kind and their link with semiorders go back to Stanley; the proof uses Bertrand's postulate.

↳ Used in: Theorem 3.1 (p. 8), Corollary 3.3 (p. 10), Corollary 4.2 (p. 12), Theorem 6.1 (p. 18), Theorem 6.3 (p. 19), Lemma 7.1 (p. 21), Lemma 8.2 (p. 29), Lemma 9.1 (p. 37), 11.8 (p. 44)

Proof

Let U be the cell containing x^0. It is open and w_f is constant on it. Points with pairwise distinct coordinates are dense, so pick one, $x^1 \in U$. Then pick $\delta > 0$ such that the box $B = \prod_i (x_i^1 - \delta, x_i^1 + \delta)$ lies in U, $x_i^1 - \delta > 0$ for every i, and the r intervals $(x_i^1 - \delta, x_i^1 + \delta)$ are pairwise disjoint. We use Bertrand's postulate in the form: for every real $y \ge 1$ there is a prime p with $y < p \le 2y$ [2] . Choose e so large that, for each i,

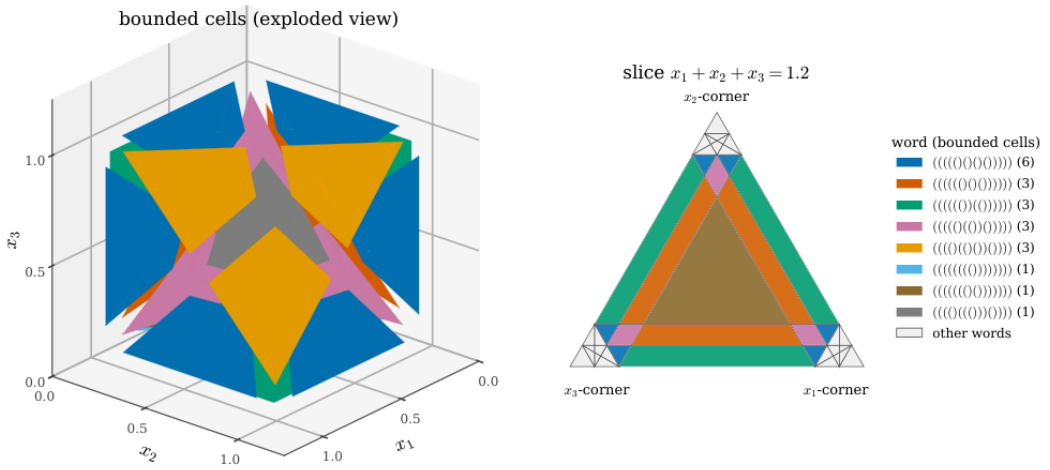
$$J_i = \left(2^{e(x_i^1 - \delta)},\ 2^{e(x_i^1 + \delta)}\right)$$

has left end $y_i \ge 2$ and ratio $2^{2e\delta} > 2$. Then $(y_i, 2y_i] \subset J_i$ contains a prime $p_i > 2$. The J_i are pairwise disjoint, so the p_i are distinct. Moreover $\log p_i / \log 2^e \in (x_i^1 - \delta, x_i^1 + \delta)$, so $x(n) \in B \subset U$ and $\langle\langle n \rangle\rangle_2 = w_f(x(n)) = w_f(x^0)$.

Evidence & scope

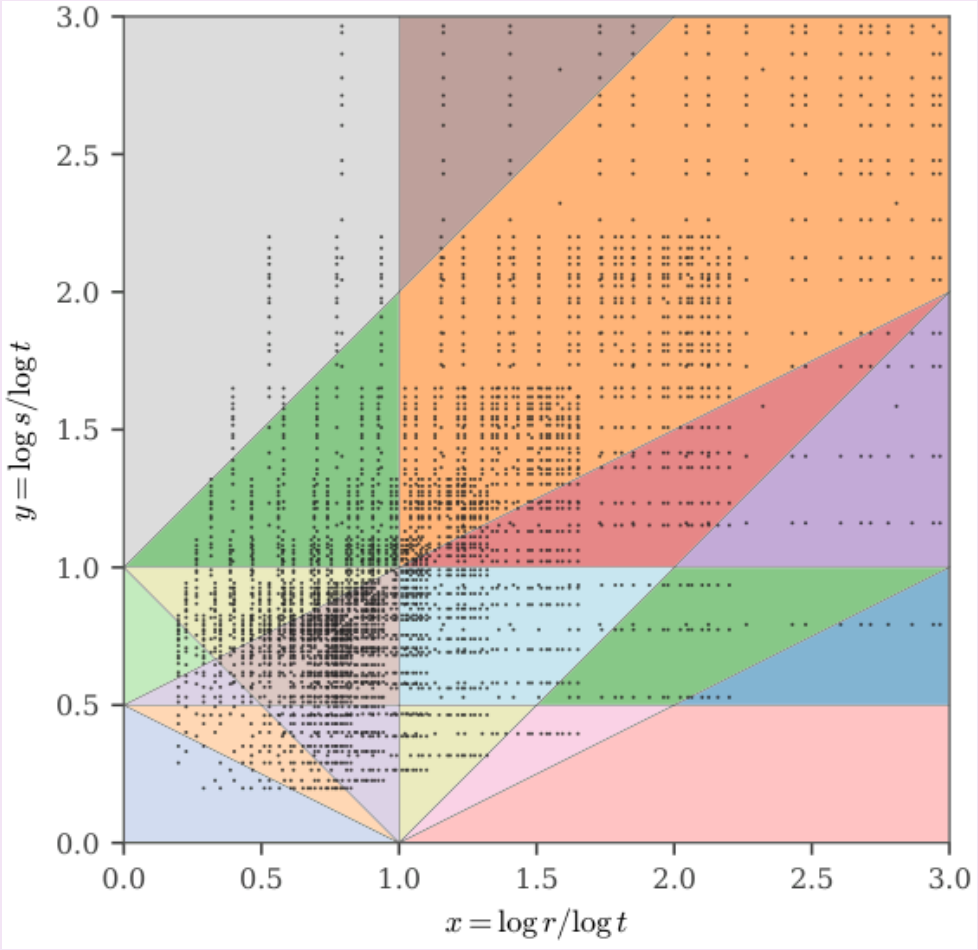
- **Program:** Figure 3: for all 18 098 points with r, s ≤ 200 and e ≤ 11 in the window, the word of n = 2^(e−1) r s² is the word of its cell (the figure script stops if a value disagrees); in the figure every one of the 19 cells of the window [0,3]² contains one of the 3590 points with r, s ≤ 100, e ≤ 8. working script `fig_primos_a1.py`, not published

Figure 2



The arrangement $\mathcal{A}_{(1,1,1)}$ (pattern $pp'p''$, $k = 8$) of the 19 planes $\alpha \cdot x = 1$, $\alpha \in \{-1, 0, 1\}^3$, that meet the positive octant. Left: its 21 bounded cells, each shrunk towards its centre so that all are visible, coloured by word (8 words; the number of cells of each word is in the legend). Permuting the primes does not change the word, and the multiplicities 6, 3, 1 are consistent with orbits under permutation of the coordinates. Right: the slice $x_1 + x_2 + x_3 = 1.2$, cut exactly by the planes (43 pieces, 11 words); pieces of the eight words on the left keep their colours.

Figure 3



Pattern $(1, 2)$ ($m = rs^2$, $k = 6$): the cells of the lines $\alpha x + \beta y = 1$, $|\alpha| \le 1$, $|\beta| \le 2$, in the window $[0, 3]^2$, coloured by word (19 cells, 16 words; cells of equal colour have the same word), and the points $(\log r, \log s)/\log 2^e$ for distinct primes $3 \le r, s \le 100$ and $e \le 8$ (3590 points). Every one of the 19 cells in the window contains such a point.

3 Maximal refinement

Theorem 3.1. p. 6

> *Let $k \ge 2$ with prime factorisation $k = q_1 q_2 \cdots q_r$, repetitions allowed. Then L_k is the set of words $\langle\langle n \rangle\rangle_2$ whose odd part has pattern $(q_1 - 1, \ldots, q_r - 1)$:*
>
> $$L_k = \big\{ \langle\langle n \rangle\rangle_2 : \text{ the odd part of } n \text{ has pattern } (q_1 - 1, \ldots, q_r - 1) \big\}.$$
>
> *Equivalently, every word of every pattern f with $\prod(f_i + 1) = k$ is also a word of the finest pattern $(q_1 - 1, \ldots, q_r - 1)$.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the combinatorial core: the word is unchanged when x_1 is split into $(x_1, (u+1)x_1)$; the realisation by primes (Lemma 2.3) is not formalized

`word_refinement` · `Isabelle/CaballeroDyckWords/Caballero_Radix/Caballero_Radix.thy`, line 257

```
theorem word_refinement:
  assumes "finite R"
  shows "wordF (idxS u v R) (posS x u \<rho>) = wordF (idxF ((u + 1) * (v +
1) - 1) R) (posF x \<rho>)"
```

comparison_transfer · Isabelle/CaballeroDyckWords/Caballero_Radix/Caballero_Radix.thy, line 239

```
theorem comparison_transfer:
  assumes "t \<in> idxS u v R" "t' \<in> idxS u v R"
  shows "posS x u \<rho> t < 1 + posS x u \<rho> t' \<longleftrightarrow>
posF x \<rho> (phi u t) < 1 + posF x \<rho> (phi u t')"
```

no_unit_difference_transfer · Isabelle/CaballeroDyckWords/Caballero_Radix/Caballero_Radix.thy, line 244

```
theorem no_unit_difference_transfer:
  assumes "\<And>a b. a \<in> idxF ((u + 1) * (v + 1) - 1) R \
<Longrightarrow> b \<in> idxF ((u + 1) * (v + 1) - 1) R \<Longrightarrow>
      posF x \<rho> a - posF x \<rho> b \<noteq> 1"
    and "a \<in> idxS u v R" "b \<in> idxS u v R"
  shows "posS x u \<rho> a - posS x u \<rho> b \<noteq> 1"
```

Trust base: ⚠️ not checked. Written proof, audited internally; the combinatorial core (the word is unchanged when x_1 is split into $(x_1, (u+1)x_1)$) compiles in Isabelle/HOL (session Caballero_Radix, public); the realisation step (Lemma 2.3) is not formalized

🆕 **Classical and new:** the paper counts it among its new results («The novelty claim is limited to the counts, to Theorem 3.1 and its consequence Corollary 4.2, …»); the mixed-radix decomposition is classical.

↳ Used in: Corollary 3.3 (p. 10), Corollary 4.2 (p. 12), 4.3 (p. 13), 11.6 (p. 44)

Proof

The words of the finest pattern lie in L_k by definition. For the converse it is enough to show: if $f = (f_1, \dots, f_s)$ has $f_1 + 1 = (u+1)(v+1)$ with $u, v \geq 1$, and $f' = (u, v, f_2, \dots, f_s)$, then every word of pattern f is a word of pattern f'. Repeating this step splits every composite $f_i + 1$ into primes. The step terminates because each application increases the number of parts.

Let $\langle\langle n \rangle\rangle_2$ have pattern f, and let $x = x(n) \in \mathbb{R}^s_{>0}$, so x lies on no hyperplane of $\mathcal{A}_f$ and $\langle\langle n \rangle\rangle_2 = w_f(x)$. Define $y = (x_1, (u+1)x_1, x_2, \dots, x_s) \in \mathbb{R}^{s+1}_{>0}$ and $\varphi : I_{f'} \to I_f$, $\varphi(i, j, a_2, \dots, a_s) = (i + (u+1)j, a_2, \dots, a_s)$. By Lemma 3.2, φ is a bijection and $u_{a'}(y) = u_{\varphi(a')}(x)$ for all $a' \in I_{f'}$.

Now y lies on no hyperplane of $\mathcal{A}_{f'}$. Indeed, $u_{a'}(y) - u_{b'}(y) = 1$ would give $u_{\varphi(a')}(x) - u_{\varphi(b')}(x) = 1$, which is impossible. Moreover $c'_{b'}(y) = \#\{a' : u_{a'}(y) < 1 + u_{b'}(y)\} = c_{\varphi(b')}(x)$, so the two sorted multisets coincide and $w_{f'}(y) = w_f(x) = \langle\langle n \rangle\rangle_2$. The point y is degenerate, since its second coordinate is exactly $u+1$ times its first and no pair of primes has that property. But Lemma 2.3 only needs y to avoid $\mathcal{A}_{f'}$. It supplies n' with odd part of pattern f' and $\langle\langle n' \rangle\rangle_2 = w_{f'}(y) = \langle\langle n \rangle\rangle_2$.

💡 **In plain words (commentary, not part of the paper)**

Many integers have odd parts with k divisors: for $k = 12$ the odd part can be p^{11}, $p^5 p'$, $p^3 p'^2$ or $p^2 p' p''$. The theorem says that only the last kind matters: every word of semilength k is already the word of some n whose odd part has the finest pattern, one prime for each prime

factor of k (counted with multiplicity). So $|L_{12}|$ is the number of words of the pattern $p^2p'p''$, 252 by exact enumeration (Figure 4(a)).

Why it is true: in logarithmic units the divisors of p^F, $F+1=(u+1)(v+1)$, sit at Nx_1, $0 \le N \le F$; by division with remainder these are the points $ix_1 + j(u+1)x_1$, so the point $(x_1, (u+1)x_1)$ of the finer pattern makes exactly the same comparisons $d<td'$ and gives the same word. That point is not given by two primes, but it lies off every hyperplane, so it sits in an open cell, and Lemma 2.3 (Bertrand's postulate) finds primes in that cell.

The paper counts this theorem among its new results; the mixed-radix decomposition it uses is classical (Lemma 3.2).

Evidence & scope

- **Audit:** machine-checked in Isabelle/HOL session Caballero_Radix (task 291): word_refinement (the word w_f of the comparisons u_a < 1 + u_b is the same for the pattern (u, v, …) at (x₁, (u+1)x₁, …) as for (F, …) at (x₁, …), for every number of further primes), with comparison_transfer and no_unit_difference_transfer. Compiles without sorry; out of scope: the realisation by primes. working script `Caballero_Radix.thy`, not published
- **Program:** k ≤ 18: the number of words of the finest pattern equals |L_k| over all patterns (exact enumeration with SageMath polyhedra over $\mathbb{Q}$, e.g. k = 12: 12, 69, 80, 252 for p^{11}, p^5p', $p^3p'^2$, $p^2p'p''$), and the one-prime words lie inside the two-prime words for k = pq ≤ 49. working scripts `formas_exactas.sage`, `tabla_pq.py`, not published

Lemma 3.2. Isabelle **(mixed radix; classical)** p. 6

> *Let $u, v \ge 1$ and $F = (u+1)(v+1) - 1$. Every integer $N \in \{0, \dots, F\}$ can be written in exactly one way as $N = i + (u+1)j$ with $0 \le i \le u$, $0 \le j \le v$. Hence, for every real x_1,*
> *$\{Nx_1 : 0 \le N \le F\} = \{ix_1 + j(u+1)x_1 : 0 \le i \le u,\ 0 \le j \le v\}$, and the map $(i, j) \mapsto i + (u+1)j$ is a bijection between the index sets.*

Isabelle · ours **show**

`radix_bij` · `Isabelle/CaballeroDyckWords/Caballero_Radix/Caballero_Radix.thy`, line 74

```
theorem radix_bij:
  "bij_betw (radix u) ({0..u} \<times> {0..v}) {0..(u + 1) * (v + 1) - 1}"
```

NEW **Classical and new:** the paper: «mixed radix; classical».

↳ Used in: Theorem 3.1 (p. 8)

Proof

> Division with remainder by $u+1$; $j = \lfloor N/(u+1) \rfloor \le v$.

Evidence & scope

- **Audit:** machine-checked in Isabelle/HOL session Caballero_Radix (task 291): radix_bij proves the bijection (i, j) ↦ i + (u+1)j from 0..u×0..v onto 0..(u+1)(v+1)−1. Compiles without sorry; the control with factor u+2 fails. working script `Caballero_Radix.thy`, not published

Corollary 3.3. p. 7

1. $L_1 = \{()\}$. *For q prime, $L_q = \{O_q(N) : 0 \le N \le q-1\}$ with $O_q(N)$ the word whose j-th letter) is preceded by $\min(q, N+j)$ letters (($j = 1, \dots, q$). Hence $|L_q| = q$.*
2. *For primes $p \le q$, L_{pq} is the set of words of pattern $(p-1, q-1)$. So $|L_{pq}| = W(p-1, q-1)$, where $W(a,b)$ denotes the number of words of pattern (a,b).*

↳ Used in: Theorem 8.12 (p. 34), Theorem 9.4 (p. 40)

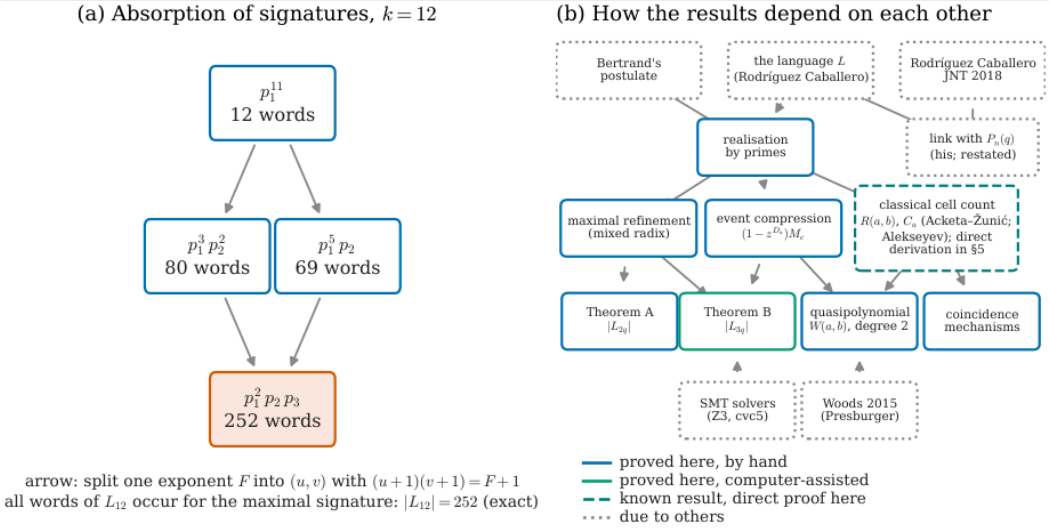
Proof

(b) is Theorem 3.1. For (a), let $m = p^{q-1}$ and $\Lambda = \log t / \log p \notin \mathbb{Z}$. In units of $\log p$, the letters (sit at $0, \dots, q-1$ and the letters) at $\Lambda, \dots, \Lambda + q - 1$. The j-th) (at $\Lambda + j - 1$, $j \ge 1$) is preceded by $\#\{i \le q-1 : i < \Lambda + j - 1\} = \min(q, \lfloor \Lambda \rfloor + j)$ letters (, so the word is $O_q(\min(\lfloor \Lambda \rfloor, q-1))$. The words $O_q(N)$, $0 \le N \le q-1$, are pairwise distinct: the first) is preceded by $N+1$ letters (. Each is realised, by Lemma 2.3 applied to $x^0 = 1/(N + \frac{1}{2})$.

