

Inversion height two in the free field of the free group: strict hierarchies and a non-membership criterion



Associative rings · division rings

16K40

16S85

16S10

16S34

16E60

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At a glance

What the paper contributes

The free field D of $A = \mathbb{C}[F_3]$ is built in stages: E_1 inverts the nonzero elements of A , E_2 those of E_1 , and so on. The stages never stabilise (Reutenauer; Herbera–Sánchez). The paper works **inside the single stage E_2** , where one restricts which elements are inverted.

- **A non-membership criterion** (Theorem A = Theorem 3.4). If H^{-1} has its entries in the ring $\mathcal{U}_2$ generated by the inverses of a family Σ of full matrices, each atomic factor of H is stably associated with an atomic factor of some $Q \in \Sigma$ (Lemma 3.1). To refute that, it suffices that one atomic factor of H be more singular than every Q at some evaluation, measured by an invariant that does not grow on factors: Smith counts along an arc, the corank on a curve of integral quaternions, the corank at a character. The allowed denominators may themselves be singular there.
- **Two hierarchies inside E_2 , strict at every step** (Theorem B). The rings $\mathcal{U}_r$ generated by inverses of sums of products of at most r inverse binomials: $\mathcal{U}_{r-1} \subsetneq \mathcal{U}_r$, with witness $(u^{-2}+v^{-2}+w^{-2})^{-1}$ for $r=2$; their union is a proper subring of E_2 . The matrix-size chain $A_{d-1} \cap E_2 \subsetneq A_d \cap E_2$ for every $d \geq 2$. The two hierarchies are independent, and the size chain is already strict inside $\mathcal{U}_2$ (Corollary 7.10).
- **Infinitely many stopping steps of the Dubrovin–Gräter–Hanke series inside E_2** (Theorem C): for every $k \geq 3$ the stratum of sums of k elements of D_π , with explicit witnesses f_k . That there are infinitely many stopping steps follows from Herbera–Sánchez, but no infinite family had been located within a fixed stage. The rings reached at these steps form a strict chain whose union is $\mathcal{U}_2$.

Main tools: Cohn's theory of firs, atoms and stable association, and of rational closures; Smith normal forms over $\mathbb{C}[[t]]$ along an exact arc of 2×2 matrices; the free group generated by $1+2i, 1+2j, 1+2k$; a grading argument that gives an atom for every k .

 **Start here: Theorem A:** the non-membership criterion: one atomic factor against the allowed denominators.

Main results:

- **Theorem A:** the non-membership criterion: one atomic factor against the allowed denominators
- **Theorem B:** the binomial hierarchy and the size chain are strict inside E_2
- **Theorem C:** every $k[\pi]$ is a stopping step of the Dubrovin–Gräter–Hanke series

47 results

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● 18 with a part in Lean



22 checked by a program

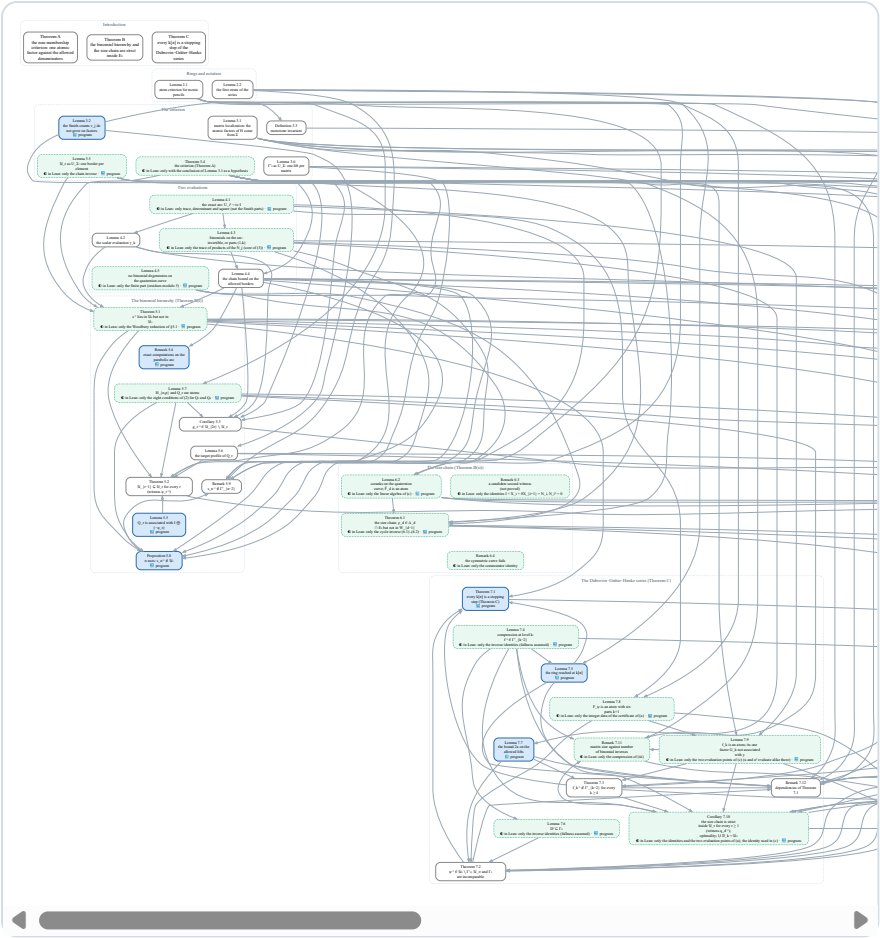
written proof only

(a result can carry several tags)

Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow $A \rightarrow B$ means that this proof of B uses A . Every result rests on its written proof; the tags record what was checked besides, and each statement's “Evidence & scope” block

says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



 **Setting (copied from the paper, §2.1)**

$P = \mathbb{C}\langle x, y, z \rangle \subset A = \mathbb{C}[F_3] \subset D$, where D is the free field, the universal field of fractions of P and of A . We write $u = 1 + x, v = 1 + y, w = 1 + z$. $E_1 = A[s^{-1} : 0 \neq s \in A] \subset D$ and $E_2 = E_1[e^{-1} : 0 \neq e \in E_1]$. A *binomial* is $\ell = 1 - \beta h, h \in F_3 \setminus \{1\}, \beta \in \mathbb{C}^\times$. For $r \geq 0$, $\mathcal{F}_r$ is the span of the products $a_0 \ell_1^{-1} a_1 \cdots \ell_q^{-1} a_q$ with $q \leq r$, ℓ_i binomials and $a_i \in A$, and $\mathcal{U}_r = E_1[\mathcal{F}_r^{-1}]$, the subring of D generated by E_1 and the inverses of the nonzero elements of $\mathcal{F}_r$; $\mathcal{U}_\infty = \bigcup_r \mathcal{U}_r$. A_b is the subring of D generated by A and the entries of G^{-1} for all full $a \times a$ matrices G over A with $a \leq b$. A square matrix over a subring of D is *full* if it is invertible over D .

1 Introduction

Theorem A. (Theorem 3.4)  **p. 2**

Let Σ be a family of full matrices over P and H a full square matrix over P with an atomic factor G in a complete factorization. If for every $Q \in \Sigma$ there is a monotone invariant μ_Q (Definition 3.3) with $\mu_Q(G) > \mu_Q(Q)$, then H^{-1} does not have entries in U_Σ .

