Integer frieze patterns and Tabachnikov's four sign changes

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Abstract

Tabachnikov asked for which rows the differences between the entries of a row of two frieze patterns of the same width must change sign at least four times along the period, and left open whether this holds for all rows when the entries are positive integers. We show that it does not. For every even $n \geq 14$ there are two integer (Conway–Coxeter) frieze patterns of width $n-3$ whose middle rows are strictly ordered, and for every odd $n \geq 15$ there are two whose rows $\lfloor (n-3)/2 \rfloor$ differ with no zero and exactly two sign changes. The second family lies inside the range $1 \leq k \leq [w/2]$ in which the question is stated. The differences in both families never vanish, so they are counterexamples however zeros are counted. When zeros may count as sign changes, an exhaustive search shows that 14 is the smallest n with strictly ordered rows and 15 the smallest with fewer than four sign changes in a row of the stated range. Both families are verified in Isabelle/HOL for all n, and the base pairs also in Lean.

Companion site. `https://karlesmarin.github.io/blueprints/integer-frieze-sign-changes/`: every statement with its dependencies, its formal proofs and its checks, and labs in the browser (Section 8). This note: doi:10.5281/zenodo.23656979 (Zenodo), with its supplementary material.

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1 The question

A frieze pattern of width w is an array of w bi-infinite rows of numbers, each offset by half a step from the previous one, bordered above and below by a row of 1s and a row of 0s, in which every elementary diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfies $ad - bc = 1$ [3]. Put $n = w + 3$. Conway and Coxeter showed that the frieze patterns whose entries are positive integers are exactly those whose first row $(a_0, \ldots, a_{n-1})$, the *quiddity*, counts the triangles at the vertices of a triangulation of an n-gon [1, 2]; see [4] for a survey.

Following [5], write $v_{i,j}$ for the entries, with $a_i = v_{i-1,i+1}$, so that row k consists of the entries $v_{i,i+k+1}$, $i \in \mathbb{Z}$; it is n-periodic, and by the glide symmetry $v_{i,j} = v_{j,i+n}$ rows k and $w+1-k$ carry the same values. Given two frieze patterns u and v of the same width, Tabachnikov considers the cyclic sequence $d_i = v_{i,i+k+1} - u_{i,i+k+1}$ and asks:

> *Problem 1* [5]. For which k must the cyclic sequence d_i have at least four sign changes?

The range considered there is $1 \le k \le [w/2]$, by the glide symmetry. Four sign changes must occur for $k = 1$ and $k = 2$, and for every k in an infinitesimal version of the problem. For real entries the answer is not "all k": M. Cuntz gave two frieze patterns of width 5 with positive rational entries whose third rows are strictly ordered. Tabachnikov notes that the statement might still hold for every k when the entries are positive integers.

Zeros. The differences may vanish, and the question counts sign changes "not excluding the case when either of these differences vanishes". We use three readings of the number of sign changes of a non-zero cyclic sequence d: $s^-(d)$, the sign changes after deleting the zeros; $s^+(d)$, the largest number of sign changes obtainable by giving signs to the zeros; and $E(d) = \#\{i : d_i d_{i+1} \le 0\}$, indices modulo the length n of d. Two distinct frieze patterns may have a whole row in common (for $n = 6$, the quiddities $(1, 2, 2, 2, 1, 4)$ and $(2, 1, 4, 1, 2, 2)$ both have row 2 equal to $(3, 3, 1, 3, 3, 1)$); for $d = 0$ we put $s^-(0) = 0$, $s^+(0) = 2\lfloor n/2 \rfloor$ and $E(0) = n$. With Z the number of zeros of d and $c = \#\{i : d_i d_{i+1} < 0\}$,

$$Z + c \;\le\; s^+(d) \;\le\; E(d) \qquad (d \neq 0),$$

and $s^- \le s^+$. Indeed, giving every zero the sign of the non-zero entry before it leaves exactly the s^- changes of the sequence without zeros; a block of ℓ zeros between two non-zero entries can be signed so as to give at least ℓ sign changes, whatever the signs at its ends, while the c strict changes between non-zero neighbours remain; and a sign change after signing the zeros is a pair of neighbours with $d_i d_{i+1} \le 0$. All three readings agree when d has no zero. Our counterexamples have no zero, so they are counterexamples in every reading.

