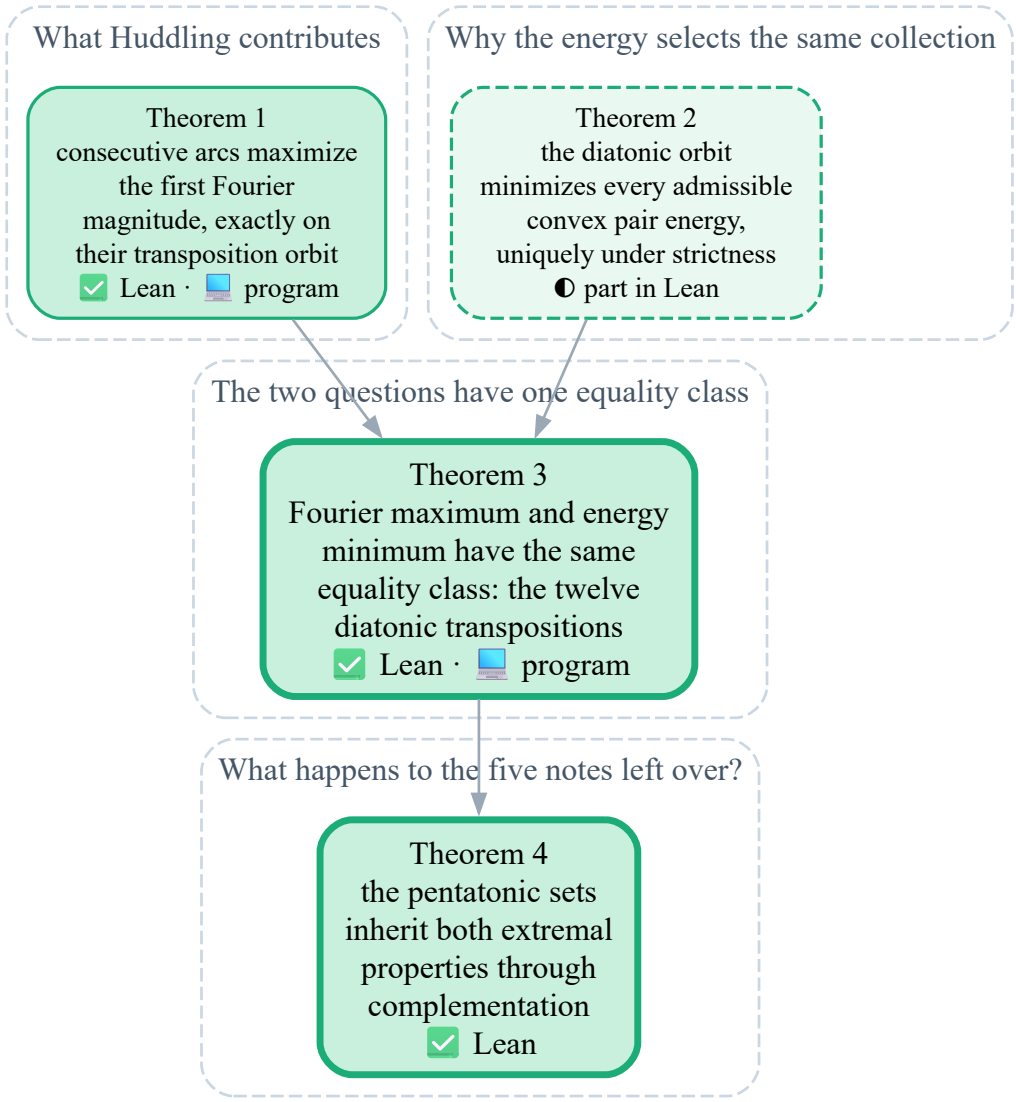
Map of *Huddling, diatonic and pentatonic scales in Z12: Fourier maxima and convex-energy minima formalized in Lean 4*

Mathematical music theory · discrete Fourier analysis on $\mathbb{Z}_{12}$

4 results | ✅ **3** entirely in Lean | ◐ **1** partly in Lean | 💻 **2** checked by a program | written proof only | (a result can carry several tags)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



What the paper contributes

- A connected Lean 4 formalization, with complete twelve-tone equality conditions, of why the seven white keys solve two optimization problems: they maximize $|\widehat A(5)|$ and minimize every strictly admissible convex pair energy (Theorem 3).
- The route: Huddling in $\mathbb{Z}_{12}$ (Theorem 1), transported to frequency five by M_5; double Abel summation for the energy (Theorem 2); complementation carries both to the pentatonic sets (Theorem 4).
- Exact arithmetic throughout: comparisons in $\mathbb{Z}[\sqrt3]$, no rounded trigonometric values. Two finite censuses use `native_decide` (trusted compiled evaluation), stated in the axiom audit.
- The mathematics is classical; the paper claims the connected formalization, not a new characterization.

🚀 **Start here: Theorem 1**: consecutive arcs maximize the first Fourier magnitude, exactly on their transposition orbit.

Main results:

- **Theorem 3**: Fourier maximum and energy minimum have the same equality class: the twelve diatonic transpositions
- **Theorem 4**: the pentatonic sets inherit both extremal properties through complementation

🧭 Fields: where the paper lives and what it connects

Tools taken from other branches

- **Discrete Fourier analysis** → the Huddling lemma for sums of distinct roots of unity (Amiot), the index permutation M_5, exact comparison in $\mathbb{Z}[\sqrt3]$ — Theorem 1
- **Summation by parts** → double Abel summation of the interval counts against a discretely convex potential — Theorem 2
- **Formal verification** → Lean 4 + Mathlib: two finite censuses (native_decide) and symbolic transport and complement bridges — Theorem 1, Theorem 2, Theorem 3, Theorem 4

What the results serve

- **Maximal evenness and scale theory** ← the diatonic and pentatonic collections as one equality class of a Fourier maximum and a convex-energy minimum —

Theorem 3, Theorem 4

- **Energy of vertex sets in graphs** ← the C_{12} instance of the complement-energy theorem of Bushaw, Cody and Leffler, checked in Lean — Theorem 4