 **Classical and new:** new, as far as the paper found: no use of Smith profiles along an arc, nor of coranks of atoms at a representation where the allowed denominators are singular, to

prove non-membership (§8.8). It rests on Lemma 3.1, which the paper does not claim (Remark 8.6); kernel dimensions already separate stable association classes [SV26].

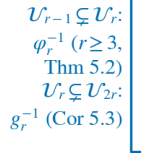
↳ Not used by another result of the paper.

Theorem B. ■ p. 3

(i) $\mathcal{U}_{r-1} \subsetneq \mathcal{U}_r$ for every $r \geq 1$; for $r = 2$ a witness is $(u^{-2} + v^{-2} + w^{-2})^{-1}$ (Theorems 5.1, 5.2). (ii) $A_{d-1} \cap E_2 \subsetneq A_d \cap E_2$ for every $d \geq 2$, and $\bigcup_r \mathcal{U}_r \subsetneq E_2$ (Theorem 6.1).

NEW **Classical and new:** the rings $\mathcal{U}_r$ have not been studied, to the paper's knowledge; nor the strictness of the size chain inside E_2 for $d \geq 3$. The case $d = 2$, and the strictness of the chain A_b in $E(b)$, follow from [HS15, Thm 4.6] by pivoting (§8.8). The separation of the whole stages $E(n) \subsetneq E(n+1)$ is [Reu96], [HS15].

↳ Not used by another result of the paper.



Theorem C. p. 3

For every $k \geq 3$, the stratum of sums of k elements of D_π is a stopping step of the Dubrovin–Gräter–Hanke series of D for the monoid $\mathbb{C}^\times F_3$ of scalar multiples of group elements, in Jakobs’ formulation [Jak19, Def. 1.1.4]. Witnesses are $u^{-2} + v^{-2} + w^{-2}$ for $k = 3$ and $f_k = u^{-2} + w^{-2} + \sum_{i=1}^{k-2} y^i u^{-2} y^{-i}$ for $k \geq 4$ (Theorems 7.1, 7.3).

NEW Classical and new: infinitely many stopping steps follow from [HS15], but none had been located within a fixed stage; the first stopping step (sums of two monomials) is implicit in [Gra20, §4].

↳ Not used by another result of the paper.

2 Rings and notation

Lemma 2.1. (atom criterion) p. 6

Let $N \geq 1$ and let $L = A_0 + \sum_i A_i x_i$ be an $N \times N$ monic pencil with A_0 invertible. Then L is full and not invertible, and L is an atom iff the matrices $M_i = A_0^{-1} A_i$ ($1 \leq i \leq g$) have no common invariant subspace other than 0 and k^N .

NEW **Classical and new:** classical: Cohn [Coh06, Thm 5.8.8], in the form of [HKV22, Rem. 2.6].

↳ Used in: [Lemma 5.7](#) (p. 18), [Proposition 5.8](#) (p. 20), [Lemma 6.2](#) (p. 22), [Lemma 7.8](#) (p. 30), [7.12](#) (p. 34), [8.4](#) (p. 38)

Proof

$L(0) = A_0$ is invertible, so L is full (§2.2(a)). A right monic matrix is not a unit [Coh06, Lemma 5.8.1, p. 312]. For $N = 1$, [Coh06, Thm 5.8.8, p. 317] says that L is an atom, and the subspace condition holds trivially. Let $N > 1$. By [Coh06, Thm 5.8.8], L is not an atom iff there are $P, Q \in GL_N(k)$ and $r, s > 0$ with $r + s = N$ such that every PA_jQ ($j = 0, \dots, g$) has a zero $r \times s$ block in the upper right corner.

- Given such P, Q , put $W = Q \text{span}(e_{r+1}, \dots, e_N)$ and $W' = P^{-1} \text{span}(e_{r+1}, \dots, e_N)$. Then $A_j W \subseteq W'$ for all j . Since A_0 is injective and $\dim W = \dim W'$, we get $A_0 W = W'$, hence $M_i W \subseteq A_0^{-1} W' = W$, with $0 < \dim W = s < N$.
- Conversely, let W be invariant with $0 < s = \dim W < N$. Take Q whose last s columns span W , and $P = Q^{-1} A_0^{-1}$. Then $PA_0 Q = I$ and $PA_i Q = Q^{-1} M_i Q$ maps $\text{span}(e_{r+1}, \dots, e_N)$ into itself, so the upper right $r \times s$ block vanishes for all j . $\square$

Lemma 2.2. (the first strata) p. 7

1. For $1 \leq n < \omega$, D_n is the set of sums of at most n elements of D_1 , and $D_\omega = A$.
2. The first Case 3 occurs at $\alpha' = \omega$, with $\alpha_1 = 2$; $D_{\omega+1} = A \cup D_2^{-1}$, and the nonzero elements of D_2^{-1} are the monomials and the unit multiples $\ell^{-1}m$ of inverse binomials.
3. From $\omega + 1$ on, Case 1 applies up to the first index σ at which the set is additively closed, and $D_\sigma = V_1$.
4. If $D_{\omega+1} D_{\omega+1} \not\subseteq V_1$, Case 2 applies at σ with the multiset $2[\omega + 1]$, so $\pi = \sigma + 1$ and $D_\pi = V_1 \cup D_{\omega+1} D_{\omega+1}$.
5. Let $k \geq 2$ and suppose $D_\pi^{+j} \not\subseteq D_\pi^{+(j-1)} + V_1$ for $2 \leq j \leq k$. Then the strata born after π and before $k[\pi]$ are among the $j[\pi] + x$ with $1 \leq j < k$ and $x \in N(\sigma)$, each contained in $D_\pi^{+j} + V_1$, no Case 2 or Case 3 occurs in between, and $D_{k[\pi]} = D_\pi^{+k}$.

↳ Used in: [5.9](#) (p. 21), [Lemma 7.5](#) (p. 28), [Corollary 7.10](#) (p. 32), [7.12](#) (p. 34)

Proof

(1) For $n < \omega$, D_n is not additively closed (it misses sums of $n + 1$ distinct monomials), so Case 1 applies. For $n < \omega$, every multiset smaller than $(n + 1)[1]$ involves only the

ordinals 0 and 1, with at most n copies of 1, so its sum lies in D_n ; and $(n+1)[1]$ gives sums of $n+1$ distinct monomials, which are not in D_n .