2 Results

Theorem A. *For every even $n \ge 14$ there are two integer frieze patterns of width $n - 3$ whose middle rows ($k = n/2 - 1$, the diameters of the n-gon) are strictly ordered: $d_i > 0$ for all i.*

Theorem B. *For every odd $n \ge 15$ there are two integer frieze patterns of width $w = n - 3$ whose rows $k = \lfloor w/2 \rfloor = (n - 3)/2$ differ with no zero and exactly two sign changes.*

Corollary 1. *Let $w = n - 3$.*

(a) For every odd $n \ge 15$, four sign changes fail for integer frieze patterns of width w in a row of the printed range $1 \le k \le \lfloor w/2 \rfloor$.

(b) For every $n \ge 14$, they fail in a row of the range $1 \le k \le \lceil w/2 \rceil$.

For even n the middle row is row $\lceil w/2 \rceil = \lfloor w/2 \rfloor + 1$, just outside the printed range; Cuntz's example is a middle row as well (width 5, row 3), which is why (b) counts it. For even n the constructions here say nothing about the rows $k \le \lfloor w/2 \rfloor$: by Proposition 1 there is no counterexample there for $n = 14$, and the question is open for larger even n.

Proposition 1. *For $w \le 10$, every pair of distinct integer frieze patterns of width w has, in every row $k \le \lceil w/2 \rceil$, $Z + c \ge 4$; hence $s^+ \ge 4$ and $E \ge 4$. For $w = 11$ the same holds in rows $k \le 5$, and in the middle row there are 672 pairs (a, b) with $d_i > 0$ for every i (each such unordered pair counted once, in the order that makes d positive). For $w = 12$ the same holds in rows $k \le 5$; in row 6 there are 480 unordered pairs with no zero and exactly two sign changes, and 60 more with one zero and $s^+ = 2$. Frieze patterns are labelled: a rotated or reflected triangulation counts as a different frieze pattern.*

So $n = 14$ is the smallest n with strictly ordered rows, and $n = 15$ the smallest with fewer than four sign changes in some row $k \le [w/2]$, in the readings s^+ and E. Proposition 1 is the

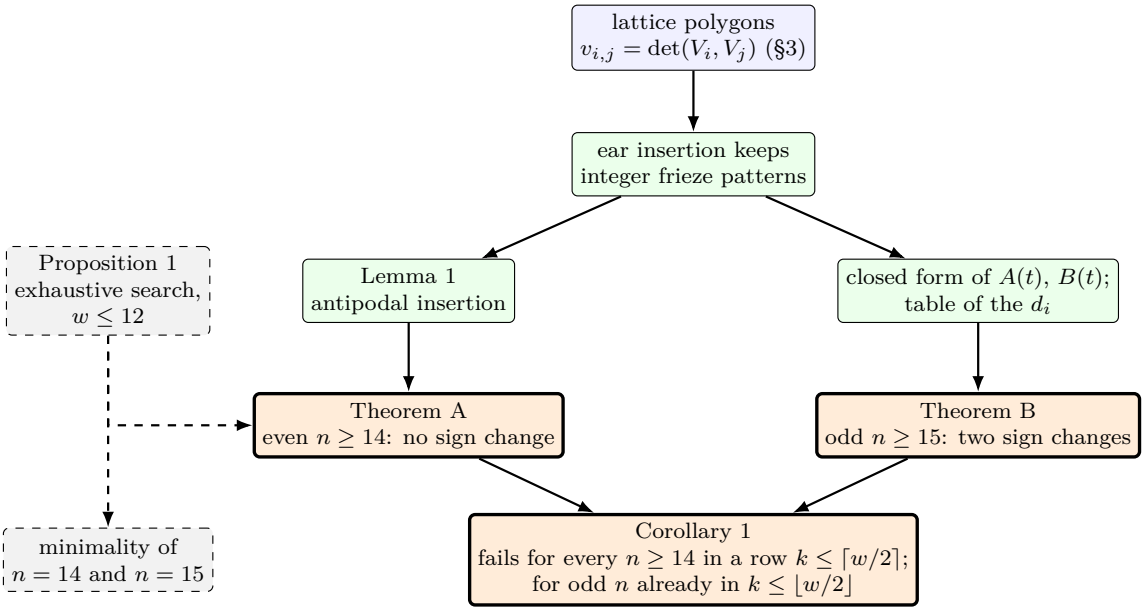


Figure 1: Map of the note. Blue: model; green: lemmas; orange: results; grey: computation. The arrow from Proposition 1 to Theorem A records where the base pair of $n = 14$ was found.