Evidence & scope

- **Program:** the lab «Figure 1 live» computes N and checks O_q(N) against the word of n for one-prime odd parts (JavaScript, BigInt; equal to the Python version on 32 values of n, labjs/control_js_python.js). `labjs/nucleo.js`, `lab/dyck_divisors_lab.py`

Figure 4



(a) Maximal refinement for $k = 12$: each arrow splits one exponent F into (u, v) with $(u+1)(v+1) = F+1$, and every word of the coarser pattern is a word of the finer one. The counts 12, 69, 80 and 252 of words of the patterns p^{11}, p^5p', $p^3p'^2$ and $p^2p'p''$ are exact enumerations (SageMath over $\mathbb{Q}$), so $|L_{12}| = 252$. (b) How the results depend on each other. Boxes in dotted grey are due to others (Bertrand's postulate, the language L and the link with $P_n(q)$ of Rodríguez Caballero, Woods' theorem) or are external tools (the SMT solvers). The dashed teal box is a known result with a direct proof here (the cell count of Section 5, classical: Acketa–Žunić, Alekseyev). The colour of the frame of the other boxes gives the status of our results (proved by hand, or computer-assisted).

4 Decidability and complexity

Proposition 4.1. p. 8

> *Let $\mathcal{A}$ be an arrangement of hyperplanes $\alpha \cdot x = 1$ in the positive orthant of $\mathbb{R}^r$, with $\alpha \in \mathbb{Z}^r$ and $|\alpha_i| \le M$ ($M \ge 1$). Every open cell of $\mathcal{A}$ contains a point with $0<x_i<r!\,M^{r-1}+1$ for all i.*

Trust base: ⚠️ not checked. Written proof; not machine-checked. Table 1, which this bound makes exact, is an exact enumeration in rational arithmetic (SageMath polyhedra over $\mathbb{Q}$ for three or more primes, exact fractions for two), not a formula

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** the whole statement

↳ Used in: Corollary 4.2 (p. 12), 10.1 (p. 42)

Proof

> The closure P of a cell U is a full-dimensional polyhedron in the closed orthant, hence pointed; let v be a vertex. At v there are r linearly independent active constraints $Av = d$, each a hyperplane $\alpha \cdot x = 1$ or a face $x_i = 0$. So A is an integer matrix with entries of absolute value at most M, and $d \in \{0,1\}^r$. By Cramer's rule $v_i = \det A_i / \det A$ with $|\det A| \ge 1$, and expanding $\det A_i$ along the column d gives $|\det A_i| \le r!\,M^{r-1}$. Points of U arbitrarily close to v satisfy the bound.

Evidence & scope

- **By hand:** the bound for the four maximal patterns of Table 1, (1,1,1), (2,1,1), (1,1,1,1), (2,2,1): 7, 25, 25, 25, all below the box x_i ≤ 50 of the enumerations (paper, last section of Part III)
- **Program:** Table 1 (exact enumeration): k = 4, 6, 8, 9, 10: 6, 16, 40, 28, 47 and k = 12: 252, 16: 1083 (patterns 15; 7,1; 3,3; 3,1,1; 1,1,1,1: 16, 127, 95, 682, 1083), 18: 1222 (17; 8,1; 5,2; 2,2,1: 18, 163, 200, 1222), SageMath polyhedra over $\mathbb{Q}$ in the box x_i ≤ 50. working script `formas_exactas.sage`, not published
- **Program:** Table 1 (exact enumeration): two primes by exact fractions: k = 4: 6, 6: 16, 9: 28, 10: 47, 14: 96, 15: 136, 21: 288, 25: 255, 35: 971, 49: 1083. working script `tabla_pq.py`, not published
- **Program:** Table 1 (exact enumeration): the lab recount: |L_4|, |L_6|, |L_9|, |L_10|, |L_14|, |L_15|, |L_21|, |L_25|, |L_35| = 6, 16, 28, 47, 96, 136, 288, 255, 971. `lab/comprueba_lab.py`

Corollary 4.2. (decidability) p. 8

> *Membership in L is decidable. The empty word is not in L, and $L_1 = \{()\}$. Given a balanced word w of semilength $k \ge 2$, let $k = q_1 \cdots q_r$. Then $w \in L$ if and only if w is the word of some cell of the arrangement of the pattern $(q_1 - 1, \ldots, q_r - 1)$, and these finitely many cells can be listed exactly.*

Trust base: ⚠️ not checked. Written proof (audited); not machine-checked

Proof: in this note · **Covers:** not formalized

> 🆕 **Classical and new:** the paper: it answers algorithmically the decision problem posed by Rodríguez Caballero in 2020 (FOM), not his question about first-order Peano arithmetic.

↳ Used in: 4.3 (p. 13), 11.7 (p. 44)

Proof

By Theorem 3.1, L_k is the set of words of the cells of $\mathcal{A}_{(q_1-1,\dots,q_r-1)}$, and every cell is realised (Lemma 2.3). The arrangement is finite. Each sign vector of the hyperplanes and of the coordinate faces defines a strict rational linear system, whose feasibility is decided exactly, and a feasible system yields a rational point and hence the word of its cell. The search can be confined to the box of Proposition 4.1.

Remark 4.3. (complexity) p. 8

Let $k = q_1 \cdots q_r$, so $r \le \log_2 k$. The maximal pattern has $f_i = q_i - 1$, and its arrangement has at most $H = \prod_i (2q_i - 1) - 1 \le k^{\log_2 3}$ hyperplanes, since $2q - 1 \le q^{\log_2 3}$ for $q \ge 2$. An arrangement of H hyperplanes in $\mathbb{R}^r$ has at most $\sum_{j \le r} \binom{H}{j} \le (H+1)^r$ cells, and they can be listed incrementally with one rational linear program per candidate. So membership is decided in time $\exp\big(O((\log k)^2)\big)$, and in time polynomial in k when the number r of prime factors of k (with multiplicity) is bounded. Whether a uniformly polynomial algorithm exists is Question 11.7.

↳ Not used by another result of the paper.

5 The number of cells

Theorem 5.1. p. 9

For $a, b \ge 1$,

$$R(a,b) = 1 + (3a+1)b + a + \frac{1}{2}\sum_{q=1}^{2a}\Big[(2a+1-q)\,S_q(b) - B_q(a,b)\Big],$$

where

$$S_q(b) = \sum_{\substack{1 \le p \le 2b \\ \gcd(p,q)=1}} (2b - p + 1),$$

$$B_q(a,b) = \sum_{\substack{-a \le h \le a-q \\ q \mid h}} \Phi_q\Big(\Big\lfloor \frac{b}{\max(|h/q|,\,|h/q+1|)} \Big\rfloor\Big) = 2\sum_{k=1}^{\lfloor a/q \rfloor} \Phi_q\big(\lfloor b/k \rfloor\big).$$

The maximum in B_q is never 0, since h/q and $h/q+1$ are integers that do not both vanish; $B_q = 0$ for $q > a$.

Trust base: ⚠ not checked. Written proof, audited internally 25-nov (adversarial audit: survives); not machine-checked; the formula is checked against an exact independent cell enumeration

Proof: in this note · **Covers:** not formalized

> 🆕 **Classical and new:** known result, direct proof here (version 2): by Proposition 5.3, R(a,b) = (N(2b,2a) + 4a + 4b + 6)/8, where N is Alekseyev's exact count of two-dimensional threshold functions on a grid (SIAM J. Discrete Math. 24 (2010), Theorem 3); the linear partitions of the grid go back to Acketa–Žunić (1991). Version 1 presented the formula as new; the paper now claims only the direct proof in its coordinates.

↳ Used in: Corollary 5.2 (p. 15), Proposition 5.3 (p. 16), Lemma 9.2 (p. 38)

Proof

Exchanging the names of the summation variables, $\sum_{q=1}^{2a}(2a+1-q)S_q(b) = \sum(2a+1-q)(2b+1-p)$ over the coprime pairs $1 \le p \le 2b$, $1 \le q \le 2a$, which is $V(2b,2a)$. For the boundary terms, the second form of B_q gives

$$\frac{1}{2}\sum_{q=1}^{2a} B_q(a,b) = \sum_{q=1}^{a}\ \sum_{k=1}^{\lfloor a/q\rfloor} \Phi_q\big(\lfloor b/k\rfloor\big) = \#\big\{(q,k,p)\in\mathbb{Z}_{\ge 1}^3 :\ kq\le a,\ kp\le b,$$

because $(q,k,p)\mapsto(kq,kp)$ is a bijection onto $\{1,\dots,a\}\times\{1,\dots,b\}$, with inverse $(x,y)\mapsto(x/g,g,y/g)$, $g=\gcd(x,y)$. Theorem 5.1 then reads $R(a,b) = 1+(3a+1)b+a+\frac{1}{2}V(2b,2a) - ab = 1+a+b+2ab+\frac{1}{2}V(2b,2a)$. Since $(4b+1)(4a+1)+1+4a+4b+6 = 8(1+a+b+2ab)$, this is $(N(2b,2a)+4a+4b+6)/8$. The symmetry follows from $V(m,n)=V(n,m)$.

💡 In plain words (commentary, not part of the paper)

For two primes, $n = 2^{e-1}r^a s^b$, the word depends only on where the point $(\Lambda,\theta) = (\log t, \log r)/\log s$ lies with respect to the lines $\Lambda + h\theta = \rho$, $|h|\le a$, $|\rho|\le b$. The theorem counts the pieces (cells) into which these lines cut the quadrant: an exact formula, valid for all $a,b\ge 1$. For $a=1$ it gives $(5b+4)(b+1)/2$, and for $(a,b)=(3,3)$ it gives 191 cells, of which only 95 carry different words (Section 2).

Why it is true: adding the lines one at a time, each new line raises the number of cells by one plus the number of points where it meets earlier lines (Euler's relation). Every crossing has $\theta = p/q$ with $q\le 2a$, and the lines through one crossing form an arithmetic progression of slopes without gaps; counting the pairs $(h, h+q)$ and halving with an involution produces the Euler-function sums S_q and the correction B_q.

The count itself is classical (version 2 of the paper): by Proposition 5.3, $R(a,b) = (N(2b,2a)+4a+4b+6)/8$, where N is Alekseyev's exact count of two-dimensional threshold functions on a grid, and the linear partitions of the grid go back to Acketa and Žunić. What the theorem adds is a direct proof in the coordinates of the paper. Counting cells through vertices is the planar case of Zaslavsky's face-count formula. The formula counts cells, not words: $W(a,b)\le R(a,b)$, with equality failing exactly where cells coincide (Section 7).

🛡️ Evidence & scope

- **Program:** block T1: direct exact vertex count (fractions) for a ≤ 5, b ≤ 9 (number of lines, no gaps in the progressions, N_q = (T_q − B_q)/2, Euler = formula), and the formula = an independent exact enumeration in 56 cases up to a = 6, b = 4; controls (without B_q, without ½, Euler with m_v, all lines) fail. working scripts `teoria_general.py`, `independiente.py`, not published

- **Program:** the formula against R(1,b), R(2,b) (Lemmas 8.3, 9.2) for b < 40 and the exact row a = 3 for b ≤ 110. working script `formula_R_v369.py`, not published
- **Program:** the lab's exact recount = formula for a ≤ 4 (a = 1: b ≤ 20, a = 2: b ≤ 14, a = 3: b ≤ 9, a = 4: b ≤ 7); 191 cells for (3,3), 38 for (1,3); control without B_q fails. `lab/comprueba_lab.py`

Corollary 5.2. p. 11

1. *For each $a \geq 1$, $b \mapsto R(a,b)$ is a quasipolynomial of degree 2 on all of $b \geq 1$ (no threshold), with a period dividing $\operatorname{lcm}(1,2,\ldots,2a)$, and*

$$R(a,b) = C_a b^2 + O_a(b), \qquad C_a = \sum_{q=1}^{2a} (2a+1-q)\frac{\varphi(q)}{q}.$$

   So $C_1,\ldots,C_4 = 5/2,\ 22/3,\ 73/5,\ 1709/70$. The leading coefficient is C_a in every residue class. Equivalently $C_a = (2a+1)\Psi(2a) - \sum_{q\leq 2a}\varphi(q)$, with $\Psi(x) = \sum_{q\leq x}\varphi(q)/q$.

2. $C_a = \dfrac{12}{\pi^2}\,a^2 + O(a\log a)$ *as* $a \to \infty$.

Trust base: ⚠️ not checked. Written proof (audited 25-nov); not machine-checked

Proof: in this note · **Covers:** not formalized

🆕 **Classical and new:** known result, direct proof here (version 2): C_a = (2a+1)Ψ(2a) − Σ_{q≤2a} φ(q) is the leading coefficient of Alekseyev's Theorem 25 with m = 2b, n = 2a, and C_a ~ 12a²/π² follows from the asymptotics of Ψ and Φ recalled there (Remark 5.4). The quasipolynomial structure in b (no threshold, period bound) is an elementary consequence, not stated by Alekseyev.

↳ Used in: 5.4 (p. 16), Theorem 6.3 (p. 19), Corollary 6.4 (p. 21), 11.1 (p. 43), 11.2 (p. 43), 11.3 (p. 43), 11.4 (p. 43)

Proof

(a) Write $\Phi_q(x) = \frac{\varphi(q)}{q}x + \pi_q(x)$ for integers $x \geq 0$. Since $\Phi_q(x+q) = \Phi_q(x) + \varphi(q)$, π_q is q-periodic, and $|\pi_q| \leq \varphi(q)$. Then $S_q(b) = \sum_{y=1}^{2b}\Phi_q(y) = \frac{\varphi(q)}{q}\,b(2b+1) + \sum_{y=1}^{2b}\pi_q(y)$. The last sum is $2b\,\bar{\pi}_q + ($ a $q-$*periodic function of* 2b), where $\bar{\pi}_q$ is the mean of π_q over a period. So S_q is a quasipolynomial in b of degree 2, leading coefficient $2\varphi(q)/q$, and period dividing q (in fact $q/\gcd(q,2)$). Next, $g(b) = \Phi_q(\lfloor b/k\rfloor)$ satisfies $g(b+kq) = g(b) + \varphi(q)$, because $\lfloor (b+kq)/k\rfloor = \lfloor b/k\rfloor + q$; so $g(b) - \frac{\varphi(q)}{kq}b$ has period kq, and g is a quasipolynomial of degree ≤ 1 with period dividing $kq \leq a$. Every term of Theorem 5.1 is therefore a quasipolynomial for all $b \geq 1$ with period dividing $\operatorname{lcm}(1,\ldots,2a)$, and the leading coefficient is $\frac{1}{2}\sum_q (2a+1-q)\cdot 2\varphi(q)/q = C_a$; the terms $B_q \leq 2\lfloor a/q\rfloor b$ and $(3a+1)b$ are $O_a(b)$. The proof gives the slightly finer period $\operatorname{lcm}\big(\operatorname{lcm}(1,\ldots,a),\ \text{odd } q \leq 2a\big)$. (Numerically, the minimal periods for $a = 1,\ldots,5$ are $1, 3, 15, 105, 105$; we do not prove them.)

The second expression for C_a is $\sum_q (2a+1-q)\varphi(q)/q = (2a+1)\sum_q \varphi(q)/q - \sum_q \varphi(q)$.

(b) Since $\#\{x : q \leq x \leq 2a\} = 2a+1-q$, we have $C_a = \sum_{x=1}^{2a}\Psi(x)$. By $\varphi(q)/q = \sum_{d\mid q}\mu(d)/d$,

$$\Psi(x)=\sum_{d\le x}\frac{\mu(d)}{d}\left\lfloor\frac{x}{d}\right\rfloor=x\sum_{d\le x}\frac{\mu(d)}{d^2}+O\Big(\sum_{d\le x}\frac{1}{d}\Big)=\frac{6}{\pi^2}x+O(1+\log x),$$

using $\sum_d \mu(d)/d^2 = 6/\pi^2$ and $\sum_{d>x} d^{-2} = O(1/x)$. Summing over $x \le 2a$, $C_a = \frac{6}{\pi^2}\cdot\frac{2a(2a+1)}{2} + O(a\log a) = \frac{12}{\pi^2}a^2 + O(a\log a)$.

Proof

Evidence & scope

- **Program:** block T2: C_1 … C_8 exact; for a ≤ 5 vanishing third differences of step lcm(1..2a) and second differences 2C_a L² in every class (control with φ(q) replaced by q fails); minimal periods 1, 3, 15, 105, 105 observed, not proved; growth of (C_a − 12a²/π²)/(a log a) for a ≤ 2000. working script `teoria_general.py`, not published

Proposition 5.3. p. 12

For $a, b \ge 1$,

$$R(a,b) = 1 + a + b + 2ab + \tfrac{1}{2}V(2b,2a) = \frac{N(2b,2a)+4a+4b+6}{8}.$$

In particular $R(a,b) = R(b,a)$, and $R(a,b)$ is the count of Alekseyev [16] after the affine rescaling $N \mapsto (N+4a+4b+6)/8$.

Trust base: ⚠️ not checked. Two written proofs (algebraic from Theorem 5.1; geometric from Alekseyev's Theorem 3); not machine-checked; the identity is checked in exact arithmetic

Proof: in this note · **Covers:** not formalized

NEW **Classical and new:** the identity is a direct consequence of Alekseyev's Theorem 3 (SIAM J. Discrete Math. 24 (2010)); the paper claims no novelty for the count. The second proof (the cells of the quadrant are the regions of Alekseyev's arrangement in one octant; deletion–restriction for the two extra planes) was not found in Alekseyev's paper (read in full) nor in OEIS A335678, A115009, A114999, A306302; it is elementary. Added in version 2.