1. $D_\omega = A$ is a ring and $D_1^{-1} \subseteq D_1$. A sum $\alpha g + \beta h$ of two monomials with $g \neq h$ is not a unit of A [Jak19, Thm 1.3.10], so $D_2^{-1} \not\subseteq A$; and $(\alpha g + \beta h)^{-1} = \ell^{-1}(\alpha g)^{-1}$ with $\ell = 1 + \alpha^{-1}\beta g^{-1}h$.
2. Case 2 needs an additively closed set and Case 3 a ring. Sums of elements of $D_{\omega+1}$ lie in V_1 , which is the additive closure of $D_{\omega+1}$ (§2.1: $m\ell^{-1}m'$ is a unit multiple of an inverse binomial). So the first additively closed set is V_1 .
3. In $N(\sigma)$ the multisets below $2[\omega+1]$ are those over $\{0, \dots, \omega\}$ and the $[\omega+1] + x$ with x over $\{0, \dots, \omega\}$; any multiset containing an ordinal in $(\omega+1, \sigma]$ is larger. The corresponding products lie in $A \cdot (A \cup D_2^{-1}) \cdot A \subseteq V_1$.
4. As long as Case 1 applies, the indices are taken in increasing order; every multiset containing an ordinal born after π exceeds every multiset of $N(\pi)$ [Jak19, p. 3], and $j[\pi] + x < (j+1)[\pi]$. The strata over $N(\sigma)$ lie in V_1 , so $D_{j[\pi]+x} \subseteq D_\pi^{+j} + V_1$ (recall $V_1 \subseteq D_\pi$). Before $k[\pi]$ the current set lies in $D_\pi^{+(k-1)} + V_1$, which by hypothesis does not contain D_π^{+k} ; it is therefore not additively closed, so neither Case 2 nor Case 3 can occur, and $k[\pi]$ is eventually the least multiset with a new sum. $\square$

3 The criterion

Lemma 3.1. (matrix localization) p. 8

Let R be a semifir with universal field of fractions D , and Σ a set of full matrices over R , each with a complete factorization (this holds for every Σ if R is fully atomic, for instance a fir such as P or A). Let $U_\Sigma \subseteq D$ be the subring generated by R and the entries of Q^{-1} , $Q \in \Sigma$. If H is a full matrix over R and H^{-1} has entries in U_Σ , then every atomic factor of H is stably associated with an atomic factor of a complete factorization of a single member $Q \in \Sigma$. Conversely, if H has a complete factorization whose factors are all stably associated with atomic factors of members of Σ , then H^{-1} has entries in U_Σ .

NEW **Classical and new:** not claimed: the analogue, for U_Σ and without factor-completeness, of Cohn's results on localization at factor-complete sets [Coh06, Thm 7.11.9] (Remark 8.6).

↳ Used in: [Theorem 3.4](#) (p. 9), [Lemma 7.9](#) (p. 31), [7.12](#) (p. 34), [8.1](#) (p. 34), [8.4](#) (p. 38), [8.5](#) (p. 38), [8.6](#) (p. 39)

Proof

1. By (T3) the torsion submodules of M form a sublattice $L(M)$ of the submodule lattice, so $L(M)$ is modular.
2. For $L \subseteq L'$ in $L(M)$, the quotient L'/L is torsion by (T2). The interval $[L, L']$ has no interior point exactly when L'/L is 0 or Tor-simple: the torsion submodules of L'/L are the L''/L with $L'' \in [L, L']$, by (T2).
3. By the Schreier refinement theorem [Coh06, App. Thm A.3, p. 521], applied in the modular lattice $L(M)$ of torsion submodules (not in the lattice of all submodules),

the chains $0 \subseteq N \subseteq M$ and (M_j) have refinements in $L(M)$ whose intervals pair off into projective intervals.

4. Projective intervals are isomorphic as lattices and give isomorphic quotient modules [Coh06, p. 521]. A refinement of (M_j) only repeats terms, so its nontrivial intervals are the $[M_{j-1}, M_j]$.
5. Hence every nontrivial interval of the refinement of $0 \subseteq N \subseteq M$ has a Tor-simple quotient isomorphic to some M_j/M_{j-1} . The part between 0 and N is a $\mathcal{C}$ -filtration of N .
6. The part between N and M is a chain $N = L_0 \subseteq \cdots \subseteq L_t = M$. The L_i/N are torsion submodules of M/N with $(L_i/N)/(L_{i-1}/N) \cong L_i/L_{i-1}$, which gives a $\mathcal{C}$ -filtration of M/N .
7. If M/N is Tor-simple, a $\mathcal{C}$ -filtration of it with distinct terms is $0 \subset M/N$, so M/N is isomorphic to a member of $\mathcal{C}$. $\square$

Lemma 3.2. (cf. [Tho79]; we give a direct proof) ■ p. 11

Let $j \geq 1$.

1. (Factors.) If $Y = G_1 F G_2$ with all matrices square of the same size, then $\nu_j(F^\gamma) \leq \nu_j(Y^\gamma)$, whether or not G_1^γ, G_2^γ are singular.
2. (Upper interlacing.) If $Y = \begin{pmatrix} L & \omega \\ \alpha & \delta \end{pmatrix}$ with a bottom block of size m after evaluation, then $\nu_j(Y) \leq \nu_j(L) + m$.
3. (Lower interlacing.) In the same situation, $\nu_j(L) - m \leq \nu_j(Y)$.
4. (Chains.) If L is block upper triangular with diagonal blocks $\Delta_1, \dots, \Delta_q$, each with $\det \Delta_i \neq 0$ and largest part e_i , then every part of L is at most $e_1 + \cdots + e_q$. If L is block diagonal, its largest part is the largest over the blocks.

NEW **Classical and new:** cf. [Tho79]; the paper gives a direct proof.

↳ Used in: 3.3 (p. 8), Lemma 4.4 (p. 13), Lemma 5.6 (p. 18)

Proof

(1) $\text{im}(G_1 F G_2) \subseteq \text{im}(G_1 F)$, so $\text{coker}(G_1 F)$ is a quotient of $\text{coker } Y$; likewise $\text{coker } F^\top$ is a quotient of $\text{coker}(F^\top G_1^\top)$, and transposition does not change Smith forms; a quotient does not increase $\mu(t^{j-1} \cdot)$. (2) $\text{coker } Y$ maps onto $\text{coker}[L \ \omega]$ with kernel generated by m elements, and $\text{coker}[L \ \omega]$ is a quotient of $\text{coker } L$; over a discrete valuation ring a submodule of an m -generated module is m -generated. (3) $\text{coker}[L \ \omega] = \text{coker } L / \text{im } \omega$ and $t^{j-1}(\mathcal{X}/\mathcal{X}') \cong t^{j-1}\mathcal{X}/(t^{j-1}\mathcal{X} \cap \mathcal{X}')$ with $t^{j-1}\mathcal{X} \cap \mathcal{X}'$ at most m -generated, so $\mu(t^{j-1} \cdot)$ drops by at most m ; then $\text{coker } Y \twoheadrightarrow \text{coker}[L \ \omega]$. (4) Since each Δ_i is injective, $\text{coker } L$ is an iterated extension of the $\text{coker } \Delta_i$, and $t^{e_1 + \cdots + e_q}$ kills it. $\square$

Evidence & scope

- **Program:** `nu_j` does not increase on factors: 480 random pairs, with a control (supplement C01). `scripts/auditoria_S23/s23_monotonia.py`

Definition 3.3. ■ p. 11

A *monotone invariant* is a function μ from square matrices over P to $\mathbb{N} \cup \{\infty\}$ such that

1. $\mu(F \oplus I) = \mu(F)$ for identity matrices I of every size, and $\mu(MFN) = \mu(F)$ for M, N invertible over P of the size of F ;
2. $\mu(F) \leq \mu(G_1FG_2)$ whenever F, G_1, G_2 are square of the same size.

Examples.