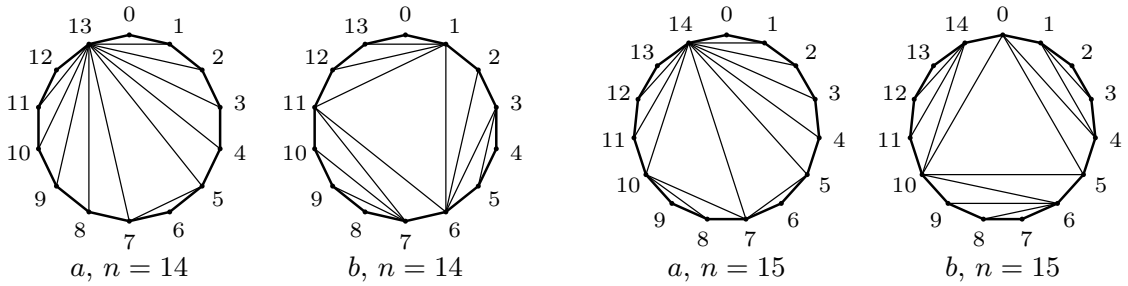


Figure 2: The triangulations of the two base pairs. Left: $n = 14$, the pair of Theorem A ($t = 0$); right: $n = 15$, the pair of Theorem B ($t = 0$). The quiddity a_i is the number of triangles at vertex i.

result of an exhaustive computer search over all unordered pairs of frieze patterns (Section 6); it is not formalized.

Figure 1 shows how the results depend on each other.

Remark 1. In the reading s^- (zeros deleted) the answer is negative much earlier: there are integer pairs with $s^- = 2$ in row 3 of width 6, and pairs whose non-zero differences all have the same sign in widths 7 and 8; for example $a = (1, 2, 2, 3, 3, 2, 1, 3, 4, 1, 5)$ and $b = (1, 3, 1, 5, 4, 1, 2, 2, 3, 2, 3)$ give, in row 3 of width 8, $d = (0, 2, 1, 0, 0, 0, 0, 1, 0, 0, 0)$, with $s^- = 0$, $s^+ = 8$ and $E = 10$. These examples all have zeros.

3 Frieze patterns as lattice polygons

A frieze pattern of width $w = n - 3$ with positive entries is the same as a sequence of vectors $V_0, V_1, \ldots, V_{n-1} \in \mathbb{R}^2$, up to $\mathrm{SL}(2, \mathbb{R})$, with $\det(V_i, V_{i+1}) = 1$, $\det(V_0, V_{n-1}) = 1$ and $\det(V_i, V_j) > 0$ for $0 \le i < j < n$; extended by $V_{i+n} = -V_i$, the entries are $v_{i,j} = \det(V_i, V_j)$ [5, 4]. The quiddity is $a_i = \det(V_{i-1}, V_{i+1})$, and $V_{i+1} = a_i V_i - V_{i-1}$. For an integer frieze pattern one

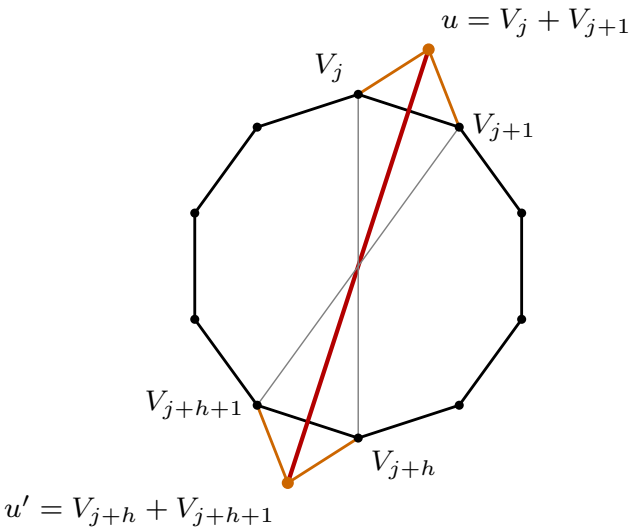


Figure 3: Antipodal ear insertion. The old diameters (grey: two of them) keep their values; the only new diameter is (u, u') (red), of value $S = \det(V_j + V_{j+1}, V_{j+h} + V_{j+h+1})$.