↳ Used in: 5.4 (p. 16), 5.5 (p. 17)

Evidence & scope

- **Program:** Σ_q B_q(a,b) = 2ab for a ≤ 12, b ≤ 40 (480 cases); R(a,b) of Theorem 5.1 = 1 + a + b + 2ab + V(2b,2a)/2 = (N(2b,2a) + 4a + 4b + 6)/8 for a ≤ 8, b ≤ 30 (240 cases, 0 failures); the cells of the drawing of K(m,n) recounted exactly (Euler) equal the first 28 terms of OEIS A335678, and (A335678(2a+1,2b+1) + 2)/2 = R(a,b) for a, b ≤ 3; controls (Σ B_q = 2ab + 1, (T+1)/2, constant 5 instead of 6) fail in every case. working script `comprobar_v4_alekseyev.py`, not published
- **Program:** Remark 5.5 (v2 after review): Alekseyev's OEIS comment T(m,n) = V(m−1,n−1) + mn − 1 and Theorem 13 of Alekseyev–Basova–Zolotykh (2015): the cells of the triangle cut by the lines x1 u + x2 v = 1 recounted exactly equal f1/4 + (m+n)/2 = V(m−1,n−1) + mn, and the drawing of K(m,n) has one cell less, for 2 ≤ m, n ≤ 7; T(7,7) = 380, R(3,3) = 191; controls fail. working script `comprobar_v2_V376.py`, not published

Remark 5.4. (the leading term) p. 13

For $m \geq n$, Alekseyev also proves $N(m,n) = 2\big((n+1)\Psi(n) - \sum_{q \leq n} \varphi(q)\big)m^2 + O(mn^3)$ [16 , Theorem 25] . With $m = 2b$, $n = 2a$ and Proposition 5.3, this is $R(a,b) = C_a b^2 + O(ba^3)$ with $C_a = (2a+1)\Psi(2a) - \sum_{q \leq 2a} \varphi(q)$, the second expression of Corollary 5.2(a). Part (b) of that corollary also follows from the asymptotics of Ψ and of $\sum_{q \leq x} \varphi(q)$ recalled there. For the asymptotics of the number of threshold functions when both sides of the grid grow, see Haukkanen and Merikoski [20] , who improve the error terms of [17] and [16 , Theorem 25] . The quasipolynomial structure of $b \mapsto R(a,b)$ (no threshold, the bound on the period) is not stated in [16] ; it follows from the exact formula by the elementary argument of Corollary 5.2(a). Nor did we find there, or in the OEIS entries cited, the linear term of $R(a,b)$ for fixed a.

↳ Not used by another result of the paper.

Remark 5.5. (the drawing of $K_{m,n}$) p. 13

Extend V to zero arguments by the empty sum, $V(m,0) = V(0,n) = 0$. Let $T(m,n)$ be the number of bounded regions of the figure formed by m equally spaced points on a line, n equally spaced points on a parallel line, and the segments joining each point of one line to each point of the other (sequence A335678 of [25]). In a comment of 28 June 2020 recorded in that entry, Alekseyev gives $T(m,n) = V(m-1,n-1) + mn - 1$ for $m,n \geq 1$, with the following justification. Place the points at $(i,0)$ and $(j,1)$, $0 \leq i<m$, $0 \leq j<n$. The projective map $(x,y) \mapsto (u,v) = \big((1-y)/(x+1),\ y/(x+1)\big)$ sends the segment from $(i,0)$ to $(j,1)$, which lies on $x = i(1-y) + jy$, onto the segment of the line $(i+1)u + (j+1)v = 1$ between the two axes. So it sends the figure to the partition of the right triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ by the lines $x_1 u + x_2 v = 1$, $(x_1,x_2) \in \{1,\dots,m\} \times \{1,\dots,n\}$, except for one cell. The image of the convex hull of the figure is $\{u,v \geq 0,\ u+v \leq 1,\ mu+nv \geq 1\}$, and no line of the partition meets the region $u,v>0$, $mu+nv<1$ (there $x_1 u + x_2 v \leq mu + nv<1$); so that region is a single triangular cell of the partition with no counterpart in the figure, which is the -1. For $m,n \geq 2$, Alekseyev, Basova and Zolotykh count the cells of the partition as $c(m,n) = f_1(m,n)/4 + (m+n)/2$ [19 , Theorem 13] , where $f_1(m,n)$ is the sum of $(m-|i|)(n-|j|)$ over $-m<i<m$, $-n<j<n$, $\gcd(i,j) = 1$. The four open quadrants contribute $4V(m-1,n-1)$ to $f_1(m,n)$, and the two axes, where the only primitive vectors are $(\pm 1,0)$ and $(0,\pm 1)$, contribute $2(m-1)n + 2m(n-1)$. Hence

$$f_1(m,n) = 4V(m-1,n-1) + 4mn - 2m - 2n, \qquad c(m,n) = V(m-1,n-$$

and $T(m,n) = c(m,n) - 1$. For $m = 1$ (or $n = 1$) the figure is a fan of n (or m) segments, and $T = n-1$ (or $m-1$) is the same formula with $V(0,\cdot) = V(\cdot,0) = 0$. For example $V(3,3) = 31$, $f_1(4,4) = 172$, $c(4,4) = 47$ and $T(4,4) = 46$. With the first expression of Proposition 5.3 this gives

$$R(a,b) = \frac{T(2a+1,2b+1)+2}{2}.$$

As an independent check, we counted exactly (rational arithmetic, Euler's relation) the cells of the figure and of the partition of the triangle for $2 \le m, n \le 7$, and the figure for $m = 1$ or $n = 1$, which agree with these formulas, and the identity for $a, b \le 3$ (for instance $T(7,7) = 380$ and $R(3,3) = 191$).

↳ Not used by another result of the paper.

6 Quasipolynomial growth for every fixed exponent

Theorem 6.1. p. 14

> *For every fixed $a \ge 1$ there are N_a, $P_a \ge 1$ and polynomials $Q_{a,0}, \dots, Q_{a,P_a-1}$ of degree at most two such that*
>
> $$W(a,b) = Q_{a,\, b \bmod P_a}(b) \qquad \text{for all } b \ge N_a.$$
>
> *In particular $\sum_{b\ge 1} W(a,b)\, t^b = A_a(t)/(1-t^{P_a})^3$ for some polynomial $A_a(t)$.*

Isabelle · ours · part show

Only a part is in Isabelle: the algebraic part of Step 3 (telescoping of progressions, cancellation of 1 − z^D in ℤ[z], multisets ↔ generating polynomials); the Presburger encoding and Woods' theorem are not formalized

`events_progression` · Isabelle/CaballeroDyckWords/Caballero_Eventos/Caballero_Eventos.thy, line 60

```
theorem events_progression:
  assumes "s \<ge> 1" "s dvd D"
  shows "(1 - zp D) * (\<Sum>n<N. zp (u + s * n))
       = (\<Sum>\<rho><D div s. zp (u + s * \<rho>) - zp (u + s * N + s * \
<rho>))"
```

`events_constant` · Isabelle/CaballeroDyckWords/Caballero_Eventos/Caballero_Eventos.thy, line 88

```
theorem events_constant:
  "(1 - zp D) * (of_nat N * zp u) = of_nat N * zp u - of_nat N * zp (u +
D)"
```

`events_cancel` · Isabelle/CaballeroDyckWords/Caballero_Eventos/Caballero_Eventos.thy, line 94

```
theorem events_cancel:
  assumes "D \<ge> 1" and "(1 - zp D) * M = (1 - zp D) * (M' :: int poly)"
  shows "M = M'"
```

`genpoly_eq_iff` · Isabelle/CaballeroDyckWords/Caballero_Eventos/Caballero_Eventos.thy, line 112

```
theorem genpoly_eq_iff: "genpoly A = genpoly B \<longleftrightarrow> A = B"
```

Trust base: ⚠️ not checked. Written proof, audited internally 25-nov (adversarial audit: survives); rests on Woods' theorem (J. Symbolic Logic 2015); the algebraic identities of Step 3 compile in Isabelle/HOL (session Caballero_Eventos, public)

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the algebraic part of Step 3 (telescoping of progressions, cancellation of 1 − z^D in ℤ[z], multisets ↔ generating polynomials); the Presburger encoding and Woods' theorem are not formalized

🆕 **Classical and new:** the paper: «All the depth is in Woods' theorem; encoding a counting problem in Presburger arithmetic to obtain quasipolynomiality is a standard route. The only point specific to this problem is that equality of words … admits an encoding whose size is bounded in terms of a». Existence only: period and threshold not determined.

↳ Used in: Theorem 6.3 (p. 19), Corollary 6.4 (p. 21), Lemma 7.1 (p. 21), Lemma 7.5 (p. 25), 11.1 (p. 43), 11.4 (p. 43), 11.6 (p. 44)

💡 **In plain words (commentary, not part of the paper)**

Fix the exponent a of the first prime. Then the number of words $W(a,b)$ is, from some b on, given by finitely many quadratic polynomials in b that repeat with a period: for $a=1$ and $a=2$ Theorems A and B give them, with periods 2 and 6. For $a\geq 3$ the theorem only says that such polynomials exist; it does not give the period, the threshold or the coefficients (for $a=3$ the paper has a fitted formula of period 60, a conjecture, Question 11.1).

Why it is true: the word is fixed by a vector c of $2a+1$ integers, whether c occurs is a linear condition, and whether two vectors give the same word can be written with a number of conditions bounded in terms of a alone (multiplying the generating polynomial by $1-z^{D_a}$ compresses each progression into a few "events"). So $W(a,b)$ is a Presburger counting function, and a theorem of Woods makes it eventually a quasipolynomial. The degree is at most two because $W\leq R=O(b^2)$; Theorem 6.3 shows it is exactly two.

The paper is explicit that all the depth is in Woods' theorem and that this encoding is a standard route; what is specific is the bounded encoding of equality of words.

🛡️ **Evidence & scope**

- **Audit:** machine-checked in Isabelle/HOL session Caballero_Eventos (task 293), the algebraic part of Step 3: events_progression ((1 − z^D)·Σ_n<N z^(u+sn) telescopes to D/s pairs of monomials when s | D), events_constant (slope 0), events_cancel (1 − z^D cancels in ℤ[z]) and genpoly_eq_iff (a finite multiset of naturals is determined by its generating polynomial). Compiles without sorry; controls (D not a multiple of s, a shifted exponent) fail. Out of scope: the Presburger encoding and Woods' theorem. working script `Caballero_Eventos.thy`, not published
- **Literature:** K. Woods, Presburger arithmetic, rational generating functions, and quasi-polynomials, J. Symbolic Logic 80 (2015), Def. 1.6 and Thm. 1.10 (numbering of arXiv:1211.0020, as the paper says)

Remark 6.2. p. 16

The proof gives no useful bound on N_a or P_a, and does not use Theorems A and B. What makes it work is that the multiset $Z(c)$, whose size grows with b, is described by a number of events bounded in terms of a alone. This is the same compression that makes the certificate of Section 9.2 independent of b. Whether the leading coefficient of $W(a,b)$ is the same on every residue class is open (Question 11.2).

↳ Not used by another result of the paper.

Theorem 6.3. p. 16

For all $a, b \geq 1$,

$$W(a,b) \;\geq\; N(b) := \#\big\{(u,v) \in [0,b-1]^2 :\; 2u+v \geq b,\; u+2v \geq b\big\} \;\geq\; \lfloor 2b/3 \rfloor^2,$$

and $N(b) = \sum_{u=0}^{b-1} \min\big(2u, \lfloor (b+u)/2 \rfloor\big) = \frac{2}{3}b^2 + O(b)$. Hence $W(a,b) = \Theta_a(b^2)$; more precisely $\big(\frac{2}{3} - o(1)\big)b^2 \leq W(a,b) \leq \big(C_a + o(1)\big)b^2$, where the lower constant does not depend on a.

Trust base: ⚠️ not checked. Written proof (audited 25-nov); not machine-checked

Proof: in this note · **Covers:** not formalized

↳ Used in: Corollary 6.4 (p. 21), 6.5 (p. 21), 11.2 (p. 43)

Proof

Fix (u,v) in the set and put $\Lambda = u + \frac{1}{3}$, $\theta = u + v + \frac{2}{3}$; both are positive. Then $\Lambda + h\theta = u + h(u+v) + \frac{1+2h}{3}$.

- $h = 0$: $\Lambda \in (u, u+1)$ and $0 \leq u \leq b-1$, so $c_0 = u$.
- $h = -1$: $\Lambda - \theta = -v - \frac{1}{3} \in (-v-1, -v)$ and $-b \leq -v - 1$, so $c_{-1} = -v - 1$.
- $h \geq 1$: $\Lambda + h\theta \geq \Lambda + \theta = 2u + v + 1 > b$, so $c_h = b$.
- $h \leq -2$: $\Lambda + h\theta \leq \Lambda - 2\theta = -u - 2v - 1 < -b$, so $c_h = -b - 1$.

The point lies on no line of $\mathcal{L}_{a,b}$: $\Lambda + h\theta$ is an integer only when $3 \mid 1 + 2h$, that is $h \equiv 1 \pmod 3$, and then $h \geq 1$ or $h \leq -2$, where $|\Lambda + h\theta| > b$ by the last two items. So the vector c above is realisable (its cell is open; by Lemma 2.3 it is realised by primes, after moving the point off the lines $\Lambda + h\theta \in \mathbb{Z}$ with $|\rho| > b$, which does not change c).

Now compute $Z(c)$. For $0 \leq i' \leq a$, the values $h = i' - i$ ($0 \leq i \leq a$) include i' values $h \geq 1$ ($f = b+1$), the value $h = 0$, and the value $h = -1$ exactly when $i' < a$; the others have $f = 0$. Since $f(-v - 1 + j') = \max(0, j' - v)$ for $0 \leq j' \leq b$,

$$H_{i'}(j') = i'(b+1) + \min(b+1,\; u + j' + 1) + [i' < a]\max(0,\; j' - v).$$

The minimum of Z is $H_0(0) = u + 1$: H_0 is nondecreasing in j', and $H_{i'} \geq b + 1 > u + 1$ for $i' \geq 1$. The sum of Z is $F(u) + a\binom{b-v+1}{2}$, where $F(u)$ does not depend on v; since $a \geq 1$ and $b - v \geq 1$, it is strictly decreasing in v. Both are functions of the word. So the word determines u, then v, and the $N(b)$ points give $N(b)$ distinct words.

Counting: for fixed u, v ranges over $[\max(b - 2u, \lceil (b-u)/2 \rceil),\; b-1]$, which has $\min(2u,\; b - \lceil (b-u)/2 \rceil) = \min(2u, \lfloor (b+u)/2 \rfloor)$ elements. Replacing the summand by $\min(2u, (b+u)/2)$ changes it by less than 1, and $\sum_{u<b} \min(2u, (b+u)/2) = \int_0^{b/3} 2u\,du + \int_{b/3}^{b} \frac{b+u}{2}\,du + O(b) = \frac{b^2}{9} + \frac{5b^2}{9} + O(b)$. The square $[\lceil b/3 \rceil, b-1]^2$ lies in the set (there $2u + v \geq 3\lceil b/3 \rceil \geq b$), which gives $N(b) \geq (b - \lceil b/3 \rceil)^2 = \lfloor 2b/3 \rfloor^2$. The upper bound is $W \leq R$ and Corollary 5.2.

Evidence & scope

- **Program:** block T3: for a ≤ 3, b ≤ 12 and a = 4, b ≤ 9 (45 cases) every point is off the lines, its vector and its word (read directly from the positions) are the predicted ones and the N(b) words are pairwise distinct; controls: on the full square the 253 extra points lie on a line, and min Z alone leaves 369 pairs unseparated. working script `teoria_general.py`, not published

Corollary 6.4. p. 17

> *In Theorem 6.1 every constituent $Q_{a,i}$ has degree exactly 2, and its leading coefficient lies in $[\frac{2}{3}, C_a]$.*

Trust base: ⚠ not checked. Written proof (audited 25-nov); not machine-checked

Proof: in this note · **Covers:** not formalized

↳ Not used by another result of the paper.

Proof

> Along $b \equiv i \pmod{P_a}$, $b \geq N_a$, we have
> $\frac{2}{3}b^2 - O(b) \leq Q_{a,i}(b) \leq C_a b^2 + O_a(b)$.

Remark 6.5. p. 17

The argument uses only the constraints $0 \leq u, v \leq b-1$, $2u+v \geq b$ and $u+2v \geq b$. Restricted to the square $[\lceil 2b/3 \rceil, \lfloor 3b/4 \rfloor]^2$, which lies in this set, it already gives $W(a,b) \geq n_b^2$ for every $b \geq 1$, with $n_b = \lfloor 3b/4 \rfloor - \lceil 2b/3 \rceil + 1 \geq (b-5)/12$; among $b \leq 5$ one has $n_b = 0$ exactly for $b \in \{1, 2, 5\}$. The larger set gives the constant $2/3$. For $a = 1, 2$ the true leading coefficients are $9/4$ and $22/3$, so the constant $\frac{2}{3}$ is far from sharp.

↳ Not used by another result of the paper.

7 Three mechanisms of coincidence

Lemma 7.1. p. 19

> *Let $a, b \geq 1$. Then $\Phi(O(0)) = \{\varphi(\theta) : \theta \in (0, \infty) \setminus F_{a,b}\}$, and*
>
> $$|\Phi(O(0))| = 1 + \sum_{h=1}^{a} \#\{1 \leq j \leq b : \ \gcd(j,h) = 1\}.$$

Trust base: ⚠ not checked. Written proof, audited 25-nov (the three lemmas of Section 7 survive); not machine-checked

Proof: in this note · **Covers:** not formalized

↳ Used in: 7.6 (p. 26), 11.5 (p. 44)

Proof

Step 1: when is the word $O(0)$. Fix (Λ,θ) off the lines and sort P as $p_1 \le \cdots \le p_K$. Then $N(p_k+\Lambda) \ge k$, and $N(p_k+\Lambda)$ is nondecreasing in k, so it is the k-th smallest element of Z. Hence $Z=\{1,\dots,K\}$ if and only if $N(p_k+\Lambda)=k$ for every k, that is $p_{k+1} \ge p_k+\Lambda$ for $k<K$. Equality is excluded: $p_{k+1}-p_k=\Lambda$ reads $\Lambda+h\theta=\rho$ with $|h|\le a$, $|\rho|\le b$. The differences of two positions with distinct indices are the numbers $\pm(h\theta-j)$ with $(h,j)\in\{0,\dots,a\}\times\{-b,\dots,b\}\setminus\{(0,0)\}$, all of which occur. So, by (2), the word at (Λ,θ) is $O(0)$ if and only if $\Lambda<g(\theta)$, where

$$g(\theta)=\min\bigl\{|h\theta-j|:\ 0\le h\le a,\ |j|\le b,\ (h,j)\ne(0,0)\bigr\}.$$

Coincident positions give $g(\theta)=0$. For $\theta>0$ we have $g(\theta)=0$ if and only if $h\theta=j$ with $h\ge 1$ and $1\le j\le b$, that is $\theta\in F_{a,b}$ (the terms with $h=0$ are $|j|\ge 1$, since $b\ge 1$).

Step 2: the vector on this region. Let $0<\Lambda<g(\theta)$. Then $\Lambda<1$, so $c_0=0$. Let $1\le h\le a$ and $m=\lfloor h\theta\rfloor$.