- 1. For an arc (a unital ring homomorphism) $\gamma : P \rightarrow M_m(\mathbb{C}[[t]])$ and $j \geq 1$, $\mu(F) = \nu_j(F^\gamma)$. Condition 1 holds because M^γ, N^γ are invertible over $\mathbb{C}[[t]]$ and padding does not change ν_j ; condition 2 is Lemma 3.2(1).
- 1. For a unital ring homomorphism $\Theta : P \rightarrow M_m(K)$ into matrices over a field (not necessarily commutative), $\mu(F) = \text{corank } F^\Theta$. Condition 1 holds because invertible matrices go to invertible matrices; condition 2 holds because $\text{rank}(XYZ) \leq \text{rank } Y$.

Neither example needs the arc or Θ to extend to D , or even to A . Example (b) is m times the nullity of the Sylvester rank function $F \mapsto \text{rank } F^\Theta / m$, and every Sylvester rank function with values in $\frac{1}{m}\mathbb{Z}$ gives a monotone invariant in the same way [Mal80], [Sch85, Ch. 7]. Example (a) is a difference of two instances of (b): if $\Theta_j : P \rightarrow M_{mj}(\mathbb{C})$ is the reduction of γ modulo t^j followed by the regular representation of $\mathbb{C}[t]/t^j$, then $\nu_j(F^\gamma) = \text{corank } F^{\Theta_j} - \text{corank } F^{\Theta_{j-1}}$, because multiplication by t^λ on $\mathbb{C}[t]/t^j$ has kernel of dimension $\min(\lambda, j)$, so $\text{corank } F^{\Theta_j} = \sum_i \min(\lambda_i, j)$. Its monotonicity is Lemma 3.2(1); it does not follow from that of (b).

↳ Used in: 8.5 (p. 38)

Theorem 3.4. p. 11

Let Σ be a family of full matrices over P and H a full square matrix over P with an atomic factor G in a complete factorization. Suppose that for every $Q \in \Sigma$ there is a monotone invariant μ_Q with $\mu_Q(G) > \mu_Q(Q)$. Then H^{-1} does not have entries in U_Σ .

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Axioms — not_inverseInU: propext, Classical.choice, Quot.sound.

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↳ Used in: Theorem 5.1 (p. 14), Theorem 5.2 (p. 16), Corollary 5.3 (p. 17), 5.9 (p. 21), Theorem 6.1 (p. 21), Theorem 7.2 (p. 27), Theorem 7.3 (p. 27), Corollary 7.10 (p. 32), 7.11 (p. 33), 7.12 (p. 34), 8.1 (p. 34), 8.2 (p. 35), 8.3 (p. 37), 8.4 (p. 38)

Proof

Suppose that H^{-1} has entries in U_Σ . By Lemma 3.1 there are $Q \in \Sigma$ and an atomic factor of Q stably associated with G . In a complete factorization, $Q = G_1 \mathcal{A} G_2$ with $\mathcal{A}, G_1, G_2$ square of the size of Q and $\mathcal{A}$ an atom stably associated with that factor, hence with G [Coh06, Cor. 0.5.5]. Then $\mu_Q(G) = \mu_Q(\mathcal{A})$ by condition 1, and $\mu_Q(\mathcal{A}) \leq \mu_Q(Q)$ by condition 2, a contradiction. $\square$

Lemma 3.5. ■ p. 13

$\mathcal{U}_r = U_{\Sigma_r}$ with $\Sigma_r = \Sigma_0 \cup \{\tilde{B}_f : 0 \neq f \in \mathcal{F}_r\}$, one border per f .

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InverseIdentities.lean (SHA-256 of the compiled file: 3e2df75becd9213c...; a source page on this site may show it with comments edited, under its own hash).

↳ Used in: Theorem 5.1 (p. 14), Corollary 5.3 (p. 17), Theorem 6.1 (p. 21), 8.2 (p. 35)

Proof

$\mathcal{U}_r \subseteq U_{\Sigma_r}$: U_{Σ_r} is a ring containing E_1 and f^{-1} , since $-f^{-1}$ is a corner entry of $\tilde{B}_f^{-1}$.
 $U_{\Sigma_r} \subseteq \mathcal{U}_r$: the entries of $\tilde{B}_f^{-1}$ are polynomials in the entries of L^{-1} (whose entries lie in E_1 , because $\ell^{-1} \in E_1$ and $a_i \in A$), in f^{-1} and in the entries of the inverse pivot, which lie in A . $\square$

Evidence & scope

- **Program:** chain inverse and Schur(B_f) = -f with random non-commuting blocks, with controls (supplement C02). scripts/auditoria_S12/s12_cadena_schur.py

Lemma 3.6. ■ p. 13

$\Gamma'_2 = U_{\Sigma'_2}$ with $\Sigma'_2 = \Sigma_1 \cup \{\tilde{B}_M : M \text{ full over } V_1, a \leq 2\}$, one lift per M .

↳ Used in: 5.9 (p. 21), Theorem 7.2 (p. 27), Theorem 7.3 (p. 27)

Proof

$\tilde{B}_M$ is full, because M is full and Δ is invertible over D . $\Gamma'_2 \subseteq U_{\Sigma'_2}$: $U_{\Sigma'_2} \supseteq U_{\Sigma_1} = \mathcal{U}_1$; the block of $\tilde{B}_M^{-1}$ complementary to the triangular pivot is B_M^{-1} , and the $(1, 1)$ block of B_M^{-1} is M^{-1} (Schur complement of Δ). $U_{\Sigma'_2} \subseteq \Gamma'_2$: the entries of $\tilde{B}_M^{-1}$ are polynomials in those of B_M^{-1} , of the inverse pivot (in A) and of elements of A , and

$$B_M^{-1} = \begin{pmatrix} M^{-1} & -M^{-1}R\Delta^{-1} \\ \Delta^{-1}CM^{-1} & \Delta^{-1} - \Delta^{-1}CM^{-1}R\Delta^{-1} \end{pmatrix}$$

with $\Delta^{-1} \in M_m(E_1) \subseteq M_m(\mathcal{U}_1)$ and R, C over A . $\square$

4 Two evaluations

Lemma 4.1. (exact arc) ■ p. 14

$\text{tr } U_i = 0$, $\det U_i = -t\varepsilon$ and $U_i^2 = t\varepsilon I$. In particular $U_i(t, t^k)^2 = t^{k+1}I$, and $U_i(t, t^k)$ has Smith parts $(1, k)$.

✓ Checked when this page was built: ExactArc.lean, ParabolicFamily.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — exactArc_trace, exactArc_det, exactArc_sq, exactArc_sq_pow, exactArc_eq_conj: propext, Classical.choice, Quot.sound.

Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, lake env lean ExactArc.lean

ParabolicFamily.lean (SHA-256 of the compiled file: 54b7df24f5d380c4...; a source page on this site may show it with comments edited, under its own hash).