may take $V_0 = (1, 0)$, $V_1 = (0, 1)$, and then all V_i lie in $\mathbb{Z}^2$; the entries are the continuants $v_{i,j} = K(a_{i+1}, \ldots, a_{j-1})$, where $K() = 1$, $K(x_1) = x_1$ and $K(x_1, \ldots, x_r) = x_r K(x_1, \ldots, x_{r-1}) - K(x_1, \ldots, x_{r-2})$ (so $K(x, y) = xy - 1$, $K(x, y, z) = xyz - x - z$). In particular row k is $v_{i,i+k+1} = K(a_{i+1}, \ldots, a_{i+k})$, $i = 0, \ldots, n-1$; we write $r_k(a)$ for this row of the frieze pattern with quiddity a, listed from $i = 0$, so that $d = r_k(b) - r_k(a)$. Two facts are used throughout, both immediate from the bilinearity of the determinant:

- *Ear insertion.* Inserting $V_j + V_{j+1}$ between V_j and V_{j+1} gives again an integer frieze pattern, of width $w + 1$; in the triangulation, this glues a triangle on the side $(j, j + 1)$.
- *New entries.* The entries between old vectors do not change, and an entry involving the new vector is a sum of two old ones: $\det(V_j + V_{j+1}, V_x) = \det(V_j, V_x) + \det(V_{j+1}, V_x)$.

4 Proof of Theorem A

Let $n = 2h$. The middle row consists of the diameters $D_i = \det(V_i, V_{i+h})$, $0 \le i < h$.

Lemma 1 (antipodal insertion)**.** *Insert ears after the positions j and $j + h$. The diameters of the new polygon are the old ones, in the same cyclic order, together with one new diameter between the two new vertices, of value*

$$S(V, j) = \det(V_j + V_{j+1},\, V_{j+h} + V_{j+h+1}).$$

Hence if $D_i(U) < D_i(V)$ for all i and $S(U, j) < S(V, j)$, the strict order survives the insertion.

Proof. A new diameter between two old vertices x, x' has one new vertex on each side, so x' is the old antipode of x and the value is the old one. The only other diameter joins the two new vertices (Figure 3). □

Start from the pair of $n = 14$,

$$a(0) = (1, 2, 2, 2, 2, 3, 1, 3, 2, 2, 2, 2, 1, 11), \qquad b(0) = (1, 5, 2, 3, 1, 2, 5, 4, 1, 2, 2, 4, 2, 2),$$

whose middle rows $D_0, \ldots, D_6$ are $(6, 6, 6, 6, 6, 6, 2)$ and $(7, 7, 8, 7, 7, 7, 3)$ (Figure 2, left; each repeated twice along the period 14), and insert antipodal ears at $j = 0$, t times ($t \in \mathbb{Z}_{\ge 0}$). Write

$n_t = 14 + 2t$ and $h_t = 7 + t$. Along the family V_0 and V_{h_t} keep their coordinates (the index h_t grows), while V_1 and V_{h_t+1} move by V_0 and V_{h_t} at each step, so S is a quadratic polynomial in t:

$$S(a(t), 0) = 6(t+2)^2, \qquad S(b(t), 0) = 7t^2 + 40t + 40,$$

and $S(b(t), 0) - S(a(t), 0) = t^2 + 16t + 16 > 0$. Lemma 1 and induction on t give Theorem A. In terms of quiddities, for $t \geq 1$ (with 2^r standing for r consecutive 2s),

$$a(t) = (t+1,\ 1,\ 2^{t-1},\ 3, 2, 2, 2, 3, 1,\ t+3,\ 1,\ 2^{t-1},\ 3, 2, 2, 2, 1, 11),$$
$$b(t) = (t+1,\ 1,\ 2^{t-1},\ 6, 2, 3, 1, 2, 5,\ t+4,\ 1,\ 2^{t+2},\ 4, 2, 2).$$