- $h\theta<b$: then $m+1\le b$ and $\Lambda<m+1-h\theta$, so $\Lambda+h\theta\in(m,m+1)$ and $c_h=m$. If $h\theta>b$: $\Lambda+h\theta>b$ and $c_h=b$. ($h\theta=b$ is excluded, since $b/h\in F_{a,b}$.) In both cases $c_h=\min(b,m)$.
- $h\theta<b+1$, so $m\le b$: if $m\ge 1$, then $\Lambda<h\theta-m$; if $m=0$, then $\Lambda<h\theta$ (the term $(h,0)$ of g). Either way, since also $\Lambda>0$ and $h\theta<m+1$, we get $-m-1<-h\theta<\Lambda-h\theta<-m$, so $\Lambda-h\theta\in(-m-1,-m)$ and $\lfloor\Lambda-h\theta\rfloor=-m-1\ge -b-1$, so $c_{-h}=-\min(b,m)-1$. If $h\theta\ge b+1$: $\Lambda-h\theta<1-(b+1)=-b$, so $c_{-h}=-b-1=-\min(b,m)-1$.

So $c(\Lambda,\theta)=\varphi(\theta)$ at every such point, on the lines or not. The case $\theta>b$, where every c_h with $h\ne 0$ is clipped, is included.

Step 3: the fibre. If c is realisable with word $O(0)$, take a point of its cell; by Step 1 it has $\Lambda<g(\theta)$ and $\theta\notin F_{a,b}$, and by Step 2, $c=\varphi(\theta)$. Conversely let $\theta_0\notin F_{a,b}$, and let I be the component of $(0,\infty)\setminus F_{a,b}$ containing it. Since φ is constant on I and g is continuous and positive on I, the set $\{(\Lambda,\theta):\theta\in I,\ 0<\Lambda<g(\theta)\}$ is open and nonempty. The lines $\Lambda+h\theta\in\mathbb{Z}$ ($|h|\le a$) form a locally finite family, so this set contains a point off all of them. By Steps 1 and 2 that point has word $O(0)$ and vector $\varphi(\theta_0)$, which is therefore realisable.

Step 4: counting. The coordinates φ_h with $h>0$ are nondecreasing in θ (those with $h<0$ are nonincreasing, and determined by them). Let $x\in F_{a,b}$, written in lowest terms as $x=j/h$. Then $h\le a$ and $j\le b$, because any representation J/H with $H\le a$, $J\le b$ has $h\le H$ and $j\le J$. When θ crosses x, $\lfloor h\theta\rfloor$ goes from $j-1$ to $j\le b$, so φ_h increases strictly. Hence φ takes different values on different components of $(0,\infty)\setminus F_{a,b}$, and $|\Phi(O(0))|=|F_{a,b}|+1$. Writing each element of $F_{a,b}$ in lowest terms is a bijection onto the pairs (h,j) with $1\le h\le a$, $1\le j\le b$, $\gcd(j,h)=1$.

Evidence & scope

- **Program:** column L1: the size formula and the fibre φ(θ) as a set, a = 1..4, b = 1..7 (28 cases, exact cell enumeration), 0 failures; controls: dropping gcd = 1 differs in 18 of 28 cases, dropping the clip changes the set in 28 of 28. working script `colisiones_lemas.py`, not published

- **Program:** fibre sizes outside O(0) by exact enumeration (rational critical points, independent of Z(c)) for a ≤ 5, b ≤ 9: largest fibre 2 for a = 1, 3 for a = 2, 4 for a = 3, 4, 5. working scripts `fibras_v367.py`, `fibras_a4.py`, not published

Lemma 7.2. p. 19

Let $b \geq 1$, and let $\beta \geq 0$, $d \geq 0$, $\iota \in \{0,1\}$ be integers with

$$\beta + 2d + \iota \leq b - 1 \qquad \text{and} \qquad 2\beta + 2d + \iota \geq b - 1.$$

Put $\Delta = b - \beta - 2d - \iota \ (\geq 1)$ and

$$\xi = (\beta + d + \iota,\ \beta,\ \beta + 2d + 1), \qquad T(\xi) = (\nu + \Delta,\ \beta - \Delta,\ b) = (b - d,\ 2\beta +$$

Then ξ and $T(\xi)$ are distinct realisable triples and $Z(\xi) = Z(T(\xi))$.

Trust base: ⚠️ not checked. Written proof, audited 25-nov; not machine-checked

Proof: in this note · **Covers:** not formalized

↳ Used in: Corollary 7.3 (p. 24), 7.4 (p. 24), 11.5 (p. 44)

Proof

Write $\xi = (\nu, \beta, \gamma)$, $\nu' = \nu + \Delta = b - d$ and $\beta' = \beta - \Delta \geq -1$.

Realisability. For ξ take $\nu_0 = \nu$, $u = d + \iota$ and $\varepsilon = 1 - 2\iota$ in Lemma 8.2: $u + \varepsilon = d + 1 - \iota \geq 0$ and $\nu + u + \varepsilon = \beta + 2d + 1 \leq b$. For $T(\xi)$ take $\nu_0 = b - d$, $u = \nu' - \beta' = \nu - \beta + 2\Delta \geq 0$ and $\varepsilon = 1$: then $\nu_0 + u + 1 = 3b - 4d - 2\beta - \iota + 1 \geq b$, because $2\beta + 4d + 2\iota \leq 2b - 2$; so $\gamma = b$. All coordinates lie in $[-1, b]$, so nothing else is clipped.

Distinct. If $\gamma < b$ the third coordinates differ. If $\gamma = b$, then $\iota = 0$ (since $\beta + 2d + 1 + \iota \leq b$) and $\Delta = 1$, so the first coordinates differ.

Equal multisets. $Z(\nu, \beta, \gamma) = S(\nu, \beta) \uplus S(\nu, \gamma)$ and S is symmetric. Put $\tau = \nu + \beta + 2$, $\sigma = \nu + \gamma + 2$, $U = b - \nu$, $V = b - \gamma$. By (3), using $0 \leq \beta \leq \nu \leq \gamma \leq b$,

$$Z(\xi) = \mathrm{AP}(\tau; U) \uplus [\tau + 2U, 2b + 1] \uplus \mathrm{AP}(\sigma; V) \uplus [\sigma + 2V, 2b + 1] \uplus \{T\}^{\nu+\beta+2}.$$

For $T(\xi)$ we have $-1 \leq \beta' \leq \nu' \leq b$, $\nu' + \beta' = \nu + \beta$ and $b - \nu' = d = U - \Delta$, so

$$Z(T(\xi)) = \mathrm{AP}(\tau; d) \uplus [\tau + 2d, 2b + 1] \uplus [\nu' + b + 2, 2b + 1] \uplus \{T\}^{\nu+\beta+2},$$

where the second row $S(\nu', b) = [\nu' + b + 2, 2b + 1] \uplus \{T\}^{\nu'+1}$ is (3) with $p' = b$. Now $\tau + 2d + 1 = \nu + \beta + 2d + 3 = \sigma$, and $\tau + 2U = 2b + 2 + \beta - \nu \leq 2b + 2$. So the interval $[\tau + 2d, \tau + 2U - 1]$ splits into its even and odd offsets:

$$[\tau + 2d, 2b + 1] = \mathrm{AP}(\tau + 2d; \Delta) \uplus \mathrm{AP}(\sigma; \Delta) \uplus [\tau + 2U, 2b + 1],$$

and $\mathrm{AP}(\tau; d) \uplus \mathrm{AP}(\tau + 2d; \Delta) = \mathrm{AP}(\tau; U)$. After cancelling the common part it remains to show

$$\mathrm{AP}(\sigma;\Delta)\uplus[\nu'+b+2,2b+1]=\mathrm{AP}(\sigma;V)\uplus[\sigma+2V,2b+1].$$

Here $\sigma+2V=2b+1-d+\iota$ and $\nu'+b+2=2b+2-d$. If $\iota=1$, then $\Delta=V$ and $\nu'+b+2=\sigma+2V$: the two sides coincide. If $\iota=0$, then $\Delta=V+1$, the left side is $\mathrm{AP}(\sigma;V)\uplus\{\sigma+2V\}\uplus[\sigma+2V+1,2b+1]$, and $\sigma+2V=2b+1-d\le 2b+1$, so this is the right side.

Evidence & scope

- **Program:** column L2, b = 2..7: every ξ of the domain and T(ξ) are realisable cells with the same word, |domain| = 2, 4, 6, 9, 12, 16, 0 failures; controls (dropping $\beta-\Delta\ge-1$; $\Delta+1$ in place of Δ) give bad pairs. working script `colisiones_lemas.py`, not published

Corollary 7.3. p. 20

Let $b\ge 2$. The map T of Lemma 7.2 is injective, and:

1. *for $d\ge 1$, ξ is the triple I_1 ($\iota=0$) or I_2 ($\iota=1$) of Lemma 8.7 with parameters (β,d), and $T(\xi)=J(\tau,d)$ with the same $\tau=2\beta+d+2+\iota$. The pairs $\{\xi,T(\xi)\}$ with $d\ge 1$ are exactly the pairs $\{I,J(\tau,d)\}$ with $I\ne J$ of Lemma 8.7; there are $\sum_{d=1}^{\lfloor(b-1)/2\rfloor}(b-2d)=\lfloor(b-1)^2/4\rfloor$ of them;*
2. *for $d=0$, $\xi=(\beta,\beta,\beta+1)$ or $(\beta+1,\beta,\beta+1)$ (Lemma 8.6(b),(c)) and $T(\xi)=J_0(s)$ with $s=2\beta+1+\iota$; these are the b pairs of $O(s)$ with $b\le s\le 2b-1$.*

Trust base: ⚠ not checked. Written proof, audited 25-nov; not machine-checked

Proof: in this note · **Covers:** not formalized

↳ Not used by another result of the paper.

Proof

From $T(\xi)=(b-d,2\beta+2d+\iota-b,b)$ one reads d, then $2\beta+\iota$, hence β and ι by parity: T is injective. (a) The formulas of Lemma 8.7 give $I_1=(\beta+d,\beta,\beta+2d+1)$, $I_2=(\beta+d+1,\beta,\beta+2d+1)$, and $J(\tau,d)=(b-d,\tau-b+d-2,b)$ with $\tau-b+d-2=2\beta+2d+\iota-b=\beta-\Delta$. So the hypothesis $\beta-\Delta\ge-1$ is exactly $\tau\ge b-d+1$, the lower end of the J-range; the upper end $\tau\le 2b-3d+2$ holds because $\tau\le 2b-3d+1$ on the I-range. The hypothesis $\Delta\ge 1$ is automatic for I_1 ($\Delta=b-\gamma+1$) and, for I_2 ($\Delta=b-\gamma$), excludes only $\gamma=b$, the case $I_2=J$ noted in the proof of Theorem 8.12. The count is the one in that proof. (b) $J_0(s)=(b,s-1-b,b)$ for $s\ge b$, and $s-1-b=2\beta+\iota-b$. The two hypotheses read $b\le s$ and $s\le 2b-1-\iota$; as (β,ι) varies, s covers $[b,2b-1]$ once.

Evidence & scope

- **Program:** for b = 2..7 the pairs ξ, T(ξ) with d ≥ 1 are both the list of Lemma 8.7 (I ≠ J) and all two-element fibres outside the one-prime words; the d = 0 pairs give s = b..2b−1. working script `colisiones_lemas.py`, not published

Remark 7.4. (what the exchange adds) p. 21

Lemma 7.2 finds no coincidence that Lemmas 8.6 and 8.7 did not already list, and by Lemma 8.8 there is none to find. What it adds is a single affine map, $(\nu,\beta,\gamma)\mapsto(b-d,\ \beta-\Delta,\ b)$ on two sheets $\iota\in\{0,1\}$, that produces all the pairs of the family Q and the b pairs of the classes $O(s)$ with $b\le s\le 2b-1$, together with a proof that does not go through the profiles. Its domain has two free parameters, β and d, for each b; heuristically, this is why the excess for $a=1$ is quadratic. The pairs of $O(s)$ with $1\le s\le b-1$, whose partner $J_0(s)$ has $\beta\le -2$, and the class $O(0)$ are not of this form.

↳ Not used by another result of the paper.

Lemma 7.5. p. 21

Let $a\ge 2$, and let p,k be integers with $1\le p\le a-1$, $k\ge 1$ and $(p+1)k\le b$. Define $u=u^{(p,k)}$ and $v=v^{(p,k)}$ in $\mathbb{Z}^{2a+1}$ by

$$u_h=\operatorname{clip}\bigl(hk-[h<-p]\bigr),\qquad v_h=\operatorname{clip}\bigl(hk-[h>p]\bigr)\qquad(-a\le h\le a).$$

Then:

1. *u and v are realisable and $u\ne v$;*
2. *$Z(u)=Z(v)$;*
3. *the common word begins with $()^k(($ and its height is $p+1$. Hence different pairs (p,k) give different words, and none of them is $O(0)$.*

So the lemma gives $\sum_{p=1}^{a-1}\lfloor b/(p+1)\rfloor=\sum_{j=2}^{a}\lfloor b/j\rfloor$ coincident pairs, in pairwise distinct fibres different from $\Phi(O(0))$. For $a=2$ ($p=1$) they are the pairs A_k, $1\le k\le\lfloor b/2\rfloor$, of Section 9.

Trust base: ⚠️ not checked. Written proof, audited 25-nov; not machine-checked

Proof: in this note · **Covers:** not formalized

↳ Used in: Lemma 7.7 (p. 26), 7.8 (p. 27), 11.1 (p. 43), 11.5 (p. 44)

Proof

Choose $\eta>0$ with $(a+p+1)\eta<1$ and Λ with $p\eta<\Lambda<(p+1)\eta$ (for instance $\Lambda=(p+\tfrac12)\eta$); then $\Lambda+a\eta<1$. Let $\theta_\pm=k\pm\eta>0$.

(i) $\Lambda+h\theta_+=hk+(\Lambda+h\eta)$. For $h\ge -p$ we have $\Lambda+h\eta\in(0,1)$, and for $h<-p$, $\Lambda+h\eta\in(-1,0)$, because $\Lambda<(p+1)\eta\le|h|\eta$ and $|h|\eta-\Lambda<1$. So no $\Lambda+h\theta_+$ is an integer and $c(\Lambda,\theta_+)=u$. Symmetrically $\Lambda+h\theta_-=hk+(\Lambda-h\eta)$ gives $c(\Lambda,\theta_-)=v$. Both points are off the lines, so u and v are realisable. At $h=p+1\le a$: $u_{p+1}=(p+1)k\le b$, while $v_{p+1}=(p+1)k-1$, so $u\ne v$.

(ii) Use (2). The position (i,j) sits at $q\pm i\eta$ with $q=ik+j\in\mathbb{Z}$, the *level* of (i,j). For two positions of levels $q\ne q'$, $|q-q'|\ge 1$ while all the perturbations, together with Λ, are smaller than 1 in absolute value: $|(i-i')\eta\mp\Lambda|\le(a+p+1)\eta<1$. So the closing letter of (i',j'), at level q', is preceded by all positions of lower levels and by none of higher levels. Within level q' it is preceded, for θ_+, by the (i,j) with $(i-i')\eta<\Lambda$, that is $i\le i'+p$; for θ_-, by those with $i\ge i'-p$. The indices i of the positions on level q form an integer interval $I_q=[i_0,i_1]$ (the conditions $0\le i\le a$ and $0\le q-ik\le b$). The reflection $i\mapsto i_0+i_1-i$ of I_q carries $\#\{i\in I_q:i\ge i'-p\}$ to

$\#\{i \in I_q : i \le (i_0 + i_1 - i') + p\}$. So, level by level, the multisets of counts for θ_+ and θ_- coincide, and $Z(u) = Z(v)$.

(iii) Take θ_+. Levels $0, \dots, k-1$ contain only the positions $(0, q)$, since $i \ge 1$ gives $q \ge k$; level k contains $(0, k)$ and $(1, 0)$, at distance $\eta < \Lambda$. Positions of different levels are more than Λ apart, so the word begins with $()^k(($. The height of the word is the largest number of positions in a half-open window of length Λ. Such a window meets one level only, and within a level consecutive positions are η apart, so it holds at most $p+1$ of them; level pk contains the $p+1$ positions $(i, (p-i)k)$, $0 \le i \le p$ (here $pk \le b$ and $p \le a$), so the height is $p+1$. Thus the word determines k and p. It is not $O(0)$, whose height is 1.

Evidence & scope

- **Program:** column L3, a = 2..4, b = 1..7: Σ_j ⌊b/j⌋ pairs in all 21 cases, realisable with equal words, words beginning ()^k((with height p+1 and pairwise distinct, 0 failures; controls ([h ≥ p] for [h > p]; dropping (p+1)k ≤ b) fail. working script `colisiones_lemas.py`, not published

Remark 7.6. p. 21

The lemma does not say that the fibres of $u^{(p,k)}$ have exactly two elements; for $a \le 5$, $b \le 8$ they do, except in the square patterns $a = b \in \{4, 5\}$, where the pair $(p, k) = (1, 2)$ lies in a fibre of size 4 together with its image under the exchange $r \leftrightarrow s$. Together with Lemma 7.1 it gives, for $a \ge 2$, the lower bound

$$R(a,b) - W(a,b) \;\ge\; \sum_{h=1}^{a} \#\{1 \le j \le b : \gcd(j,h) = 1\} + \sum_{j=2}^{a} \left\lfloor \frac{b}{j} \right\rfloor,$$

which is linear in b. For $a = 2$ it gives $2b$, against the exact excess $E(b) \sim 43b/6$ of Theorem 9.4 ($b \ge 3$; its upper half rests on the computer-assisted Lemma 9.3).

↳ Not used by another result of the paper.

Lemma 7.7. p. 22

Let $m, q \ge 1$ be coprime and $p \ge 1$, with $(p+1)q \le a$ and $(p+1)m \le b$. Let $\eta > 0$ satisfy $(a + (p+1)q)\eta < 1/q$ (which implies $\eta < m/q$), let $pq\eta < \Lambda < (p+1)q\eta$, and put $\theta_\pm = m/q \pm \eta$. Then the vectors $c^\pm = c(\Lambda, \theta_\pm)$ are realisable and distinct, $Z(c^+) = Z(c^-)$, and the common word has height $p+1$. Explicitly,

$$c_h^+ = \operatorname{clip}\big(\lfloor hm/q \rfloor - [\,q \mid h,\ h < -pq\,]\big), \qquad c_h^- = \operatorname{clip}\big(\lfloor hm/q \rfloor - [\,q \mid h,\ h > pq$$

so the vectors do not depend on the choice of η and Λ. For $q = 1$, $m = k$, this is Lemma 7.5.

