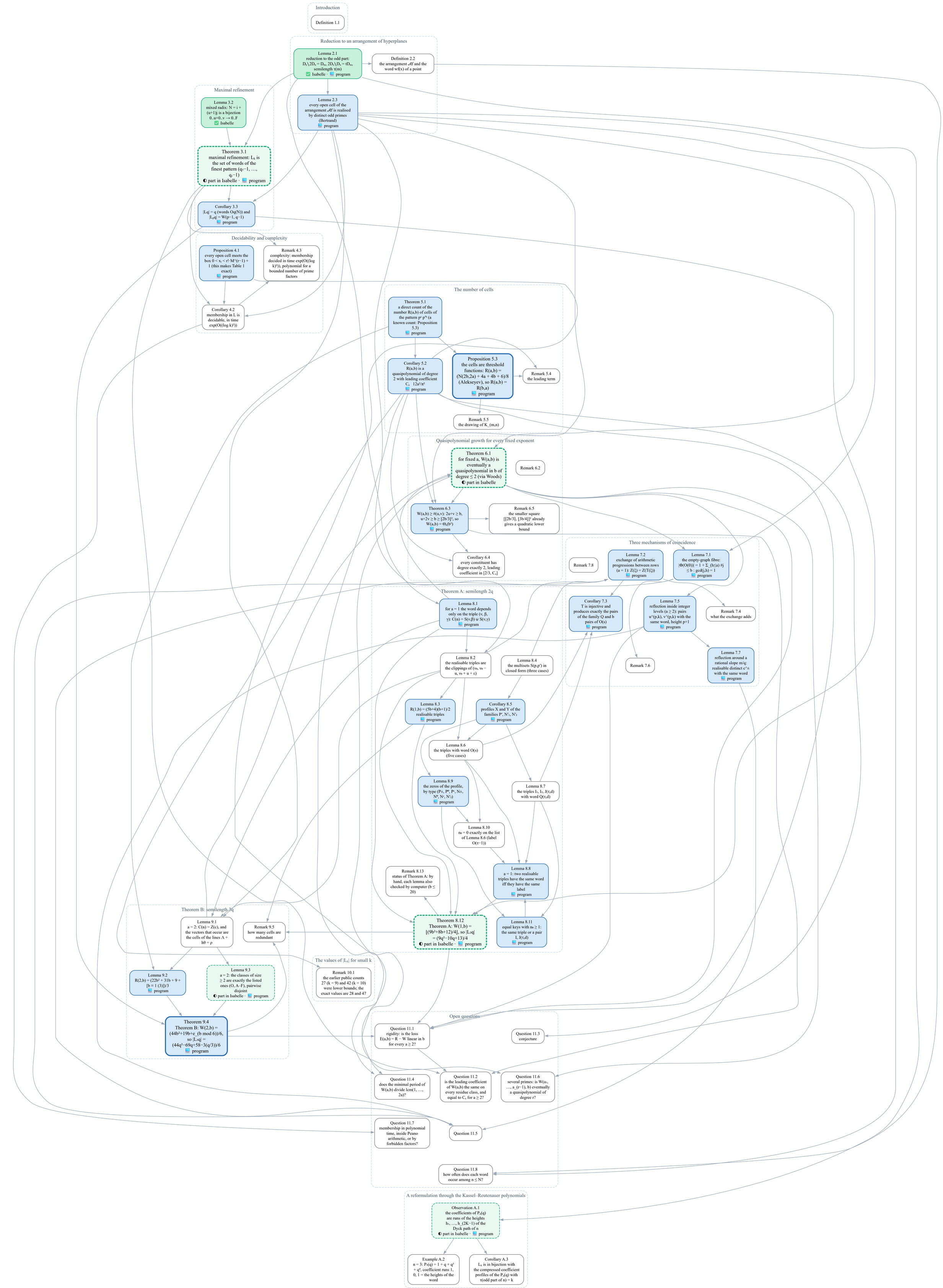
Map of *How Many Dyck Words Do the Divisors of an Integer Make? Exact Counts and Quasipolynomial Growth*

Combinatorics on words · divisors of integers · enumeration of hyperplane arrangements

58 results | 0 entirely in Lean | 5 partly in Lean | 2 in Isabelle | 25 checked by a program | written proof only | (a result can carry several tags)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



What the paper contributes

The language L of the words $\langle\!\langle n\rangle\!\rangle_2$ and the question of describing it are Rodríguez Caballero's. Starting from the reduction to the odd part, the paper counts the words of each semilength:

- **Part I (structure).** The words of a pattern are the words of the cells of a hyperplane arrangement in the logarithms of the primes, and every cell is realised by primes (Lemma 2.3). L_k is produced by the single finest pattern $p_1^{q_1-1}\cdots p_r^{q_r-1}$, $k=q_1\cdots q_r$ (Theorem 3.1), so membership in L is decidable (Corollary 4.2). Written proofs.
- **Part II (general enumeration).** The number $R(a,b)$ of cells of the pattern $p^ap'^b$ is a known count: $R(a,b)=(N(2b,2a)+4a+4b+6)/8$, with N Alekseyev's number of two-dimensional threshold functions on a grid (Proposition 5.3; Acketa–Žunić, Alekseyev); the paper gives a direct proof, with leading coefficient $C_a\sim12a^2/\pi^2$ (Theorem 5.1, Corollary 5.2); $W(a,b)$ is eventually a quasipolynomial in b of degree exactly two (Theorem 6.1, an application of Woods' theorem, existence only; Theorem 6.3, Corollary 6.4); three mechanisms by which distinct cells give the same word (Section 7). Written proofs.
- **Part III (explicit counts).** $|L_{2q}|=(9q^2-10q+13)/4$ for odd primes q (Theorem A, written proof) and $|L_{3q}|=(44q^2-69q+58-3(\tfrac q3))/6$ for primes $q\ge5$ (Theorem B: one lemma is decided by exact SMT queries with Z3, without a checked proof object); a table of $|L_k|$ for small k by exact enumeration.

The paper claims no novelty for the Kassel–Reutenauer reformulation of the appendix, which is implicit in Rodríguez Caballero's work (JNT 2018). The arrangements and their link with semiorders go back to Stanley; counting cells through vertices is the planar case of Zaslavsky's face-count formula; the quasipolynomiality rests on a theorem of Woods.

🚀 **Start here:** **Lemma 2.1**: reduction to the odd part: $D_n\backslash 2D_n = D_m$, $2D_n\backslash D_n = tD_m$, semilength $\tau(m)$.

Main results:

- **Theorem 3.1**: maximal refinement: L_k is the set of words of the finest pattern $(q_1-1, \ldots, q_r-1)$
- **Proposition 5.3**: the cells are threshold functions: $R(a,b) = (N(2b,2a) + 4a + 4b + 6)/8$ (Alekseyev), so $R(a,b) = R(b,a)$
- **Theorem 6.1**: for fixed a, $W(a,b)$ is eventually a quasipolynomial in b of degree ≤ 2 (via Woods)
- **Theorem 8.12**: Theorem A: $W(1,b) = \lfloor(9b^2+8b+12)/4\rfloor$, so $|L_{2q}| = (9q^2-10q+13)/4$
- **Theorem 9.4**: Theorem B: $W(2,b) = (44b^2+19b+e_{(b \bmod 6)})/6$, so $|L_{3q}| = (44q^2-69q+58-3(q/3))/6$

🧭 Fields: where the paper lives and what it connects

Tools taken from other branches

- **Language of divisor words (Rodríguez Caballero)** → the word $\langle\langle n\rangle\rangle_2$, its Dyck property and symmetry, the reduction to the odd part, the decision problem (FOM 2020) and the link with the Kassel–Reutenauer polynomials are his — Lemma 2.1, Corollary 4.2, A.1
- **Hyperplane arrangements (Stanley; Zaslavsky; Beck–Zaslavsky)** → the word is the cell of a point in an arrangement $\alpha \cdot x = 1$; cells are counted through vertices (Euler's relation, the planar case of Zaslavsky's face count) — Lemma 2.3, Theorem 5.1
- **Two-dimensional threshold functions (Acketa–Žunić 1991; Alekseyev 2010)** → the number of cells R(a,b) is (N(2b,2a) + 4a + 4b + 6)/8, with N Alekseyev's exact count of threshold functions on a grid; its leading term is Alekseyev's too — Proposition 5.3
- **Presburger arithmetic (Woods 2015)** → W(a,b) as a Presburger counting function is eventually a quasipolynomial (existence only) — Theorem 6.1
- **Prime numbers (Bertrand's postulate) and SMT solving (Z3, cvc5)** → every open cell is realised by primes; the completeness and disjointness of the coincidence classes for a = 2 are decided by exact SMT queries — Lemma 2.3, Lemma 9.3

What the results serve

- **Counting the language L** ← $|L_q| = q$, $|L_{2q}|$ and $|L_{3q}|$ in closed form, $|L_k|$ for small k, the growth of W(a,b) and open questions on rigidity, periods and fibres — Corollary 3.3, Theorem 8.12, Theorem 9.4, Corollary 6.4