5 Proof of Theorem B

Let $t \geq 0$, $n = 15 + 2t$, $h = t + 7$, so $n = 2h + 1$ and row $k = h - 1 = \lfloor w/2 \rfloor$ consists of the n entries

$$e_i = \det(V_i, V_{i+h})\ (0 \leq i \leq h), \qquad e_i = \det(V_{i-h-1}, V_i)\ (h < i < n),$$

the second form by the glide symmetry. Consider the two lattice polygons, each made of five pieces (a block of fixed vectors, an arithmetic progression of $t+1$ vectors, five fixed vectors, a progression of $t+1$ vectors, five fixed vectors), with $0 \leq s \leq t$:

$$\begin{aligned}
A(t) :\ & (1,0), (0,1), (-1,2); \quad \big(-(m+1),\ 2m+1\big)_{m=t+1-s}; \\
& (-3,4), (-4,5), (-9,11), (-5,6), (-11,13); \quad \big(-6-11m,\ 7+13m\big)_{m=t+1-s}; \\
& (-6,7), (-7,8), (-8,9), (-9,10), (-1,1), \\
B(t) :\ & (1,0), (0,1), (-1,3); \quad \big(-m,\ 3m-1\big)_{m=t+1-s}; \\
& (-1,1), (-2,1), (-5,2), (-18,7), (-13,5); \quad \big(-8-13m,\ 3+5m\big)_{m=t-s}; \\
& (-3,1), (-7,2), (-11,3), (-15,4), (-4,1).
\end{aligned}$$

For $t = 0$ their quiddities are

$$(1, 2, 2, 2, 2, 3, 1, 4, 2, 1, 4, 2, 2, 1, 10), \qquad (4, 3, 1, 2, 3, 3, 4, 1, 2, 2, 5, 2, 2, 1, 4)$$

(Figure 2, right), and inserting ears first after the position $t + 8$ and then after the position 2 turns $A(t)$ into $A(t+1)$ and $B(t)$ into $B(t+1)$: each progression gains a new first vector. Hence all $A(t)$, $B(t)$ are integer frieze patterns, by induction. For $t \geq 1$ their quiddities are

$$(1,\ 2,\ t+2,\ 1,\ 2^{t-1},\ 3, 2, 3, 1, 4,\ t+2,\ 1,\ 2^{t},\ 4, 2, 2, 1, 10),$$
$$(4,\ 3,\ t+1,\ 1,\ 2^{t-1},\ 3, 3, 3, 4, 1,\ t+2,\ 1,\ 2^{t-1},\ 3, 5, 2, 2, 1, 4).$$

Each e_i is the determinant of two vectors that depend affinely on t and on the position, so the differences $d_i = e_i(B(t)) - e_i(A(t))$ are explicit. For every $t \geq 1$ they are given by the following table, in which s is a parameter local to each line and $u = t - s$:

positions i	d_i
$0,\ 1$	$1,\ 2$
2	$25t+7$
$3+s,\ 0\le s<t$	$25u^2-4u-12\quad(u\ge 1)$
$t+3,\ \ldots,\ t+7$	$1,1,1,1,9$
$t+8$	$5-13=-8$
$t+9$	$2t-9$
$t+10$	$25t-18$
$t+11+s,\ 0\le s<t-1$	$25u^2-29u-26\quad(u\ge 2)$
$2t+10,\ \ldots,\ 2t+14$	$4,7,3,2,1$

Table 1: The differences d_i of Theorem B for every $t\ge 1$ (s local to each line, $u=t-s$).

The two quadratics are positive in their ranges:

$$25u^2-4u-12=25(u-1)^2+46(u-1)+9\ge 9,\qquad 25u^2-29u-26=25(u-2)^2+71(u-2)+16\ge 16,$$

and $25t-18>0$ for $t\ge 1$, while $2t-9$ is negative for $t\le 4$, positive for $t\ge 5$ and never zero. So for $t\ge 5$ the only negative entry is $d_{t+8}=-8$; for $1\le t\le 4$ the only negative entries are d_{t+8} and d_{t+9}, which are adjacent; and for $t=0$ a direct computation gives

$$d=(1,2,7,1,1,1,1,9,-8,-9,3,7,3,2,1).$$

In every case d has no zero and one block of negative entries, hence exactly two cyclic sign changes. Figure 4 shows the pattern, and Figure 5 the whole family for $t\le 8$.