Trust base: ⚠ not checked. Written proof, audited 25-nov; not machine-checked

Proof: in this note · **Covers:** not formalized

↳ Used in: 11.5 (p. 44)

Proof

Put $\varepsilon = (a+(p+1)q)\eta < 1/q$. Note $\Lambda < \varepsilon$ and $a\eta < \varepsilon$.

Off the lines, and the vectors. $\Lambda + h\theta_+ = hm/q + (\Lambda + h\eta)$ with $|\Lambda + h\eta| < \varepsilon < 1/q$. If $q \nmid h$, then hm/q is at distance at least $1/q$ from $\mathbb{Z}$ (as $\gcd(m,q)=1$), so $\Lambda + h\theta_+$ is not an integer and its floor is $\lfloor hm/q \rfloor$. If $h = sq$, then $\Lambda + h\theta_+ = sm + (\Lambda + sq\eta)$, and $\Lambda + sq\eta \in (0,1)$ for $s \geq -p$ (since $\Lambda > pq\eta$), while $\Lambda + sq\eta \in (-1,0)$ for $s \leq -p-1$ (since $\Lambda < (p+1)q\eta \leq |s|q\eta$ and $|s|q\eta \leq a\eta < 1$). For θ_- replace $h\eta$ by $-h\eta$: the case $q \mid h$ now gives $(0,1)$ for $s \leq p$ and $(-1,0)$ for $s \geq p+1$. Both points therefore lie on no line $\Lambda + h\theta \in \mathbb{Z}$, $\theta_- > 0$ because $\eta < m/q$, and $c^{\pm}$ are as displayed, hence realisable. At $h = (p+1)q \leq a$: $c^+_h = (p+1)m \leq b$ and $c^-_h = (p+1)m - 1$, so $c^+ \neq c^-$.

Levels. The position (i,j) $(0 \leq i \leq a,\ 0 \leq j \leq b)$ is at $i\theta_\pm + j = \ell/q \pm i\eta$, where $\ell = mi + qj \in \mathbb{Z}$ is its *level*. For positions (i,j), (i',j') of levels $\ell \neq \ell'$, the difference between the closing letter of (i',j') and the opening letter of (i,j) is $(\ell' - \ell)/q + \big(\Lambda \pm (i'-i)\eta\big)$, where $|\ell' - \ell|/q \geq 1/q$ and $|\Lambda \pm (i'-i)\eta| < \Lambda + a\eta < \varepsilon < 1/q$. So by (2) the count of the closing letter of (i',j') is the number of positions of level $< \ell'$, plus the number of positions (i,j) of level ℓ' with $\pm(i-i')\eta < \Lambda$.

Inside a level. On level ℓ, $mi \equiv \ell \pmod q$; since m is invertible modulo q, i lies in one residue class modulo q. The conditions $0 \leq i \leq a$ and $0 \leq (\ell - mi)/q \leq b$ define an interval of i. So the indices i on level ℓ are $i_0 + tq$, $t = 0, \ldots, N_\ell - 1$, without gaps. For two of them, $i - i' = q(t-t')$, and $pq\eta < \Lambda < (p+1)q\eta$ gives: $(i-i')\eta < \Lambda$ if and only if $t \leq t' + p$; $-(i-i')\eta < \Lambda$ if and only if $t \geq t' - p$. The reflection $t \mapsto N_\ell - 1 - t$ maps $\#\{t : t \geq t' - p\}$ to $\#\{t : t \leq (N_\ell - 1 - t') + p\}$. So, level by level, the multisets of counts at θ_+ and at θ_- coincide, and $Z(c^+) = Z(c^-)$.

Height. At a point off the lines, the height of the prefix of the word ending just before x is $\#\{\text{positions} < x\} - \#\{\text{positions} < x - \Lambda\}$, the number of positions in $[x - \Lambda, x)$; so the height of the word is the largest number of positions in a half-open window of length Λ. Positions of different levels are more than $1/q - a\eta > \Lambda$ apart, so a window meets one level, where consecutive positions are $q\eta$ apart; since $\Lambda < (p+1)q\eta$ it contains at most $p+1$ of them. The level pmq contains the $p+1$ positions $(tq,\ m(p-t))$, $0 \leq t \leq p$ (here $pq \leq a$, $pm \leq b$), spanning $pq\eta < \Lambda$. So the height is $p+1$.

Evidence & scope

- **Program:** block T4: all 127 admissible (m, q, p) for a ≤ 5, b ≤ 8: equal words, distinct realisable vectors equal to the closed form, height p+1, independent of (η, Λ); controls (Λ = (p+3/2)qη: 127/127 fail; gcd(m,q) > 1: 12/12; separation violated: 20/20). working script `teoria_general.py`, not published

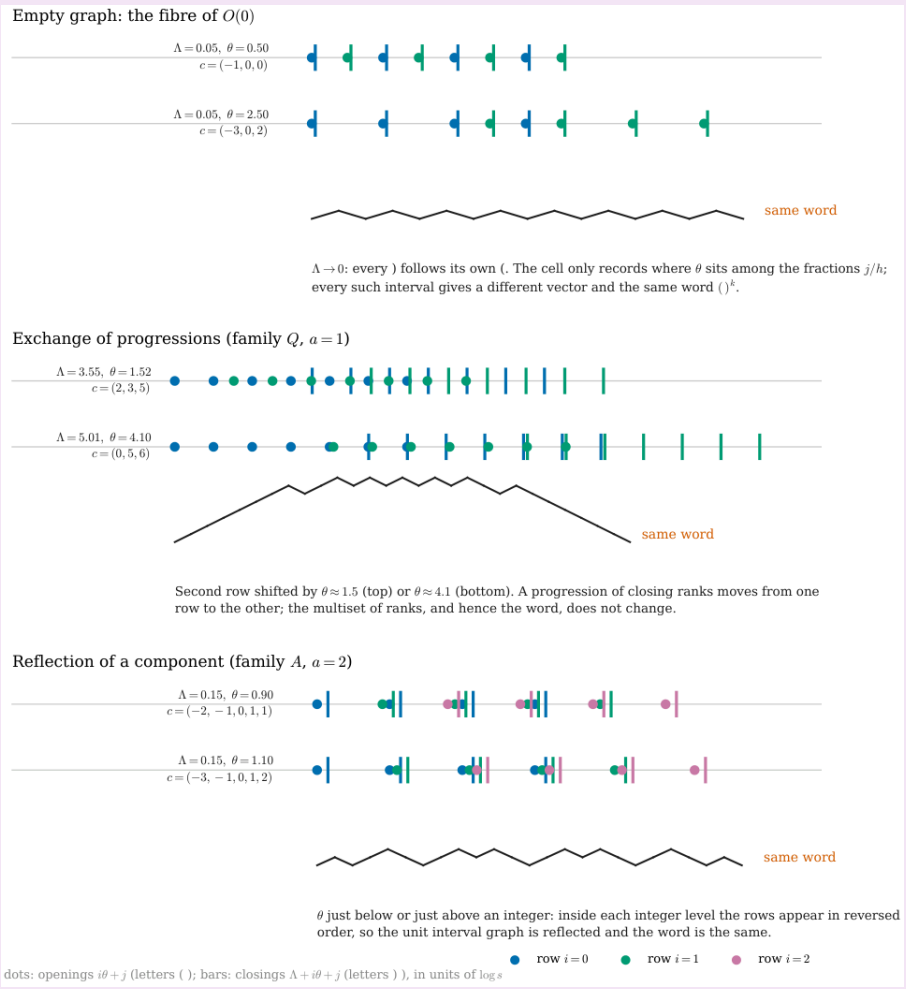
Remark 7.8. p. 22

The height separates different p, but we have not proved that different (m,q) with the same p give different words, nor that the words for $q \geq 2$ differ from those of Lemma 7.5. So the lemma is not added to the lower bound for $R(a,b) - W(a,b)$ given above. In the exact data for $a \leq 5$, $b \leq 8$, the pairs with $q \geq 2$ give words that are not words of pairs with $q = 1$ (one or two per (a,b) for $a \in \{4,5\}$, except $a = b$; none for $a \leq 3$, where $(p+1)q \leq a$ forces $q = 1$), and different triples (m,q,p) give different words, with one exception: for the square patterns

$a = b \in \{4, 5\}$, $(m, q, p) = (2, 1, 1)$ and $(1, 2, 1)$ give the same word, in a fibre of size 4 (the exchange $r \leftrightarrow s$, $(\Lambda, \theta) \mapsto (\Lambda/\theta, 1/\theta)$, pairs the four vectors $c^{\pm}(2, 1, 1)$, $c^{\pm}(1, 2, 1)$ two by two). Every other fibre met has size 2.

↳ Not used by another result of the paper.

Figure 5



The three mechanisms of coincidence. In each row, two cells with different vectors c and the same word: the positions $i\theta + j$ (dots, coloured by the row i) and $\Lambda + i\theta + j$ (bars) on the logarithmic line, and the common Dyck path below; vectors are written as $(c_{-a}, \dots, c_a)$. Top: the empty graph (Lemma 7.1), $a = 1$, $b = 3$. Middle: exchange of an arithmetic progression of closing ranks between rows (Lemma 7.2), $a = 1$, $b = 6$: the vectors $(2, 3, 5)$ and $(0, 5, 6)$ are the triples $(\nu, \beta, \gamma) = (3, 2, 5) = I_1$ ($\beta = 2$, $d = 1$) and $(5, 0, 6) = J(7, 1)$ of the family Q (Lemma 8.7, Corollary 7.3(a)). Bottom: reflection of the rows inside each integer level (Lemma 7.5), $a = 2$, $b = 3$, $p = k = 1$: the pair A_1 of Section 9.

8 Theorem A: semilength $2q$

Lemma 8.1. (triples) p. 23

> *Put* $\nu = \mathrm{clip}\lfloor\Lambda\rfloor$, $\beta = \mathrm{clip}\lfloor\Lambda - \theta\rfloor$, $\gamma = \mathrm{clip}\lfloor\Lambda + \theta\rfloor$. *The multiset* $C(n)$ *of (1) is* $Z(\nu, \beta, \gamma) := S(\nu, \beta) \uplus S(\nu, \gamma)$. *In particular the word depends only on the triple* (ν, β, γ).

↳ Used in: Theorem 6.1 (p. 18), Lemma 8.2 (p. 29), Theorem 8.12 (p. 34), Lemma 9.1 (p. 37)

Proof

None of Λ, $\Lambda \pm \theta$ is an integer. Each such equality has the form $2^e \cdot (\text{odd}) = \text{odd}$:

- $t = s^k$;
- $t = rs^k$, or $ts^{|k|} = r$ when $k<0$;
- $tr = s^k$.

All are impossible by parity. Take the letter) at $\Lambda + j'$. It is preceded by $\#\{i : i<\Lambda + j'\} = f(\lfloor\Lambda\rfloor + j')$ letters (of the row $\{0, \dots, b\}$, and by $\#\{i : \theta + i<\Lambda + j'\} = f(\lfloor\Lambda - \theta\rfloor + j')$ of the row $\theta + \{0, \dots, b\}$. Similarly, the letter) at $\Lambda + \theta + j'$ is preceded by $f(\lfloor\Lambda + \theta\rfloor + j') + f(\lfloor\Lambda\rfloor + j')$ letters (. Clipping does not change $f(v + j)$ for $0 \le j \le b$: if $v \le -b - 1$, then $f(v + j) = 0$, and if $v \ge b$, then $f(v + j) = b + 1$.

Evidence & scope

- **Program:** 0 clashes between word and triple at 4000 sampled points per b ≤ 10. `supplement_B/scripts/toro_a1.py` in `supplement.zip` of Zenodo record 10.5281/zenodo.24536373

Lemma 8.2. (realisable triples) p. 23

The triples that occur are exactly the clippings of $(\nu_0, \nu_0 - u, \nu_0 + u + \varepsilon)$ with integers $\nu_0 \ge 0$, $u \ge 0$, $\varepsilon \in \{-1, 0, 1\}$ and $u + \varepsilon \ge 0$.

↳ Used in: Lemma 7.2 (p. 23), Lemma 8.3 (p. 29), Lemma 8.6 (p. 30), Theorem 8.12 (p. 34), Lemma 9.1 (p. 37)

Proof

Write $\Lambda = \nu_0 + \lambda$ and $\theta = \kappa + \phi$ with $\nu_0, \kappa \ge 0$ integers and $\lambda, \phi \in (0, 1)$ (both fractional parts are nonzero: $\Lambda \notin \mathbb{Z}$ by Lemma 8.1, $\theta \notin \mathbb{Z}$ as noted). Then

$$\lfloor\Lambda - \theta\rfloor = \nu_0 - \kappa - [\lambda<\phi], \qquad \lfloor\Lambda + \theta\rfloor = \nu_0 + \kappa + [\lambda + \phi>1].$$

The four open regions $\{\lambda \gtrless \phi\} \cap \{\lambda + \phi \gtrless 1\}$ of $(0, 1)^2$ are nonempty. They give $(u, \varepsilon) = (\kappa, 0), (\kappa, 1), (\kappa + 1, -1), (\kappa + 1, 0)$. As κ ranges over $\mathbb{Z}_{\ge 0}$, these are all pairs with $u \ge 0$ and $u + \varepsilon \ge 0$. Each region is an open set of (Λ, θ); in the coordinates $x = (\log r, \log s)/\log t = (\theta/\Lambda, 1/\Lambda)$ it is an open set avoiding the lines of $\mathcal{A}_{(1,b)}$. Lemma 2.3 then realises it by primes $r \ne s$.

Lemma 8.3. p. 24

There are $R(1, b) = (5b + 4)(b + 1)/2$ realisable clipped triples.

↳ Used in: Theorem 8.12 (p. 34), 9.5 (p. 41)

Proof

We count clipped triples according to the unclipped parameters (ν_0, u, ε) of Lemma 8.2. If $\nu_0 \ge b$, then $\nu = b$ and the clipped triple is (b, β, b) with $\beta \in [-b - 1, b]$, giving $2b + 2$ triples. Fix $\nu_0 = \nu \le b - 1$ (no clipping of the

first coordinate) and write $\beta = \nu - u$ before clipping. The case $u = 0$ gives $\gamma \in \{\nu, \nu+1\}$. Each $1 \le u \le b-\nu-1$ gives three values $\gamma = \nu + u + \varepsilon$. The case $u = b - \nu$ gives $\gamma \in \{b-1, b\}$. The range $b - \nu + 1 \le u \le \nu + b$ gives $\gamma = b$, for 2ν triples. All $u \ge \nu + b + 1$ give the single triple $(\nu, -b-1, b)$. The total is $3b - \nu + 2$, and $\sum_{\nu=0}^{b-1}(3b-\nu+2) + 2b + 2 = (5b+4)(b+1)/2$.

Evidence & scope

- **Program:** equals the exact cell count (b ≤ 10 by sampling, and the formula of Theorem 5.1). `supplement_B/scripts/toro_a1.py` in `supplement.zip` of Zenodo record 10.5281/zenodo.24536373, `lab/comprueba_lab.py`

Lemma 8.4. p. 24

1. *If* $0 \le p \le p' \le b$, *then* $S(p,p') = \{p+p'+2, p+p'+4, \dots, 2b+p-p'\} \uplus [2b+2+p-p', 2b+1] \uplus \{T\}^{p+1}$.
2. *If* $p = -\rho < 0 \le p' \le b$ *and* $\rho \le b - p'$, *then* $S = [p'+1, p'+\rho] \uplus \{p'+\rho+2, p'+\rho+4, \dots, 2b-p'-\rho\} \uplus [2b+2-\rho-p', 2b+2-\rho]$.
3. *If* $p = -\rho$ *and* $\rho > b - p'$, *then* $S = [p'+1, 2b+2-\rho] \uplus \{b+1\}^{\rho-1-b+p'}$.

↳ Used in: Lemma 7.2 (p. 23), Corollary 8.5 (p. 30)

Proof

$f(p'+j) = p'+j+1$ for $j \le b-p'$ and $b+1$ afterwards; $f(-\rho+j) = 0$ for $j < \rho$ and $j - \rho + 1$ afterwards. Add the two and split the range of j at the break points; a range that is empty contributes nothing, by the conventions above.

Corollary 8.5. (profiles) p. 24

- *In* P': $\tau = \nu + \beta + 2$, $\ell = 2g + A$, $X = (10)^g 1^A$ *and* $Y = 0^{A+c}(10)^{g-c}1^c$.
- *In* N': $\tau = \nu + 1$, $\ell = 2g + \nu + 1$ *and* $Y = 0^{\nu+c+1}(10)^{g-c}1^c$. *Moreover* $X = 1^\rho(01)^{g-\rho}0\,1^{\nu+1}0^{\rho-1}$ *in* N_1', *and* $X = 1^{\ell-\rho+1}0^{\rho-1}$ *plus* $\rho - g - 1$ *at position* $x = g$ *in* N_2'.

↳ Used in: Lemma 8.6 (p. 30), Lemma 8.7 (p. 31), Lemma 8.9 (p. 32)

Proof

Translate Lemma 8.4 by $-\tau$. When $\beta = -1$ the N' formula with $\rho = 1$ coincides with the P' formula with $A = \nu + 1$, so the triples with $\beta = -1$ may be placed in P'.

Evidence & scope

- **Program:** part A, b = 1..20, 0 failures; the control fails on 363 cases. `supplement_B/scripts/prueba_a1.py` in `supplement.zip` of Zenodo record 10.5281/zenodo.24536373

Lemma 8.6. p. 24

Each of the following triples has word $O(s)$:

1. $s = 0$: $(0, -\rho, \rho - 1)$ *for* $\rho = 1, \dots, b+1$;
2. $s = 2k+1$: $(k, k, k+1)$ *for* $0 \le k \le b-1$;
3. $s = 2k+2$: $(k+1, k, k+1)$ *for* $0 \le k \le b-1$;
4. $s = 2b+1$: (b, b, b);
5. $J_0(s) := (\min(b, s), s - 1 - b, b)$ *for* $0 \le s \le 2b$. *This triple is realisable: take* $u = \nu - \beta$ *and* $\varepsilon = 1$ *in Lemma 8.2; then* $\nu + u + 1 = 2\nu - \beta + 1 \ge b$, *so* $\gamma = b$.

↳ Used in: Corollary 7.3 (p. 24), Lemma 8.8 (p. 31), Lemma 8.10 (p. 33)

Proof

All cases are read off Corollary 8.5.

- (a): the union of $S(0, -\rho)$ and $S(0, \rho - 1)$ is $[1, 2b+1]$ plus copies of T, so $\tau = 1$ and the profile is 1^ℓ.
- (b): P' with $A = 0$, $c = 1$: $X + Y = (10)^g + 0(10)^{g-1}1 = 1^{2g}$.
- (c): P' with $A = 1$, $c = 0$: $(10)^g 1 + 0(10)^g = 1^{2g+1}$.
- (d): $\ell = 0$.
- (e) with $s \ge b$: P' with $g = c = 0$ gives $X + Y = 1^A + 0^A$ and $\tau = s + 1$.
- (e) with $s<b$: N_2' with $\nu = s$ and $\rho = g + 1$ gives $X + Y = 1^{\nu+g+1}0^g + 0^{\nu+g+1}1^g = 1^\ell$.