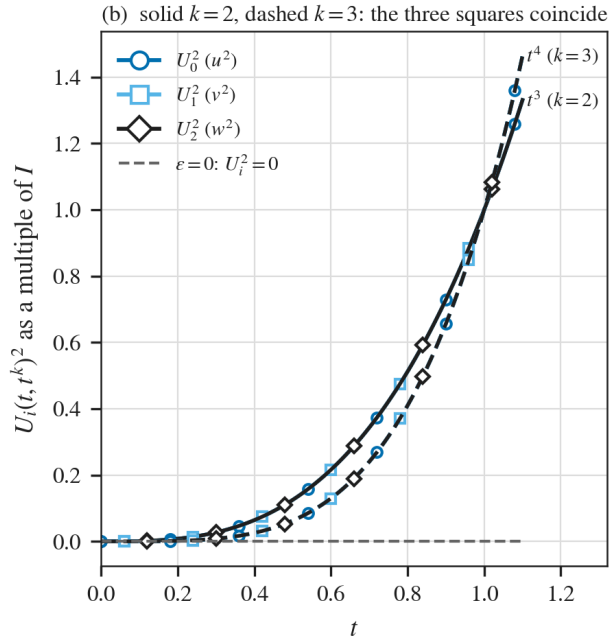
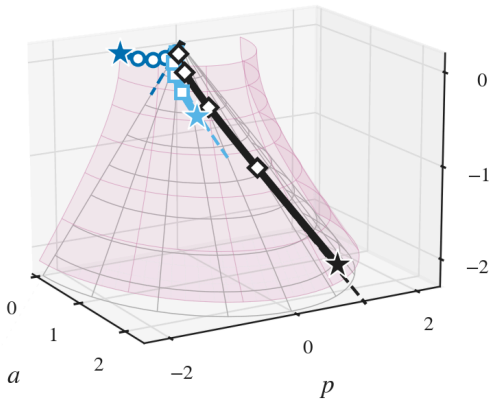
↳ Used in: [Lemma 4.2](#) (p. 11), [Lemma 4.3](#) (p. 12), [Proposition 5.8](#) (p. 20), [Lemma 7.8](#) (p. 30), [Lemma 7.9](#) (p. 31)

Proof

U_0 has zero trace, determinant $-\varepsilon$ and $U_0^2 = \varepsilon I$, and conjugation by K^i preserves all three. The parts: every entry is divisible by t (since $k \geq 1$), the $(1, 2)$ entry is $-t$, and $\det = -t^{k+1}$. $\square$

Figure

(a) (a, p, q) : $a = U_{11}$, $p = (U_{12} + U_{21})/2$, $q = (U_{12} - U_{21})/2$



$\text{---} \circ \text{---} u \mapsto U_0(t, t^2)$ $\text{---} \diamond \text{---} w \mapsto U_2(t, t^2)$ $\text{---} \text{cone } a^2 + p^2 = q^2 \text{ (det } U=0)$
 $\text{---} \square \text{---} v \mapsto U_1(t, t^2)$ $\text{---} \text{dashed, each colour: } \varepsilon=0, \text{ parabolic family, } U^2=0$ $\text{---} a^2 + p^2 - q^2 = 1; \star: \text{ the three points at } t=1$

The exact arc. A traceless 2×2 matrix U is a point (a, p, q) with $a = U_{11}$, $p = (U_{12} + U_{21})/2$, $q = (U_{12} - U_{21})/2$, and $U^2 = (a^2 + p^2 - q^2)I$. (a) The images of u, v, w for $\varepsilon = t^2$ (solid) leave the cone $a^2 + p^2 = q^2$ of nilpotent matrices, on which the parabolic family $\varepsilon = 0$ lies (dashed); at each t the three points lie on the same quadric $a^2 + p^2 - q^2 = t^3$, shown at $t = 1$ (stars). (b) The scalar λ with $U_i(t, t^k)^2 = \lambda I$ for $k = 2, 3$: the three curves coincide, $\lambda = t^{k+1}$ (Lemma 4.1). Live version, in 3D and for any k , in the [blueprint](#).

Evidence & scope

- **Program:** `U_i(t,0) = I - K^i J K^{-i}`, trace, determinant and U_i^2 , exactly (supplement C03). `scripts/smith_v1/t158_arco_exacto.py`

Lemma 4.2. (scalar evaluation) p. 14

Let $\gamma = \gamma_k$.

1. If F is a $d \times d$ matrix with entries in $\mathcal{J}^r$, every entry of F^γ is divisible by $t^{r(k+1)}$; hence $\nu_{r(k+1)}(F^\gamma) = 2d$.
2. $H_{3,2r}^\gamma = t^{r(k+1)} \left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \otimes I_2 \right)$. Its four parts all equal $r(k+1)$.

Proof

(1) A monomial of $\mathcal{J}^r$ is $a_0 u_{i_1}^2 a_1 \cdots u_{i_r}^2 a_r$ with $a_i \in A$; each u_i^2 evaluates to the central scalar $t^{k+1}I$ (Lemma 4.1) and each a_i integrally. (2) Apply the same to U_i^r , whose square is $t^{r(k+1)}I$; $\det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3 \neq 0$. $\square$

Lemma 4.3. (binomials) p. 14

Let $\ell = 1 - \beta h$ be a binomial.

1. If $\beta\chi(h) \neq 1$, then ℓ^{γ_k} is invertible over $\mathbb{C}[[t]]$: no positive parts.
2. If ℓ is exceptional, ℓ^{γ_k} has parts $(1, k)$ for every $k \geq 1$.
3. Otherwise $e_\ell < \infty$, and for $k > e_\ell$ the parts of ℓ^{γ_k} equal those at $\varepsilon = 0$; in particular they are $< k$.

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`ParabolicFamily.lean` (SHA-256 of the compiled file: 70adc1a90a531fd1...; a source page on this site may show it with comments edited, under its own hash).

Proof

(1) At $t = 0$, $\ell \mapsto (1 - \beta\chi(h))I$. (2) Write $\ell = g(1 - \beta x_j^n)g^{-1}$ with $\beta(-1)^n = 1$. For $n < 0$ use $1 - \beta x_j^n = -\beta x_j^n(1 - \beta^{-1}x_j^{-n})$, a unit times a binomial of the same kind; so let $n > 0$. The roots of $1 - \beta X^n$ are simple, so $1 - \beta X^n = (1 + X)c(X)$ with $c(-1) \neq 0$; since $x_j^\gamma \rightarrow -I$, $c(x_j)^\gamma$ and g^γ are units, and the parts are those of $u_j^\gamma = U_j$, namely $(1, k)$ (Lemma 4.1). (3) After conjugation, which changes ℓ^γ by a unit, write $h = x_{j_1}^{n_1} \cdots x_{j_q}^{n_q}$ cyclically reduced. If $q = 1$, then $\beta\chi(h) = 1$ makes ℓ exceptional, so $q \geq 2$ and $j_{i+1} \neq j_i$ for all i , indices mod q . Write $N = tE_{12}$ and $N_j = K^j N K^{-j} = t c_j r_j$ with $c_j = (1, jt)^\top$ and $r_j = (-jt, 1)$. Then $r_a c_b = (b - a)t$, $N_j^2 = 0$, and $\rho_t(x_j)^n = (-1)^n(I + nN_j)$ for $n \in \mathbb{Z}$. Expanding the product,

$$\mathrm{tr} \rho_t(h) = \chi(h) \sum_S \left(\prod_{i \in S} n_i \right) \mathrm{tr} \prod_{i \in S} N_{j_i},$$

over the subsets $S = \{i_1 < \cdots < i_m\}$ of $\{1, \dots, q\}$, with the factors in increasing order. For $m \geq 1$,

$$\mathrm{tr}(N_{j_{i_1}} \cdots N_{j_{i_m}}) = t^m \prod_{l=1}^m r_{j_{i_l}} c_{j_{i_{l+1}}} = t^{2m} \prod_{l=1}^m (j_{i_{l+1}} - j_{i_l}) \quad (i_{m+1} = i_1),$$

which vanishes for $m = 1$. Hence $\mathrm{tr} \rho_t(h)$ is a polynomial in t^2 of degree q with leading coefficient $\chi(h) \prod_i n_i \prod_i (j_{i+1} - j_i) \neq 0$ (*trace law*; compare [LU69] for two parabolic generators). Since $\det \rho_t(x_j) = 1$, $\det(I - \beta \rho_t(h)) = 1 - \beta \mathrm{tr} \rho_t(h) + \beta^2$ is a nonconstant polynomial, hence $\neq 0$, and $e_\ell < \infty$. Finally $\ell^{\gamma_k} = \ell^\gamma|_{\varepsilon=0} + O(t^k)$, and a

perturbation $O(t^k)$ of a matrix whose parts are all $< k$ does not change them ($D + O(t^k) = D(I + O(t))$). $\square$