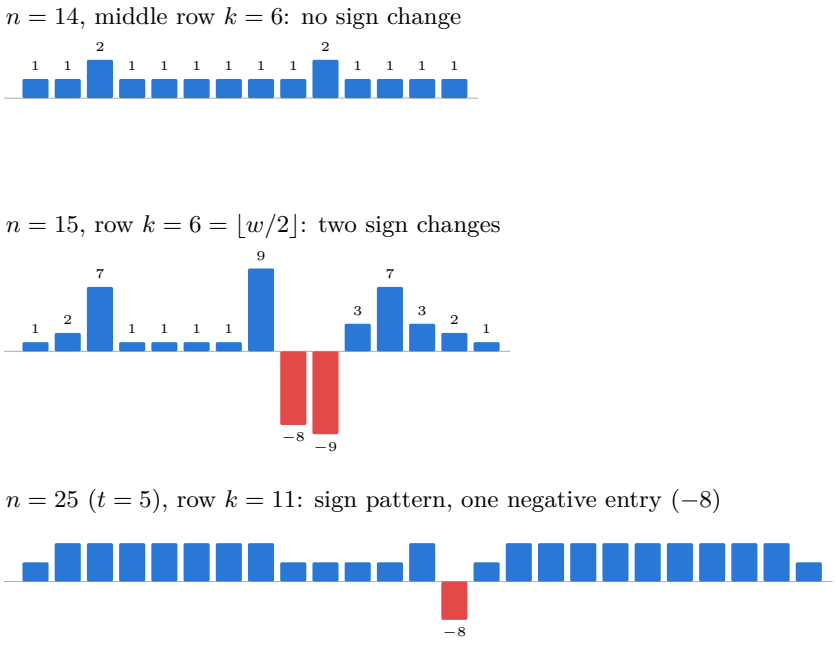


Figure 4: The differences $d_i=r_k(b)_i-r_k(a)_i$ along the period, blue for $d_i>0$ and red for $d_i<0$. Top and middle: actual values. Bottom: only the sign (bar height 1 if $|d_i|=1$, else 2), since the entries grow quadratically; the single negative entry is -8 for every $t\ge 5$.

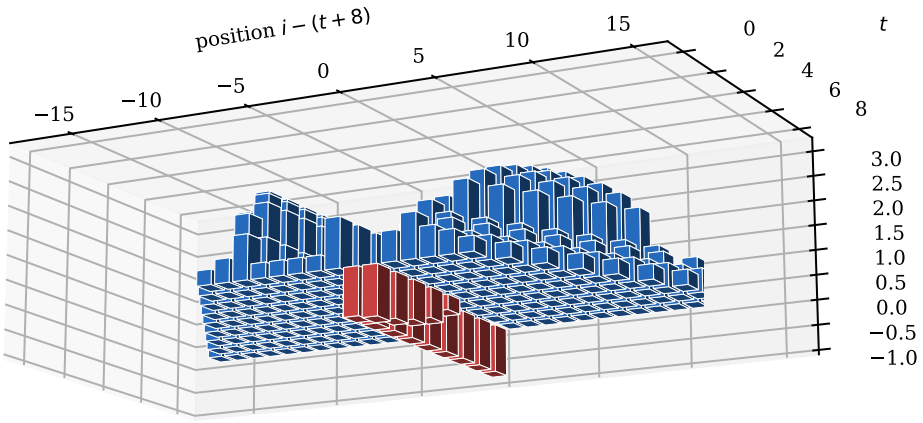


Figure 5: The differences $d_i(t)$ of Theorem B for $t = 0, \ldots, 8$, centred at the negative entry $i = t + 8$, on a signed logarithmic scale ($\operatorname{sign}(d)\log_{10}(1 + |d|)$), seen slightly from below. Blue: $d_i > 0$; red: $d_i < 0$. Along the whole family a single column (two for $t \leq 4$) stays negative.

6 The search

Proposition 1 was obtained by enumerating all integer frieze patterns of width $w \leq 12$ (one for each triangulation of the labelled n-gon, a Catalan number of them: 742 900 for $w = 12$) and, for every row $k \leq \lceil w/2 \rceil$ and every unordered pair, computing Z, c and E on a GPU; the exact value of s^+ was computed for the candidates with $Z + c \leq 3$. The pairs found were checked again by three independent constructions of the frieze patterns: from the quiddity by continuants (with $M = -\mathrm{Id}$ for the monodromy, $\sum a_i = 3n - 6$, positivity, and the closing rows of 1s and 0s), by the diamond rule in exact arithmetic, and by the search code itself. Those rechecks confirm the pairs found, not that the search missed nothing; the exhaustiveness rests on the program, which loops over every unordered pair of distinct labelled frieze patterns. As controls, the number of frieze patterns of each width is the Catalan number; for $w \leq 8$ the minima of s^-, s^+ and E in every row agree with an independent tabulation; and in rows $k \leq 5$ of width 12 the same code reports no pair with fewer than four sign changes. The rows are stored as 64-bit integers; the largest entry in the rows examined is 377 for $w = 12$ (computed over all frieze patterns), so every product $d_i d_{i+1}$ is below 377^2 in absolute value and no overflow occurs. The programs and their complete outputs are in the supplement. The family of Theorem A was found by inserting antipodal ears into a pair of $n = 14$, and that of Theorem B by inserting pairs of ears at the same positions into pairs of $n = 15$, keeping those that preserved two sign changes, and reading off a fixed rule.