Lemma 8.7. p. 25

Let $d \ge 1$. *The following triples have word* $Q(\tau, d)$:

- $I_1 = (\beta + d, \beta, \beta + 2d + 1)$, *with* $\tau = 2\beta + d + 2$;
- $I_2 = (\beta + d + 1, \beta, \beta + 2d + 1)$, *with* $\tau = 2\beta + d + 3$;
- $J(\tau, d) = (b - d, \tau - b + d - 2, b)$ *for* $1 \le d \le b$ *and* $\tau \in [b - d + 1, 2b - 3d + 2]$.

For I_1 *and* I_2 *the conditions are* $\beta \ge 0$ *and* $\beta + 2d + 1 \le b$. *They give exactly one I-triple for each* $\tau \in [d + 2, 2b - 3d + 1]$.

↳ Used in: Corollary 7.3 (p. 24), Lemma 8.8 (p. 31), Lemma 8.11 (p. 33)

Proof

For I_1, $X = (10)^g 1^d$ and $Y = 0^{2d+1}(10)^{g-d-1}1^{d+1}$ with $g \ge d + 1$, and the sum is $(10)^d 1^{2g-2d} 2^d$. I_2 is the same with one more 1. For J with $\beta \ge 0$ we have $g = c = d$ and $A \ge d$, so $X + Y = (10)^d 1^A + 0^{A+d}1^d$. For J with $\beta = -1$ the same computation applies with $A = \nu + 1 \ge d$.

Lemma 8.8. p. 25

Let $b \ge 2$. *Two realisable triples have the same word if and only if they have the same label.*

↳ Used in: Theorem 8.12 (p. 34)

Proof

Equal words give equal Z, hence equal K. If $n_0 = 0$, both labels are $O(\tau - 1)$ by Lemma 8.10. If $n_0 > 0$, Lemma 8.11 applies.

Evidence & scope

- **Program:** (⇒) also checked by Z3 on an independent probe encoding (9 cases UNSAT, controls SAT) and by the exact event encoding of §9.2 (55/55 UNSAT; dropping Q gives SAT on real collisions). `supplement_B/scripts/prueba_a1.py`, `supplement_B/scripts/certificar_k3q.py` in `supplement.zip` of Zenodo record 10.5281/zenodo.24536373

Lemma 8.9. (zeros) p. 25

Let $\mathcal{Z}_0$ be the set of zeros of the profile. Then:

tabhead type	n_0	$\mathcal{Z}_0$
P^+	A	$\{1, 3, \ldots, 2n_0 - 1\}$
P^-	$A - 1$	*same*
P^0, P^c	g	*same*
N^-	ν	$\{\rho + 2i : 0 \le i < n_0\}$
N^0, N^c	$g - \rho + 1$	*same*
N^+	$\nu + 1$	*same*
N_2'	$\rho - g - 1$	$[2g + \nu + 2 - \rho,\ \nu + g + 1)$

In particular, when $n_0 > 0$: $f_0 = 1$ on P', $f_0 = \rho\ (\ge 2)$ on N_1', and $f_0 = 2g + \nu + 2 - \rho \ge g + 1$ on N_2'.

↳ Used in: Lemma 8.10 (p. 33), Lemma 8.11 (p. 33)

Proof

P'. We have $X(x) = 0$ iff x is odd and $x < 2g$. Put $\sigma = A + c$. Then $Y(x) = 0$ iff $x < \sigma$, or $\sigma \le x < \ell - c$ with $x - \sigma$ odd. By type:

- $\delta = 1$: $\sigma = 2A + 1 \le 2g - 1$ is odd, so the zeros are the odd $x < \sigma$ and $n_0 = A$.
- $\delta = -1$: $\sigma = 2A - 1 \le 2g + 1$, so the zeros are the odd $x < \min(\sigma, 2g)$ and $n_0 = A - 1$.
- $\delta = 0$: $\sigma = 2c \le 2g$ is even and $\ell - c = 2g$, so the zeros are the odd $x < 2g$ and $n_0 = g$.
- $\delta \le -2$: $c = g$ and $\sigma \ge 2g + 2$, so the zeros are the odd $x < 2g$ and $n_0 = g$.

N_1'. The zeros of X are $x_i = \rho + 2i$ $(0 \le i \le g - \rho)$ together with $[\ell - \rho + 1, \ell)$. Since $c \ge \rho - 1$, the tail 1^c of Y covers the latter. Y vanishes on $[0, w)$, where $w = \nu + c + 1$, and at $w + 1, w + 3, \ldots, \ell - c - 1$. Write $c = \nu + \rho + \delta$. Then $x_i < w$ iff $i \le \nu + \lfloor \delta/2 \rfloor$. For $x_i \ge w$, $Y(x_i) = 0$ iff δ is even and $x_i \le 2g - \rho - \delta$. By type:

- $\delta = 0$: every $i \le g - \rho$ qualifies, so $n_0 = g - \rho + 1$.
- $\delta = 1$: only $i \le \nu$ qualifies, and $\nu \le g - \rho - 1$, so $n_0 = \nu + 1$.
- $\delta = -1$: only $i \le \nu - 1$ qualifies, so $n_0 = \nu$.

- $\delta \le -2$: $c = g$ and $\nu \ge g - \rho + 2$, so $\nu + \lfloor \delta/2 \rfloor = \lfloor (\nu + g - \rho)/2 \rfloor \ge g - \rho + 1$ and every i qualifies.

N_2'. Here $Y = 0^{\nu+g+1}1^g$. X is 1 on $[0, \ell - \rho + 1)$ (with the excess at $x = g$) and 0 afterwards. Since $\rho \le b + 1$, we have $\ell - \rho + 1 \ge g + 1$, and the zeros are $[\ell - \rho + 1, \nu + g + 1)$.

Evidence & scope

- **Program:** b = 2..20, 0 failures; the broken control is detected on 924 keys. working script `decodificar_ceros.py`, not published

Lemma 8.10. p. 27

> *$n_0 = 0$ if and only if the triple is in the list of Lemma 8.6. In that case the label is $O(\tau - 1)$.*

↳ Used in: Lemma 8.8 (p. 31)

Proof

By Lemma 8.9, $n_0 = 0$ exactly in the following cases:

- P^+ with $A = 0$: the triple $(k, k, k + 1)$, case (b).
- P^- with $A = 1$: $(k + 1, k, k + 1)$, case (c), or $(0, -1, 0)$, case (a).
- P^0 with $g = 0$: case (d).
- P^c with $g = 0$: $(b, \beta, b) = J_0(\beta + 1 + b)$.
- N^- with $\nu = 0$: case (a).
- N_2' with $\rho = g + 1$: $J_0(\nu)$.

All other types have $n_0 \ge 1$.

Lemma 8.11. p. 27

> *Let $b \ge 2$. If $K(\xi) = K(\eta)$ and $n_0 \ge 1$, then either $\xi = \eta$, or $\{\xi, \eta\} = \{I, J(\tau, d)\}$ with I an I_1/I_2-triple and $d = n_0$.*

↳ Used in: Lemma 8.8 (p. 31)

Proof

Equal τ gives equal ℓ. Between two triples of N' it also gives equal ν, hence equal g.

(1) Different families. By Lemma 8.9, $f_0 = 1$ on P', $2 \le f_0 = \rho \le g$ on N_1' and $f_0 \ge g + 1$ on N_2'. Since two triples of N' with equal τ have equal g, the ranges $[2, g]$ and $[g + 1, \infty)$ of f_0 are disjoint, so N_1' and N_2' never collide; and N_1' never collides with P', since $f_0 \ge 2$ there. So P' and N' can only collide when $f_0 = 1$ in N_2', which forces $g = 0$, $\nu = b$, $\rho = b + 1$, $\ell = b + 1$, $n_0 = b$. No P' triple has $\ell = b + 1$ and $n_0 = b$ when $b \ge 2$:

- P^0/P^c: $n_0 = g = b$ and $\ell = 2g + A$ would need $A = 1 - b < 0$;
- P^+: $n_0 = A = b$ and $\ell = 2g + A$ would need $2g = 1$;
- P^-: $n_0 = A - 1 = b$ gives $A = b + 1$, and $\ell = 2g + A = b + 1$ gives $g = 0$; then $c = A + \delta = A - 1 = b > 0 = g$, contradicting $c \le g$ in (V).

(For $b = 1$ such a pair exists, which is why $b \geq 2$ is assumed.)

(2) N_2' with N_2'. n_0 fixes ρ, and $c = g$. So $\xi = \eta$.

(3) N_1' with N_1'. ν and $\rho = f_0$ agree. The types are constrained by $N^- : \nu + \rho - 1 \leq g$, $N^0 : \nu + \rho \leq g$, $N^+ : \nu + \rho + 1 \leq g$ and $N^c : \nu + \rho \geq g + 2$. Comparing the values of n_0 in Lemma 8.9: N^- against N^0/N^c forces $\nu = g - \rho + 1$, which neither allows; N^+ against N^0/N^c forces $\nu = g - \rho$, which neither allows; and $N^\pm$ differ by 1. Within a type, c is determined. So $\xi = \eta$.

(4) P' with P'. Here $A \equiv \ell \pmod 2$ and $g = (\ell - A)/2$. P^0 and P^c both give $g = d$ and $A = \ell - 2d$, but P^0 needs $A \leq g$ and P^c needs $A \geq g + 2$, so they cannot both occur. P^+ gives $A = d$ and P^- gives $A = d + 1$, which have different parities, so these two never collide. Within one type, (ℓ, d) determines the triple.

If $\xi \in P^+$ and $\eta \in P^0 \cup P^c$, then $A_\eta = 2g_\xi - d \geq d + 2 = g_\eta + 2$, so $\eta \in P^c$ and $\beta_\eta = b - 2g_\xi \geq -1$. Then $\beta_\xi = b - g_\xi - d \geq 0$ and $\xi = I_1(\beta_\xi, d)$. Also $\eta = (b - d, b - 2g_\xi, b) = J(\tau, d)$, since $\tau - b + d - 2 = 2\beta_\xi + 2d - b = b - 2g_\xi$.

If $\xi \in P^-$ and $\eta \in P^0 \cup P^c$, then $g_\eta = d$, $A_\eta = \ell - 2d$, and $\ell = 2g_\xi + A_\xi = 2g_\xi + d + 1$, so $A_\eta = 2g_\xi + 1 - d$. In P^- we have $c_\xi = A_\xi - 1 = d \leq g_\xi$. If $g_\xi = d$, then $A_\eta = d + 1 = g_\eta + 1$, which is allowed neither in P^0 (needs $A \leq g$) nor in P^c (needs $A \geq g + 2$); so this case does not occur. Hence $g_\xi \geq d + 1$, $A_\eta \geq g_\eta + 3$ and $\eta \in P^c$. In that case $\beta_\eta = b - 2g_\xi - 1 \geq -1$, $\xi = I_2(\beta_\xi, d)$ with $\beta_\xi = b - g_\xi - d - 1 \geq 0$, and $\eta = J(\tau, d)$.

Evidence & scope

- **Program:** b = 2..20: equal keys with $n_0 > 0$ occur exactly in pairs (P+, Pc) (330) and (P−, Pc) (285); the control without the Q identification fails on 615 keys. working script `decodificar_ceros.py`, not published

Theorem 8.12. (Theorem A) p. 28

For $b \geq 2$,

$$W(1, b) = \left\lfloor \frac{9b^2 + 8b + 12}{4} \right\rfloor.$$

Consequently, for every odd prime q,

$$|L_{2q}| = \frac{9q^2 - 10q + 13}{4} \qquad (q = 3, 5, 7 : \ 16, 47, 96).$$

Isabelle · ours · part **show**

Only a part is in Isabelle: only the closing count: R(1,b), the excesses and the cell lemmas enter as given numbers

`theorem_A_algebra` · `Isabelle/CaballeroDyckWords/Caballero_Count_Algebra.thy`, line 67

```
theorem theorem_A_algebra:
  fixes b :: nat
  assumes "b \<ge> 2"
  shows "int ((5 * b + 4) * (b + 1) div 2) - (3 * int b - 1)
          - (\<Sum>d\<in>{1..(b - 1) div 2}. (int b - 2 * int d)) = int
```

```
((9 * b\<^sup>2 + 8 * b + 12) div 4)"
    and "int ((9 * b\<^sup>2 + 8 * b + 12) div 4)
           = 3 + (9 * int b * (int b + 1) - 2 * int ((b + 1) div 2)) div 4"
    and "4 dvd (9 * int b * (int b + 1) - 2 * int ((b + 1) div 2))"
```

`theorem_A_q` · `Isabelle/CaballeroDyckWords/Caballero_Count_Algebra.thy`, line 97

```
theorem theorem_A_q:
  fixes q :: nat
  assumes "odd q" "q \<ge> 3"
  shows "int ((9 * (q - 1)\<^sup>2 + 8 * (q - 1) + 12) div 4) = (9 * int q
^ 2 - 10 * int q + 13) div 4"
    and "4 dvd (9 * int q ^ 2 - 10 * int q + 13)"
```

Trust base: ⚠️ not checked. Written proof by hand, audited internally (adversarial audit: survives; ten presentation defects, all addressed); not machine-checked as a whole: only the closing arithmetic compiles in Isabelle/HOL (session Caballero_Base, public)

Proof: in this note · **Covers:** not formalized

🆕 **Classical and new:** the paper: «We know of no earlier count of L_k beyond direct enumeration»; the rows and diagonals of W(a,b) were not found in the OEIS (25 Nov 2026).

↳ Used in: 8.13 (p. 36), 9.5 (p. 41), 11.1 (p. 43)

Proof

By Lemmas 8.1, 8.2 and 8.8, $W(1,b)$ is the number of labels. This equals the number of triples (Lemma 8.3) minus the excess $\sum(|\text{class}|-1)$.

- *O-classes.* $O(0)$ has $b+1$ triples ($J_0(0)$ is case (a) with $\rho=b+1$). For $1\le s\le 2b-1$, $O(s)$ has two triples. $O(2b)$ has one, since $J_0(2b)=(b,b-1,b)$ is case (c). $O(2b+1)$ has one. The excess is $3b-1$.
- *Q-classes.* For $2d\le b-1$, the I- and J-ranges of τ overlap in $[b-d+1,2b-3d+1]$. At the right end, $I_2=J$. The excess is $b-2d$. If $2d\ge b$ there is no I-triple.

So $E(b)=3b-1+\sum_{d=1}^{\lfloor(b-1)/2\rfloor}(b-2d)$, which is $3b-1+k(k-1)$ for $b=2k$ and $3b-1+k^2$ for $b=2k+1$. Then $W(1,b)=(5b+4)(b+1)/2-E(b)$ equals $9k^2+4k+3$ for $b=2k$ and $9k^2+13k+7$ for $b=2k+1$. Equivalently,

$$W(1,b)=3+\frac{9b(b+1)-2\lceil b/2\rceil}{4}=\left\lfloor\frac{9b^2+8b+12}{4}\right\rfloor,$$

since $(9b^2+8b+12)/4$ equals $9k^2+4k+3$ for $b=2k$ and $9k^2+13k+7+\frac{1}{4}$ for $b=2k+1$. For $|L_{2q}|$, apply Corollary 3.3(b) with $b=q-1$ (even): $3+(q-1)(9q-1)/4=(9q^2-10q+13)/4$.

💡 **In plain words (commentary, not part of the paper)**

How many different words come from integers whose odd part is rs^b with two primes? Exactly $\lfloor(9b^2+8b+12)/4\rfloor$ for $b\ge 2$. Since L_{2q} is the set of words of the pattern rs^{q-1} (Theorem 3.1), this gives $|L_{2q}|=(9q^2-10q+13)/4$: 16, 47, 96 for $q=3,5,7$.

Why it is true: the word depends only on a triple (ν,β,γ) of integers; there are $(5b+4)(b+1)/2$ realisable triples (Lemma 8.3). Two triples give the same word exactly when they belong to one of two explicit families, O and Q (Lemma 8.8), proved by decoding the word from its first letter count and the positions of the zeros of a profile. Subtracting the

repetitions, $3b-1$ from the classes O and $b-2d$ from the classes $Q(\tau,d)$ for each $d \le (b-1)/2$, gives the count.

The proof is by hand; each lemma was also checked by computer for $b \le 20$ with controls that fail. Asymptotically one cell in ten repeats a word (Remark 9.5).