Evidence & scope

- **Program:** leading coefficient of $\text{tr } \rho_t(h)$ for 60 words, both signs (supplement C04).
`scripts/auditoria_S12/s12_traza.py`

Lemma 4.4. (chain bound) p. 16

Let $0 \neq f \in \mathcal{F}_q$ with border B_f and $k > e(f)$. Then every part of L^{γ_k} is at most qk , and $\nu_{qk+1}(\tilde{B}_f^{\gamma_k}) \leq 2$.

↳ Used in: [Theorem 5.1](#) (p. 14), [Theorem 5.2](#) (p. 16), [Corollary 5.3](#) (p. 17), [5.4](#) (p. 17), [Proposition 5.8](#) (p. 20), [5.9](#) (p. 21), [Theorem 7.2](#) (p. 27), [7.12](#) (p. 34), [8.1](#) (p. 34), [8.2](#) (p. 35)

Proof

Each chain C_τ has at most q binomials on its diagonal, each with $\det \neq 0$ on the arc and largest part $\leq k$ (Lemma 4.3). By Lemma 3.2(4) the parts of L^γ are at most qk , so $\nu_{qk+1}(L^\gamma) = 0$. The pivots x_j of the lift evaluate to units, so $\tilde{B}_f^\gamma$ and B_f^γ have the same parts up to zeros. The border is one row and one column, i.e. $m = 2$ after evaluation, and Lemma 3.2(2) gives $\nu_{qk+1} \leq 0 + 2$. $\square$

Lemma 4.5. (no binomial degenerates on Θ) p. 16

For every $h \in F_3 \setminus \{1\}$ and $\beta \in \mathbb{C}^\times$, $\det(I - \beta h^\Theta) \neq 0$ in $\mathbb{C}(t)$. Hence every binomial, and every element $\alpha g + \beta h$ of A with two terms, is invertible at Θ .

✓ Checked when this page was built: QuaternionResidues.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — `res_mul_res_eq_zero_iff`, `prod_res_ne_zero`: `propext`, `Classical.choice`, `Quot.sound`.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean QuaternionResidues.lean` (SHA-256 of the compiled file: 69a51a05a38ebe68...; a source page on this site may show it with comments edited, under its own hash).

NEW **Classical and new:** the freeness of the group generated by $1+2i, 1+2j, 1+2k$ is classical [LPS88].

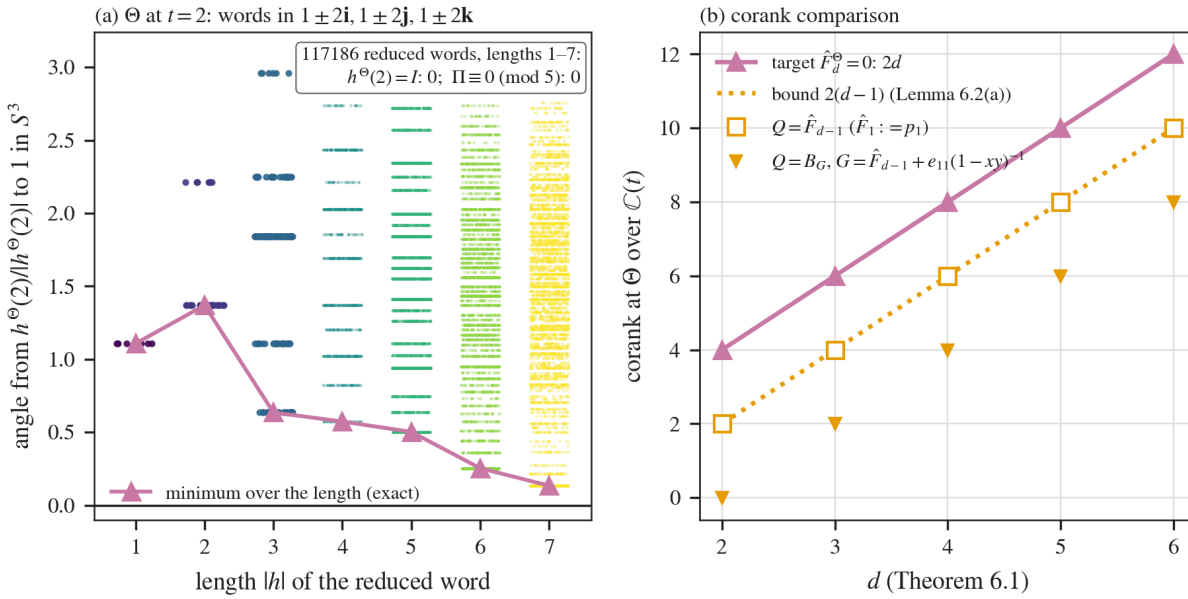
↳ Used in: [Theorem 6.1](#) (p. 21), [Lemma 6.2](#) (p. 22), [8.1](#) (p. 34)

Proof

Entries of h^Θ are regular at $t = 0$ and $t = 2$, so evaluation there is a ring homomorphism. If $\beta \neq 1$: at $t = 0$, $h^\Theta = I$ and the determinant is $(1 - \beta)^2 \neq 0$. If $\beta = 1$: at $t = 2$, $\Theta(x_j)$ is the Lipschitz quaternion $1 + 2\mathbf{q}_j$ of norm 5, and $5(1 + 2\mathbf{q}_j)^{-1} = 1 - 2\mathbf{q}_j$. For a reduced word h of length $|h|$ with n_- negative letters, $h^\Theta(2) = 5^{-n_-} \Pi$ with Π a product of $|h|$ elements of $\{1 \pm 2\mathbf{i}, 1 \pm 2\mathbf{j}, 1 \pm 2\mathbf{k}\}$, so $\text{Norm } \Pi = 5^{|h|}$. Reduce modulo 5 through the ring homomorphism from the Lipschitz order to $M_2(\mathbb{F}_5)$ with $\mathbf{i} \mapsto \text{diag}(2, 3)$, $\mathbf{j} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The six residues have rank one, $M_a = u_a v_a^\top$, and $M_a M_b = (v_a^\top u_b) u_a v_b^\top$ vanishes exactly when b is the residue of the conjugate $1 \mp 2\mathbf{q}$ of $a = 1 \pm 2\mathbf{q}$ (a check of the 36 products; Appendix A.2, supplement C05). A reduced word has no consecutive inverse pair, so $\Pi \not\equiv 0 \pmod{5}$. If $h^\Theta(2) = I$, then $\Pi = 5^{n_-} I$, $5^{2n_-} = 5^{|h|}$, $|h| = 2n_- \geq 1$, so $n_- \geq 1$ and $\Pi \equiv 0$, a

contradiction. Finally, for a real quaternion $q \neq 1$, $\det(I - q) = \text{Norm}(1 - q) > 0$. Two-term elements are $\alpha g(1 + \alpha^{-1} \beta g^{-1} h)$ with $\det \Theta(g) \neq 0$. $\square$