7 Formal verification

Theorems A and B are proved in Isabelle/HOL for all t, in the lattice-polygon model of Section 3: `Integer_Frieze_Rows` (Theorem A, with the bridge to continuants of the quiddity) and `Odd_Frieze_Rows` (Theorem B, built on the first). They check that every polygon of each family is a frieze pattern, the value of every entry of the row, and the count of sign changes; the cases $t \leq 4$ of Theorem B are evaluated by the simplifier. The pairs of $n = 14$ and $n = 15$ are

also certified in Lean 4 by kernel computation (`decide`), with `propext` as the only axiom. The search behind Proposition 1 is not formalized.

The files are in `https://github.com/karlesmarin/formalizations`, commit `b9f258c`: the Isabelle sessions `Integer_Frieze_Rows` and `Odd_Frieze_Rows` under `Isabelle/AFP/` and the base pairs in `Isabelle/FriezeDominance/` (built with Isabelle2025-2 and Isabelle2026, `isabelle build -D` *dir*); the Lean modules `Formalizations/Combinatorics/Frieze/IntegerDominance.lean` and `TwoSignChanges.lean` (Lean v4.35.0 with Mathlib, `lake build`). The formal statements are about the lattice-polygon model of Section 3; the equivalence with triangulations and with the diamond rule is the classical one and is not formalized, and the function `quid` of `Integer_Frieze_Rows` gives the quiddity at the interior positions $1, \dots, n-2$ only, which are the ones its theorems use.

8 The companion site

This note has a companion website, its *blueprint*, at `https://karlesmarin.github.io/blueprints/integer-frieze-sign-changes/`. It shows the note as a whole: what it proves, how its results depend on each other, and what exactly has been checked. There one finds:

- *The statements.* Every numbered statement, copied verbatim with its number (2 theorems, 1 proposition, 1 lemma, 1 corollary and 1 remark), each with its proof, where it has one, a click away.
- *The dependencies.* For each statement, what it uses and where it is used, as a list and as a graph.
- *The map.* One page with the main results, the tools they rest on and the fields they feed.
- *What was checked, and how.* Besides the written proof, the site says for each statement what else was checked: 3 statements are proved in Isabelle/HOL (8 theorems) and 2 have a part (the base pairs) verified in Lean 4, all linked by name to the public repository karlesmarin/formalizations; 4 statements are corroborated by programs of the supplementary material, each named with what it computes.
- *Labs.* Next to Theorem B and Proposition 1, a form runs the computations of the note in the reader's browser (Python, nothing to install) for values the reader chooses: the family of Theorem B member by member (Figures 4 and 5), and any pair of quiddities, with the triangulations, the sign changes in the three readings and the two frieze patterns as relief maps in 3D.
- *The note itself.* Its PDF, opened at the right page from every statement.

The site is a reading aid, not a part of the proofs: the statements and proofs are those of this note, whose text prevails, and a formal or computational check is reported for exactly the part it covers.

9 Questions

- In which rows $k \le \lfloor w/2 \rfloor$ do four sign changes fail, besides the row $\lfloor w/2 \rfloor$ of Theorem B? For $w \le 12$ in no other (in the readings s^+ and E). In particular, is there an even n with a counterexample inside the printed range, and is there, for each fixed $k \ge 3$, a width beyond which four sign changes fail in row k?
- Both families have long runs of 2s in their quiddities. Is there a structural description of the pairs with fewer than four sign changes?

- The infinitesimal version holds for every k [5], around the constant frieze pattern with quiddity $q_n^* = (2\cos\frac{\pi}{n}, \ldots, 2\cos\frac{\pi}{n})$. How large is

$$\rho_{n,k} = \min\{\max(\|a - q_n^*\|_\infty, \|b - q_n^*\|_\infty) : \\ a, b \text{ integer quiddities of length } n,\ s^+(r_k(b) - r_k(a)) < 4\}$$

  (when such pairs exist), the distance from the constant frieze pattern at which four sign changes first fail for integer frieze patterns?

References

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