Evidence & scope

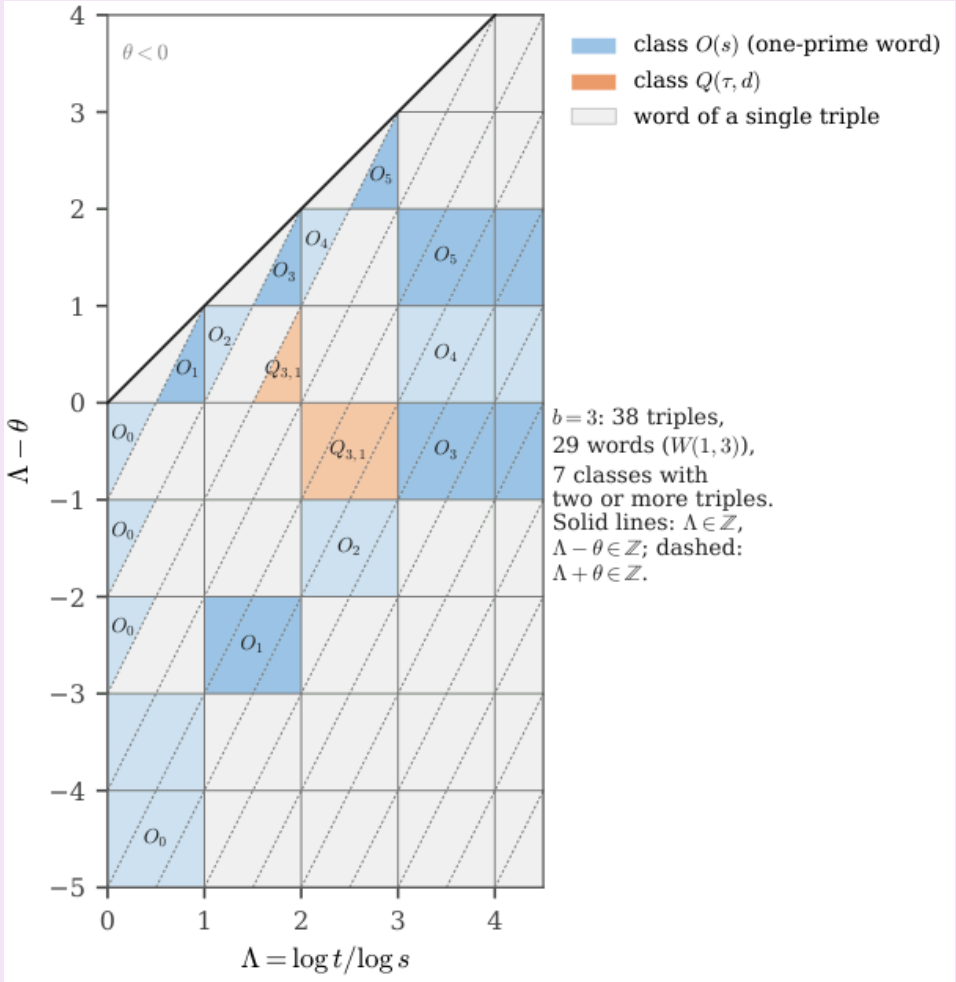
- **Audit:** machine-checked in Isabelle/HOL session Caballero_Base, theory Caballero_Count_Algebra (task 292, inside 290): theorem_A_algebra ((5b+4)(b+1)/2 − (3b−1) − Σ_d=1^⌊(b−1)/2⌋(b−2d) = ⌊(9b²+8b+12)/4⌋ = 3 + (9b(b+1) − 2⌈b/2⌉)/4 for b ≥ 2) and theorem_A_q ((9q²−10q+13)/4 for b = q−1, q odd). Only the closing count: R(1,b), the excesses and the cell lemmas enter as given numbers. Compiles without sorry; the control (+14) fails. working script `Caballero_Count_Algebra.thy`, not published
- **Program:** W(1,b) = formula for b = 2..60 by exact count, with a negative control (constant 4 in place of 3) failing in 59 values of b. working scripts `contar_a1.py`, `control_formula_a1.py`, not published
- **Program:** the lab: W(1,b) = formula for b = 2..30, |L_6|, |L_10|, |L_14| = 16, 47, 96; the control with 4 in place of 3 fails. `lab/comprueba_lab.py`

Remark 8.13. (status of Theorem A) p. 28

The proof above is by hand. Each lemma was also checked by computer for $b \le 20$, with negative controls; the direction ($\Rightarrow$) was checked by Z3 on a separate encoding (9 cases unsatisfiable, controls satisfiable) and on the event encoding of §9.2 (55/55 unsatisfiable); and $|L_6| = 16$ was confirmed from actual integers.

↳ Not used by another result of the paper.

Figure 6



Pattern $(1,3)$ ($m = rs^3$, $b = 3$) in the coordinates $(\Lambda, \Lambda - \theta)$ of Lemma 8.1: the lines $\Lambda \in \mathbb{Z}$, $\Lambda - \theta \in \mathbb{Z}$ (solid) and $\Lambda + \theta \in \mathbb{Z}$ (dashed) cut the region $\theta > 0$ into pieces; the triple (ν, β, γ) is constant on each piece. Pieces whose word is shared by two or more triples are coloured: $O(s)$ in blue (label O_s), $Q(\tau, d)$ in orange (label $Q_{\tau,d}$); one label per triple. The 38 triples give $29 = W(1,3)$ words, and every class with two or more triples is an O or a Q class (Lemma 8.8 for $b = 3$).

9 Theorem B: semilength $3q$

Lemma 9.1. (vector) p. 28

> *$C(n) = Z(c)$. The vectors c that occur are exactly the points of the open quadrant $\Lambda, \theta > 0$ that avoid the lines $\Lambda + h\theta = \rho$ ($h \in \{-2, \dots, 2\}$, $\rho \in [-b, b]$), taken up to the cell they lie in. Every such cell is realised by primes.*

↳ Used in: Theorem 6.1 (p. 18), Lemma 9.2 (p. 38), Lemma 9.3 (p. 38), Theorem 9.4 (p. 40)

Proof

> This is the argument of Lemmas 8.1, 8.2 and 2.3. A letter) at $\Lambda + i'\theta + j'$ is preceded by $f(c_{i'-i} + j')$ letters (of row i. Realisability of c is an open linear

system, decided by Fourier–Motzkin elimination of Λ. The region of a vector is convex, and distinct cells give distinct vectors.

Lemma 9.2. (cells) p. 28

The number of realisable vectors is $R(2,b) = \big(22b^2 + 31b + 9 + [b \equiv 1\ (3)]\big)/3$.

Trust base: computation (not a formal proof). Written proof by hand except the last identity (two quasipolynomials of period 6 agree on 18 values), checked by computer; the closed form also compared with the pieces for b ≤ 300

Proof: in this note · **Covers:** not formalized

↳ Used in: Theorem 9.4 (p. 40), 9.5 (p. 41)

Proof

This is Theorem 5.1 with $a = 2$. There $(3a+1)b + a = 7b + 2$, $S_1(b) = b(2b+1)$, $S_2(b) = S_4(b) = b(b+1)$ (the $p \le 2b$ coprime to 2, or to 4, are the odd ones), $B_1(2,b) = 2(b + \lfloor b/2 \rfloor)$, $B_2(2,b) = 2\lceil b/2 \rceil$ and $B_3 = B_4 = 0$. Since $\lfloor b/2 \rfloor + \lceil b/2 \rceil = b$, the formula becomes $R(2,b) = 6b^2 + 9b + 3 + S_3(b)$. With $J = \lfloor 2b/3 \rfloor$,

$$S_3(b) = \sum_{p=1}^{2b} (2b - p + 1) - \sum_{j=1}^{J} (2b + 1 - 3j) = (2b+1)(b - J) + \tfrac{3}{2} J(J+1),$$

which equals $\big(4b^2 + 4b + [b \equiv 1\ (3)]\big)/3$ in each residue class: $12m^2 + 4m$, $12m^2 + 12m + 3$ and $12m^2 + 20m + 8$ for $b = 3m$, $3m+1$, $3m+2$ (with $J = 2m$, $2m$, $2m+1$). Substituting gives the formula.

Evidence & scope

- **Program:** brute-force exact vertices = each piece for b = 1..30; Euler = pieces = closed form = number of vectors for b ≤ 30; pieces = closed form for b ≤ 300; control (drop V4) matches 0 of 30. `supplement_B/scripts/celdas_R2.py` in `supplement.zip` of Zenodo record 10.5281/zenodo.24536373

Lemma 9.3. p. 29

Let $b \ge 3$. The classes of size ≥ 2 of the relation "same word" on realisable vectors are exactly the classes listed above, and they are pairwise disjoint.

Isabelle · ours · part **show**

Only a part is in Isabelle: the algebraic identities behind the event encoding; the SMT queries themselves are not formalized

`events_progression` · `Isabelle/CaballeroDyckWords/Caballero_Eventos/Caballero_Eventos.thy`, line 60

```
theorem events_progression:
  assumes "s \<ge> 1" "s dvd D"
  shows "(1 - zp D) * (\<Sum>n<N. zp (u + s * n))
      = (\<Sum>\<rho><D div s. zp (u + s * \<rho>) - zp (u + s * N + s * \
<rho>))"
```

`events_constant` · `Isabelle/CaballeroDyckWords/Caballero_Eventos/Caballero_Eventos.thy`, line 88

```
theorem events_constant:
  "(1 - zp D) * (of_nat N * zp u) = of_nat N * zp u - of_nat N * zp (u +
D)"
```

`events_cancel` · `Isabelle/CaballeroDyckWords/Caballero_Eventos/Caballero_Eventos.thy`, line 94

```
theorem events_cancel:
  assumes "D \<ge> 1" and "(1 - zp D) * M = (1 - zp D) * (M' :: int poly)"
  shows "M = M'"
```

`genpoly_eq_iff` · `Isabelle/CaballeroDyckWords/Caballero_Eventos/Caballero_Eventos.thy`, line 112

```
theorem genpoly_eq_iff: "genpoly A = genpoly B \<longleftrightarrow> A = B"
```

Trust base: computation (not a formal proof). (⇐) by hand; (⇒) and disjointness decided by exact SMT queries uniformly in $b \geq 3$: Z3 4.16.0, 496/496 UNSAT (reverse 496/496 UNSAT, disjointness 231/231 UNSAT), no proof object produced or checked; cvc5 1.4.2 (also without proof object) 231/231 UNSAT on disjointness, 496/496 on the direct and 496/496 on the reverse queries; trusts the hand derivation of the event formulas and the branch coverage (validated for $b \leq 12$) and Lemma 9.1

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the algebraic identities behind the event encoding; the SMT queries themselves are not formalized

↳ Used in: Theorem 9.4 (p. 40)

Evidence & scope

- **Program:** the event encoding: for $b \leq 12$ the evaluated events equal $(1 - z^6)\cdot$(real word) in all 6091 (branch, vector) cases and the branch constraints equal realisability; 69 syntactic branches, 31 feasible; control (shifted event) detected on 327 vectors. `supplement_B/scripts/eventos_k3q.py` in `supplement.zip` of Zenodo record 10.5281/zenodo.24536373
- **Program:** the 496 direct and 496 reverse queries (Z3 4.16.0), all UNSAT, none unknown; controls: removing any family A, B, D, E, F from the label makes the query SAT on a genuine collision (e.g. $b = 159\,827$ for B); $a = 1$ with labels O, Q: 55/55 UNSAT. `supplement_B/scripts/certificar_k3q.py` in `supplement.zip` of Zenodo record 10.5281/zenodo.24536373, Supplement B to Theorem B (semilength 3q), p. 12 (Section 6, The SMT queries), Supplement B to Theorem B (semilength 3q), p. 14 (Section 6.5, Controls)
- **Program:** second solver cvc5 1.4.2 on the SMT-LIB2 files, without proof production: 496/496 direct UNSAT (4227 s), 496/496 reverse UNSAT (6005 s), 231/231 disjointness UNSAT (3 s); control: the 496 direct queries generated without the family A give 2 SAT (the real coincidences of A) and 494 UNSAT. `supplement_B/scripts/cvc5_k3q.py` in `supplement.zip` of Zenodo record 10.5281/zenodo.24536373, Supplement B to Theorem B (semilength 3q), p. 14 (Section 6.4, Results)
- **Program:** disjointness with b free: 231 pairs of cases with different tuples UNSAT; control with a false case repeating A gives SAT. `supplement_B/scripts/disjuntas_k3q.py` in `supplement.zip` of Zenodo record 10.5281/zenodo.24536373

- **Program:** (⇐) and witnesses: 13 715 realisability witnesses of A–F for b = 3..80 (control with altered D witnesses: 234 failures); B_2 by hand for u = 1..40 (control b = 3u+3: 40/40 distinct); brute-force classes of size ≥ 2 = predicted classes for b = 3..70. `supplement_B/scripts/testigos_a2.py`, `supplement_B/scripts/b2_mano.py`, `supplement_B/scripts/familias_a2.py` in `supplement.zip` of Zenodo record 10.5281/zenodo.24536373
- **Program:** Supplement to Theorem B: nine launchers (bloques_k3q.py, run_hand_checks.py, run_families.py, run_cells.py, run_events.py, run_label_and_disjoint.py, run_certificate_sample.py, run_cvc5_disjoint.py, check_recorded_logs.py), each with its output and exit code 0, and SHA256SUMS. `supplement_B/scripts/check_recorded_logs.py` in `supplement.zip` of Zenodo record 10.5281/zenodo.24536373, Supplement B to Theorem B (semilength 3q), p. 15 (Section 7, Reproducing everything)
- **Audit:** machine-checked in Isabelle/HOL session Caballero_Eventos (task 293): the algebraic identities behind the event encoding (telescoping of a progression of step s | 6 after multiplying by $1 - z^6$, the constant piece, cancellation in $\mathbb{Z}[z]$, multisets ↔ generating polynomials). The SMT queries themselves are not formalized. working script `Caballero_Eventos.thy`, not published

Theorem 9.4. (Theorem B; computer-assisted in Lemma 9.3) p. 30

For $b \geq 3$,

$$W(2,b) = \frac{44b^2 + 19b + e_{b \bmod 6}}{6}, \qquad (e_0, \ldots, e_5) = (30, 33, 8, 27, 36, 5).$$

Consequently, for primes $q \geq 5$, with $\left(\frac{q}{3}\right) = \pm 1$ according as $q \equiv \pm 1 \pmod 3$ (the Legendre symbol of q modulo 3, which equals $\left(\frac{-3}{q}\right)$ by quadratic reciprocity),

$$|L_{3q}| = \frac{44q^2 - 69q + 58 - 3\left(\frac{q}{3}\right)}{6} \qquad (q = 5, 7: \ 136, \ 288).$$

Trust base: computation (not a formal proof). Computer-assisted: written proof except Lemma 9.3, whose (⇒) and disjointness rest on exact SMT queries (Z3 4.16.0, 496/496 and 231/231 UNSAT; cvc5 1.4.2 agrees on 231/231 and on the 496 direct and 496 reverse queries), with no proof object; the class sizes are read off by hand

Proof: in this note · **Covers:** not formalized

NEW **Classical and new:** the paper: «Theorem B is not proved by hand. Its (⇒) step and the disjointness rest on exact SMT queries decided by Z3 4.16.0, without a checked proof object.»

↳ Used in: 9.5 (p. 41), 11.1 (p. 43)

Proof

$W(2,b) = R(2,b) - E(b)$, where $E(b)$ is the total excess of the classes of Lemma 9.3. Reading off the class sizes by hand: O contributes $(b + \lceil b/2 \rceil) + 3b$, A contributes $\lfloor b/2 \rfloor$, B contributes $3[b \equiv 2\ (3)]$, C contributes 2, D contributes $2\lfloor (b-2)/3 \rfloor$, E contributes $\lfloor (b-3)/2 \rfloor$ and F contributes b. Summing,

$$E(b) = 6b + \lceil b/2 \rceil + 2\lfloor (b-2)/3 \rfloor + 3[b \equiv 2\ (3)].$$

Subtracting from Lemma 9.2, one residue class mod 6 at a time, gives the formula. For $|L_{3q}|$, take $b = q - 1 \equiv 0, 4 \pmod 6$ in Corollary 3.3(b). This gives

$$|L_{3q}| = \begin{cases} (44q^2 - 69q + 55)/6, & q \equiv 1 \pmod 6, \\ (44q^2 - 69q + 61)/6, & q \equiv 5 \pmod 6, \end{cases}$$

which is the stated formula, since $\left(\frac{q}{3}\right) = 1$ for $q \equiv 1 \pmod 6$ and $\left(\frac{q}{3}\right) = -1$ for $q \equiv 5 \pmod 6$. (For $q = 3$, $W(2,2) = 28$, whereas the formula for $W(2,b)$ would give 37.)

In plain words (commentary, not part of the paper)

With odd part $r^2 s^b$ the number of words is $(44b^2 + 19b + e_{b \bmod 6})/6$ for $b \geq 3$, with $(e_0, \ldots, e_5) = (30, 33, 8, 27, 36, 5)$; so $|L_{3q}| = (44q^2 - 69q + 58 - 3(\frac{q}{3}))/6$ for primes $q \geq 5$: 136 and 288 for $q = 5, 7$. For $q = 3$ the formula does not apply: $|L_9| = W(2,2) = 28$.

How it is proved: the number of realisable vectors $c \in \mathbb{Z}^5$ is $R(2,b)$ (Lemma 9.2, by hand except one finite identity checked by computer). The coincidences are listed in seven families (O, A–F), each proved by hand. That the list is complete and that its classes are disjoint (Lemma 9.3) is *computer-assisted*: after an exact compression of the word into "events", 496 queries with b free were decided UNSAT by Z3 4.16.0, and the 231 disjointness queries also by cvc5 1.4.2. No proof object was produced or checked.

Unlike $a = 1$, here the coincidences are only linearly many in b (about $43b/6$), so $W(2,b)/R(2,b) \to 1$ (Remark 9.5).

Evidence & scope

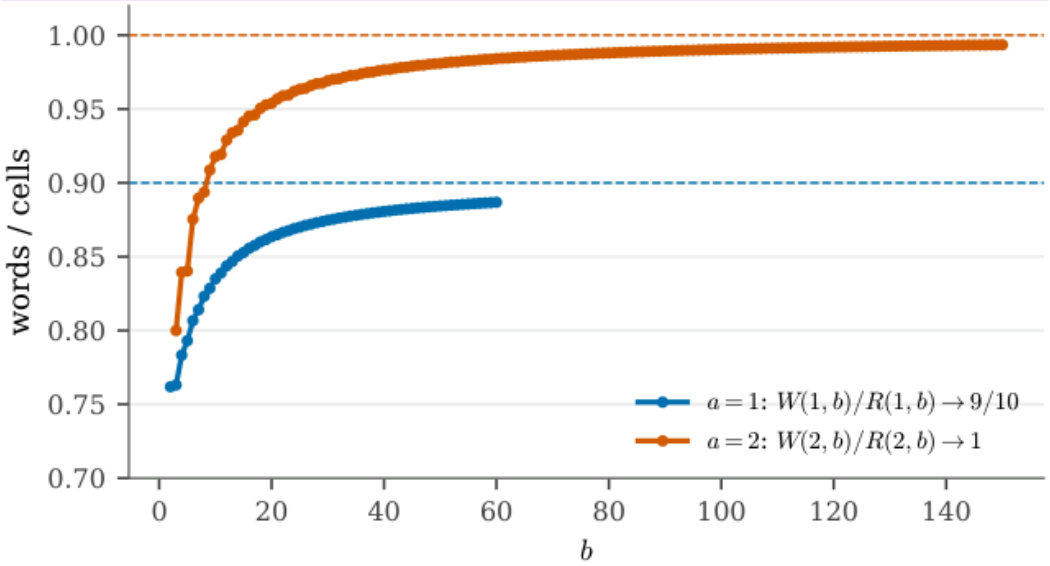
- **Program:** E(b) = real excess and W = formula for b = 3..70 (brute force); formula = data for b ≤ 150 except b = 2 (W(2,2) = 28, where the formula would give 37). `supplement_B/scripts/familias_a2.py`, `supplement_B/scripts/fila_a_c.py` in `supplement.zip` of Zenodo record 10.5281/zenodo.24536373, Supplement B to Theorem B (semilength 3q), p. 9 (Section 4.1, Class sizes and the count), Supplement B to Theorem B (semilength 3q), p. 13 (Section 6.3, From the queries to Lemma 9.3)
- **Program:** the lab: W(2,b) = formula for b = 3..18, W(2,2) = 28, |L_15| = 136, |L_21| = 288; the control with e shifted by one residue fails. `lab/comprueba_lab.py`

Remark 9.5. (how many cells are redundant) p. 30

By Lemma 8.3 and Theorem 8.12, $W(1,b)/R(1,b) \to (9/4)/(5/2) = 9/10$: asymptotically one cell in ten repeats a word. By Lemma 9.2 and Theorem 9.4, $W(2,b)/R(2,b) \to (44/6)/(22/3) = 1$: the coincidences for $a = 2$ are only linearly many in b (about $43b/6$), whereas for $a = 1$ they are quadratically many (about $b^2/4$). Figure 7 shows the two ratios.