Figure



The quaternion curve Θ . (a) For reduced words h of length at most 7 in x, y, z and their inverses, the angle on S^3 between $h^\Theta(2)/|h^\Theta(2)|$ and 1 (all words of length ≤ 5 ; for lengths 6 and 7, the 200 words of smallest angle and a random sample of 3800), with the minimum for each length taken over all 117 186 words; no word gives $h^\Theta(2) = I$ and no product Π vanishes modulo 5, as Lemma 4.5 proves for every length. (b) The corank at Θ over $\mathbb{C}(t)$ of the target $\hat{F}_d$ is $2d$ (Lemma 6.2(b)), while every $Q \in \Sigma^{(d-1)}$ has corank at most $2(d-1)$ (Lemma 6.2(a)). Two sample denominators are shown: the lift of $\hat{F}_{d-1}$ (with $\hat{F}_1 := p_1$), which attains the bound, and the border of $\hat{F}_{d-1} + e_{11}(1 - xy)^{-1}$. Live version, the words appearing on S^3 length by length, in the [blueprint](#).

Evidence & scope

- **Program:** residues mod 5 (36 products) and 3008 random reduced words, with a control (supplement C05). `scripts/auditoria_S11/s11_a1_cuaterniones.py`

5 The binomial hierarchy (Theorem B(i))

Theorem 5.1. p. 17

Let $s = u^{-2} + v^{-2} + w^{-2}$. Then $s^{-1} \in \mathcal{U}_2$ and $s^{-1} \notin \mathcal{U}_1 = E_1[V_1^{-1}]$, where $V_1 = A + \sum A\ell^{-1}A$.

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InverseIdentities.lean (SHA-256 of the compiled file: 3e2df75becd9213c...; a source page on this site may show it with comments edited, under its own hash).

↳ Used in: Theorem 5.2 (p. 16), Proposition 5.8 (p. 20), Theorem 7.1 (p. 25), Theorem 7.2 (p. 27), Corollary 7.10 (p. 32), 7.12 (p. 34), 8.1 (p. 34), 8.2 (p. 35)

Proof

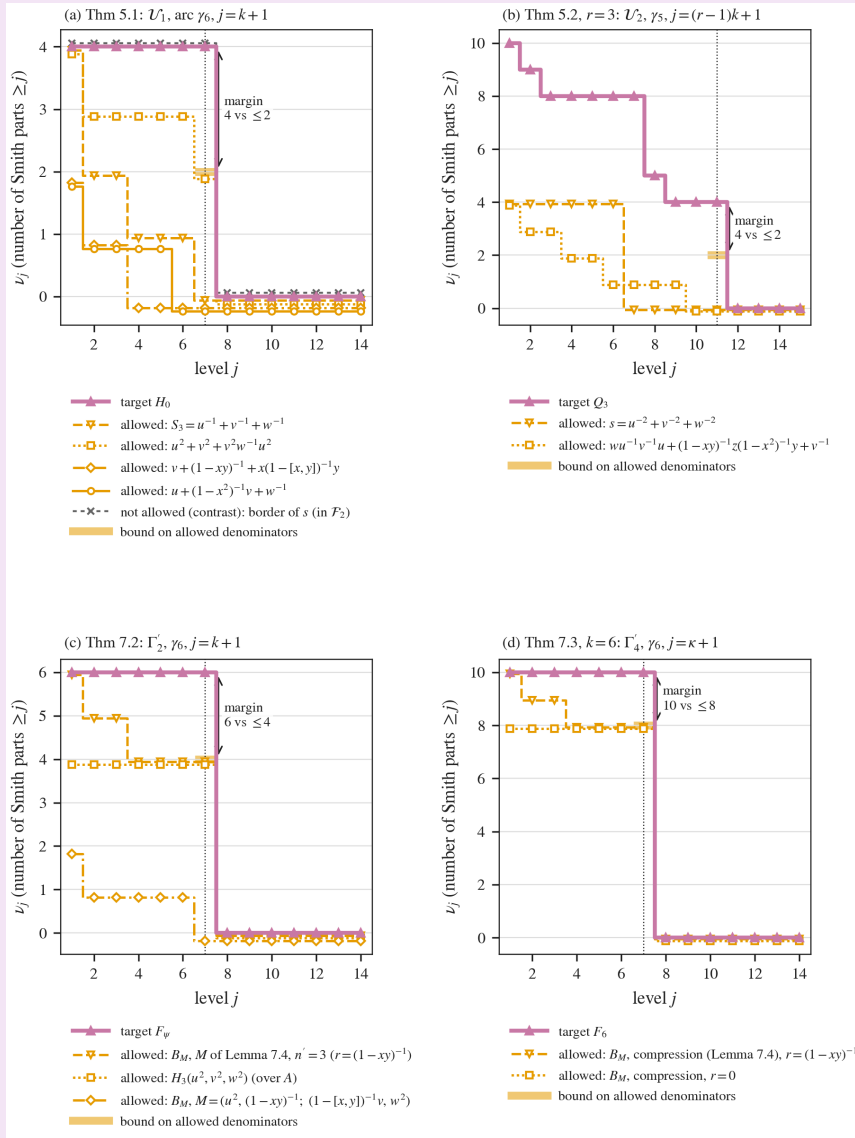
$s \in \mathcal{F}_2$ because $u^{-2} = u^{-1} \cdot 1 \cdot u^{-1}$, so $s^{-1} \in \mathcal{U}_2$. Suppose $s^{-1} \in \mathcal{U}_1$. By §5.1 and Lemma 3.5, H_0^{-1} has entries in U_{Σ_1} . We check the hypothesis of Theorem 3.4 with $G = H = H_0$ and, for each Q , $\mu_Q = \nu_{k+1}$ along γ_k (Example (a)), with k chosen after Q (the case $G = H$, $\mu = \nu_j$ of Theorem 3.4).

- *H_0 is a full atom over P .* It is $H_{3,2}$, an atom of $\mathbb{C}\langle u, v, w \rangle = P$ by Lemma 5.7 below (the case $n = 3, p = 2$).
- *Target.* By Lemma 4.2(2) with $r = 1$, $\nu_{k+1}(H_0^{\gamma_k}) = 4$ for every k .
- *Denominators.* For $Q = Q_s$ or x_j , $\nu_{k+1} \leq 2$. For $Q = \tilde{B}_f$, $0 \neq f \in V_1 = \mathcal{F}_1$, choose $k > e(f)$; Lemma 4.4 with $q = 1$ gives $\nu_{k+1}(\tilde{B}_f^{\gamma_k}) \leq 2$.