↳ Not used by another result of the paper.

Figure 7



Words per realisable cell, $W(a,b)/R(a,b)$, for $a=1$ ($2 \le b \le 60$) and $a=2$ ($3 \le b \le 150$), from exact enumeration; every value agrees with the closed forms of Lemmas 8.3, 9.2 and Theorems 8.12, 9.4.

10 The values of $|L_k|$ for small k

Remark 10.1. p. 31

Earlier counts posted in a public online discussion, 27 for $k=9$ and 42 for $k=10$, came from a search over $m \le 10^5$ and are only lower bounds. The exact values are 28 and 47.

↳ Not used by another result of the paper.

k	$\lvert L_k\rvert$	finest pattern	k	$\lvert L_k\rvert$	finest pattern
1	1	1	12	252	$p^2p'p''$
2	2	p	13	13	p^{12}
3	3	p^2	14	96	pp'^6
4	6	pp'	15	136	$p^2p'^4$
5	5	p^4	16	1083	$pp'p''p'''$
6	16	pp'^2	17	17	p^{16}
7	7	p^6	18	1222	$p^2p'^2p''$
8	40	$pp'p''$	21	288	$p^2p'^6$
9	28	$p^2p'^2$	25	255	$p^4p'^4$
10	47	pp'^4	35	971	$p^4p'^6$
11	11	p^{10}	49	1083	$p^6p'^6$

Table 1: $|L_k|$ for small k. Every value is exact; the colour gives how it is established: catWwritten proof (Corollary 3.3(a) for k prime or 1; Theorem A for $k = 2q$), catCTheorem B (computer-assisted step), catEexact enumeration in rational arithmetic. Two-prime values are exact enumerations of the cells of the arrangement of §2, and agree with Theorems A and B where those apply. The values for $k = 8, 12, 16, 18$ were enumerated exactly with SageMath polyhedra over $\mathbb{Q}$ in the box $x_i \le 50$, which meets every cell (Proposition 4.1). For $k = 16, 18$ they agree with an earlier floating-point enumeration.

Colour: blue shade = the number's size within its column.

11 Open questions

Question 11.1. p. 32

Is $E(a, b) = \Theta_a(b)$ for every fixed $a \ge 2$? Does almost every cell of the arrangement give its own word once the pattern has three rows, that is, $W(a, b)/R(a, b) \to 1$? Equivalently for the first question, by Corollary 5.2 and the linear lower bound for $a \ge 2$ after Lemma 7.5: is $W(a, b) = C_a b^2 + O_a(b)$? Where in the arrangement do the cells that share their word lie?

↳ Used in: 11.2 (p. 43)

Question 11.2. p. 33

Does $\lim_b W(a, b)/b^2$ exist for every a, that is, is the leading coefficient of $W(a, b)$ the same on all residue classes? Is it C_a for every $a \ge 2$?

↳ Not used by another result of the paper.

Question 11.3. (conjecture) p. 33

Find $W(a, b)$ for $a \ge 3$. For $a = 3$ the exact values for $b \le 110$ fit, for $b \ge 5$, the quasi-polynomial

$$W(3, b) = \tfrac{73}{5} b^2 + \tfrac{253}{30} b + c_{b \bmod 12} + u_{b \bmod 5}$$

with $(c_0, \dots, c_{11}) = (5, \frac{31}{6}, \frac{16}{3}, -\frac{25}{2}, \frac{20}{3}, \frac{17}{6}, 6, -\frac{65}{6}, \frac{13}{3}, \frac{7}{2}, \frac{23}{3}, -\frac{79}{6})$ and $(u_0, \dots, u_4) = (0, -\frac{1}{5}, \frac{2}{5}, -\frac{1}{5}, 0)$. The fit used $b \in [6, 66)$ and was validated on $b = 66, \dots, 110$ (45 points, no failure). The period is 60, with structure "12 plus 5 ", whereas for $a = 2$ the minimal period is 6. This is a **conjecture**: less than two full periods are observed. The leading coefficients are $9/4$, $22/3$ and $73/5$ for $a = 1, 2, 3$. The last two equal the leading coefficients C_2 and C_3 of the number of cells (Corollary 5.2), as rigidity predicts; $9/4 < C_1 = 5/2$ because of the family Q.

↳ Not used by another result of the paper.

Question 11.4. p. 33

Is the minimal period of the quasipolynomial $W(a,b)$ a divisor of $\operatorname{lcm}(1,\dots,2a)$? Is it equal to it for $a \geq 3$? Is the threshold N_a linear in a?

↳ Not used by another result of the paper.

Question 11.5. p. 33

Is there an absolute constant bounding $|\mathcal{F}(w)|$ for all $w \neq O(0)$, for every a and b? Is it 4? Classify the coincidence mechanisms. Three are proved in Section 7: the empty graph ($O(0)$, Lemma 7.1), exchange of arithmetic progressions between rows (family Q, Lemma 7.2), and reflection inside integer levels (family A, Lemma 7.5, with the variant of Lemma 7.7). Are there others?

↳ Not used by another result of the paper.

Question 11.6. p. 34

For fixed $a_1,\dots,a_{r-1}$, is $W(a_1,\dots,a_{r-1},b)$ eventually a quasipolynomial in b of degree exactly r? Compute $|L_{4q}|$ in closed form. Is there a formula for the square-free pattern $(1,1,\dots,1)$, that is, for $|L_{2^r}|$?

↳ Not used by another result of the paper.

Question 11.7. p. 34

Is membership in L decidable in time polynomial in k? Can the algorithm above be carried out inside first-order Peano arithmetic, which would answer Rodríguez Caballero's question [14] ? Is there an intrinsic description of L_{pq}, for instance a finite list of forbidden factors, or a characterisation of the unit interval graphs on a product of two chains that admit an additive representation with arithmetic progressions in each factor?

↳ Not used by another result of the paper.

Question 11.8. p. 34

For a fixed pattern f and a word $w \in L$, what is the asymptotic proportion of $n \leq N$ with odd part of pattern f and $\langle\langle n \rangle\rangle_2 = w$? Is it governed by a natural measure of the cells of $\mathcal{A}_f$ in the coordinates $\log p_i / \log t$, and which word is the most frequent?

↳ Not used by another result of the paper.

12 What this note does not claim

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper, p. 34.

A A reformulation through the Kassel–Reutenauer polynomials

Observation A.1.

Let $n \geq 1$ and let $h_1, \ldots, h_{2K-1}$ be the heights of the Dyck path $\langle\langle n \rangle\rangle_2$ after its first $1, \ldots, 2K-1$ letters, where $K = \tau(\text{odd part of } n)$. Then the coefficient sequence of $P_n(q)$, from q^0 to q^{2n-2}, consists of a nonempty run of h_1, then a nonempty run of h_2, and so on up to a nonempty run of h_{2K-1}. More precisely, $a_{n,i} = h_{J(i)}$, where $J(i) = \#\{k \in D_n \triangle 2D_n : k < \sqrt{2n},\ 2n/k - k \geq 2i+1\}$.

Isabelle · ours · part **show**

Only a part is in Isabelle: PARTIAL: from the coefficient formula of KR (3.2), taken as given, to the runs of the heights; not covered: that the ε_j are the first K letters of the Dyck word, the palindromy and the second half

`acoef_eq_hh` · `Isabelle/CaballeroDyckWords/Caballero_KR/Caballero_KR.thy`, line 194

```
theorem acoef_eq_hh:
  assumes "n > 0"
  shows "acoef n i = hh n (JJ n i)"
```

`dd_odd` · `Isabelle/CaballeroDyckWords/Caballero_KR/Caballero_KR.thy`, line 85

```
lemma dd_odd:
  assumes "n > 0" "k \<in> pairsK n"
  shows "odd (dd n k)"
```

`dd_gap` · `Isabelle/CaballeroDyckWords/Caballero_KR/Caballero_KR.thy`, line 104

```
lemma dd_gap:
  assumes "n > 0" "k \<in> pairsK n" "k' \<in> pairsK n" "k < k'"
  shows "dd n k' + 2 \<le> dd n k"
```

`JJ_antitone` · `Isabelle/CaballeroDyckWords/Caballero_KR/Caballero_KR.thy`, line 214

```
theorem JJ_antitone:
  assumes "i \<le> i'"
  shows "JJ n i' \<le> JJ n i"
```

`JJ_zero` · `Isabelle/CaballeroDyckWords/Caballero_KR/Caballero_KR.thy`, line 219

```
theorem JJ_zero:
  assumes "n > 0"
  shows "JJ n 0 = length (ks n)"
```

`JJ_pos` · `Isabelle/CaballeroDyckWords/Caballero_KR/Caballero_KR.thy`, line 238

```
theorem JJ_pos:
  assumes "n > 0" "i \<le> n - 1"
  shows "1 \<le> JJ n i"
```

JJ_surj · Isabelle/CaballeroDyckWords/Caballero_KR/Caballero_KR.thy, line 257

```
theorem JJ_surj:
  assumes "n > 0" "1 \<le> j" "j \<le> length (ks n)"
  shows "\<exists>i. i \<le> n - 1 \<and> JJ n i = j"
```

Trust base: ⚠️ not checked. Written proof; a part compiles in Isabelle/HOL (session Caballero_KR, public)

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: PARTIAL: from the coefficient formula of KR (3.2), taken as given, to the runs of the heights; not covered: that the ε_j are the first K letters of the Dyck word, the palindromy and the second half

🆕 **Classical and new:** the paper: «It is implicit in [RC-JNT, RC-lang]; we include a short proof for completeness and claim no novelty.» Rodríguez Caballero showed that the largest coefficient of P_n(q) is the Erdős–Nicolas function F(n) (JNT 2018); the identity used is Kassel–Reutenauer (3.2).

↳ Used in: A.2 (p. 46), Corollary A.3 (p. 47)

Proof

The pairs (k, m') in (4) are the factorisations $2n = km'$, $k<m'$, into factors of opposite parity. Let $n = 2^{e-1}m$. The odd factor o divides m and the even factor is $2n/o = t \cdot (m/o)$, so the pairs are $\{o, tm/o\}$ with $o \in D_m$. Their members are exactly $D_m \cup tD_m = D_n \triangle 2D_n$ (Lemma 2.1), and each element lies in exactly one pair. As $km' = 2n$ with $k \neq m'$, the smaller members $k_1 < \cdots < k_K$ are the K elements below $\sqrt{2n}$, that is, the first K letters of $\langle\langle n \rangle\rangle_2$. Put $\epsilon_j = (-1)^{k_j - 1}$; it is $+1$ if k_j is odd, i.e. the j-th letter is `(`, and -1 otherwise. So $h_j = \epsilon_1 + \cdots + \epsilon_j$ for $j \leq K$.

Comparing coefficients of q^i in (4) gives $a_{n,i} = \sum_{j:\, d_j \geq 2i+1} \epsilon_j$, where $d_j = m'_j - k_j = 2n/k_j - k_j$. The d_j are odd and strictly decreasing, so $\{j : d_j \geq 2i+1\} = \{1, \ldots, J(i)\}$ and $a_{n,i} = h_{J(i)}$. Here $d_1 = 2n - 1$ (from $k_1 = 1$), so $J(i) \geq 1$ for $i \leq n-1$, and $J(0) = K$. The function J takes the value $j \in [1, K]$ exactly for $i \in [(d_{j+1}+1)/2, (d_j - 1)/2]$ (with $d_{K+1} = -1$). This interval is nonempty because $d_j \geq d_{j+1} + 2$. Reading $i = n-1, \ldots, 0$ gives runs of $h_1, \ldots, h_K$. Palindromy of P_n and the symmetry $h_{2K-j} = h_j$ of the symmetric Dyck word [8 , Lemma 13] give the second half.

Evidence & scope

- **Program:** n = 1..400: the compressed coefficient profile of P_n(q) (from Kassel–Reutenauer, Theorem 1.2) equals the heights of the word, 400/400. working script `serendipia_KR.py`, not published
- **Audit:** machine-checked in Isabelle/HOL session Caballero_KR (task 294), PARTIAL: starting from the coefficient formula a_n,i = Σ_j ε_j [d_j ≥ 2i+1] (taken from KR (3.2), not rederived), acoef_eq_hh (a_n,i = h_J(i)), dd_odd and dd_gap (d_j odd, d_j ≥ d_j+1 + 2), JJ_antitone, JJ_zero, JJ_pos and JJ_surj (every value 1..K is taken: the runs are nonempty). Not covered: that the ε_j are the first K letters of the Dyck word, the palindromy and the second half. Compiles without sorry; the control fails. working script `Caballero_KR.thy`, not published

Example A.2.

For $n = 3$: $D_3 \triangle 2D_3 = \{1, 2, 3, 6\}$ and $\langle\langle 3 \rangle\rangle_2 = ()()$, with heights $h_1, h_2, h_3 = 1, 0, 1$. Here $C_3(q)/(q-1)^2 = P_3(q) = 1 + q + q^3 + q^4$, whose coefficient sequence $1, 1, 0, 1, 1$ consists of runs of 1, 0, 1. The zero coefficient is part of the profile and must not be dropped.

Proof: in this note · **Covers:** not formalized

↳ Not used by another result of the paper.

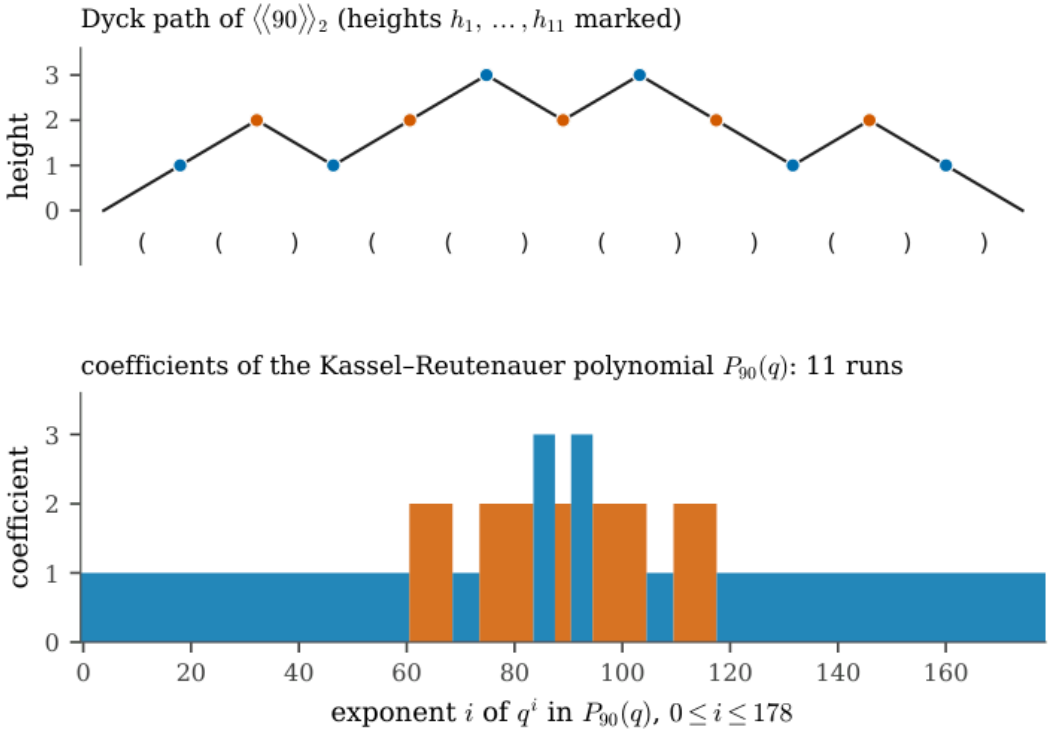
Corollary A.3.

> *L_k is in bijection with the set of compressed coefficient profiles of the polynomials $P_n(q)$ with $\tau(\text{odd part of } n) = k$.*

Proof: in this note · **Covers:** not formalized

↳ Not used by another result of the paper.

Figure 8



Top: the Dyck path of $\langle\langle 90 \rangle\rangle_2 = (()(()())())$ with the heights $h_1, \ldots, h_{11}$ marked alternately in blue and orange. Bottom: the 179 coefficients of $P_{90}(q)$ (from [4 , Theorem 1.2]); the 11 runs carry the same colours and the values $h_1, \ldots, h_{11}$.

The companion site

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper, p. 36.

Versions

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper (PDF).

Acknowledgements

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It is prose only (no statement, figure or table): read it in the paper (PDF).

📊 Tables

Every table of the paper, in its order: click one to unfold it, with where the paper cites it, how its numbers were checked, and its data as CSV. The typeset table is one click away, on its page of the PDF.

Table without a number, in the text of §8.3 Proof of (⇒)

type	n_0	$\mathcal{Z}_0$
P^+	A	$\{1,3,\dots,2n_0-1\}$
P^-	$A-1$	same
P^0, P^c	g	same
N^-	ν	$\{\rho+2i : 0\le i<n_0\}$
N^0, N^c	$g-\rho+1$	same
N^+	$\nu+1$	same
N_2'	$\rho-g-1$	$[2g+\nu+2-\rho,\ \nu+g+1)$

⬇ CSV

Table 1. $|L_k|$ for small k. Every value is exact; the colour gives how it is established: catW written proof (Corollary 3.3(a) for k prime or 1; Theorem A for $k=2q$), catC Theorem B (computer-assisted step), catE exact enumeration in rational arithmetic. Two-prime values are exact enumerations of the cells of the arrangement of §2, and agree with Theorems A and B where those apply. The values for $k=8,12,16,18$ were enumerated exactly with SageMath polyhedra over $\mathbb{Q}$ in the box $x_i\le 50$, which meets every cell (Proposition 4.1). For $k=16,18$ they agree with an earlier floating-point enumeration.

k	$\lvert L_k\rvert$	finest pattern	k	$\lvert L_k\rvert$	finest pattern
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6	16	pp'^2	17	17	p^{16}
7	7	p^6	18	1222	$p^2p'^2p''$
8	40	$pp'p''$	21	288	$p^2p'^6$
9	28	$p^2p'^2$	25	255	$p^4p'^4$
10	47	pp'^4	35	971	$p^4p'^6$
11	11	p^{10}	49	1083	$p^6p'^6$

Colour: blue shade = the number's size within its column.

p. 32 in the PDF ⬇ CSV cited in the opening, before §1 cited in §4 Decidability and complexity cited in §10 The values of |L_k| for small k cited in §12 What this note does not claim

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