Since $4 > 2$, Theorem 3.4 gives $H_0^{-1} \notin M_2(U_{\Sigma_1})$, a contradiction. $\square$

Figure 4 shows this comparison on computed examples, together with the corresponding comparisons in the proofs of Theorems 5.2, 7.2 and 7.3.

 **Figure**



Smith profiles on the exact arc: ν_j is the number of Smith parts $\geq j$ (§3.2). In each panel the target has more parts at the marked level j than any allowed denominator can have (orange band: the bound of Lemma 4.4 in (a) and (b), of Lemma 7.7 in (c) and (d)). The orange curves are sample denominators, computed exactly over $\mathbb{F}_p[[t]]/t^N$ for two primes; they illustrate the bound and are not a maximum over Σ . Curves are offset vertically by at most 0.25 for legibility; all values are integers. (a) Theorem 5.1: target H_0 on γ_6 ; the grey curve, the border of $s \in \mathcal{F}_2$, is not an allowed denominator for $\mathcal{U}_1$. (b) Theorem 5.2 with $r = 3$: target Q_3 on γ_5 , level $(r - 1)k + 1 = 11$; here the target has 4 parts, Lemma 5.6 guarantees 3. (c) Theorem 7.2: target F_ψ on γ_6 . (d) Theorem 7.3 with $k = 6$ on γ_6 : the curve is F_6 , whose count at level $\kappa + 1$ equals that of its atomic factor G_6 (Lemma 7.9(d)). Live version, each panel as a water level, in the [blueprint](#).

Evidence & scope

- **Program:** the Woodbury reductions of s^{-1} and g_r^{-1} (§5.1), with a control (supplement C06). `scripts/auditoria_S6/s6_woodbury.py`

Theorem 5.2. p. 17

For every $r \geq 1$, $\mathcal{U}_{r-1} \subsetneq \mathcal{U}_r$. For $r \geq 3$ an explicit witness is φ_r^{-1} , where $\varphi_r \in \mathcal{F}_r$ is the sum of five products of r inverses of u, v, w given in (5.1).

Proof

For $r = 1$: the element $S_3 = u^{-1} + v^{-1} + w^{-1} \in V_1 = \mathcal{F}_1$ has $S_3^{-1} \notin E_1 = \mathcal{U}_0$. This is Theorem 3.4 with μ the corank at χ (Example (b), a homomorphism $P \rightarrow M_1(\mathbb{C})$; the case $G = H$, $\mu = \text{corank}$ of Theorem 3.4), with target $H_{3,1} = \text{diag}(v, w) + u\mathbf{1}\mathbf{1}^\top$ (an atom, Lemma 5.7(1) with $n = 3, p = 1$): at χ , $H_{3,1}$ vanishes and has corank 2, while every $Q \in \Sigma_0$ has corank ≤ 1 there (the pivot T of Q_s evaluates at χ to an invertible triangular matrix, since $x_j \mapsto -1$, so Q_s^χ is equivalent to $I \oplus s^\chi$; and $x_j^\chi = -1$). For $r = 2$: Theorem 5.1, since $s \in \mathcal{F}_2$. (The case $r = 1$ also follows from [HS15, Thm 4.6] by two pivots; see §8.8.)

For $r \geq 3$: $\varphi_r \in \mathcal{F}_r$, so $\varphi_r^{-1} \in \mathcal{U}_r$. Suppose $\varphi_r^{-1} \in \mathcal{U}_{r-1}$. By Lemma 5.5 and Lemma 3.5, Q_r^{-1} has entries in $U_{\Sigma_{r-1}}$. Apply Theorem 3.4 with $G = H = Q_r$ (a full atom, Lemma 5.7(2)) and $\mu_Q = \nu_j$ along $\gamma_k, j = (r-1)k + 1$: given $Q \in \Sigma_{r-1}$, choose $k > \max(e(Q), r)$. Then $\nu_j(Q^{\gamma_k}) \geq 3$ by Lemma 5.6, while $\nu_j(Q^{\gamma_k}) \leq 2$ by Lemma 4.4 (with $q = r-1$) or by the remark after it for $Q \in \Sigma_0$. Contradiction. $\square$

Corollary 5.3. ■ p. 17

For every $r \geq 1$, $g_r = u^{-2r} + v^{-2r} + w^{-2r}$ satisfies $g_r^{-1} \in \mathcal{U}_{2r} \setminus \mathcal{U}_r$.

Proof

$g_r \in \mathcal{F}_{2r}$ (u^{-2r} is a product of $2r$ inverses of u), so $g_r^{-1} \in \mathcal{U}_{2r}$. Suppose $g_r^{-1} \in \mathcal{U}_r$. By §5.1 and Lemma 3.5, $H_{3,2r}^{-1}$ has entries in U_{Σ_r} . Apply Theorem 3.4 with $G = H = H_{3,2r}$ (an atom, Lemma 5.7(1)) and $\mu_Q = \nu_j$ along $\gamma_k, j = rk + 1$: for $Q \in \Sigma_r$ choose $k > e(Q)$. By Lemma 4.2(2) all four parts of $H_{3,2r}^{\gamma_k}$ equal $r(k+1) \geq rk + 1$, so $\nu_j = 4$, while $\nu_j(Q^{\gamma_k}) \leq 2$ by Lemma 4.4 with $q = r$, or by the remark after it for $Q \in \Sigma_0$. Contradiction. $\square$

Remark 5.4. (computations) ■ p. 18

Computed exactly (rational arithmetic, truncated series) on the parabolic arc with a generic perturbation $t^k E$; we say a border is *excluded* when its count at level $k+1$ is at most 2. The border of s is never excluded and its profile contains that of H_0 ; the control $(u^{-2} + v^{-2} + w^{-1})^{-1} = u^2 \Upsilon^{-1} v^2$ with $\Upsilon = u^2 + v^2 + v^2 w^{-1} u^2 \in V_1$ has the target $H_c = \begin{pmatrix} u^2 + v^2 & u^2 \\ u^2 & u^2 + w \end{pmatrix}$, and the evaluated matrix $H_c^{\gamma_k}$ has the four parts $(1, k, k+1, k+1)$, only two at level $k+1$ (on γ_k this follows from $H_c^{\gamma_k} \sim \text{diag}(2c I_2, W + \frac{c}{2} I_2)$ with $c = t^{k+1}$, $W = w^{\gamma_k}$ and $\det(W + \frac{c}{2} I_2) = -c + \frac{c^2}{4}$): as a target it would not exceed the bound on the allowed denominators, consistently with its inverse lying in $\mathcal{U}_1$; a border with twelve exceptional binomials, commutators of weight 2–4 that are trivial at χ , and Engel words is not excluded for $k = 10$ and is excluded for $k = 11, 12, 14$; here $e(f) = 11$ (the binomials $1 - [x, [x, y]]$ and $1 - [[x, y], [x, z]]$ have parts (5, 11) and (7, 11) under ρ_t), so the bound of Lemma 4.4 (proved on γ_k) predicts exclusion for $k \geq 12$, and at $k = 11$ the exclusion is a fact of the computation (supplement C07).

↳ Not used by another result of the paper.

Evidence & scope

- **Program:** profiles on the parabolic arc; the hard border (supplement C07).
scripts/smith_V1/parabolico_duro.py

Lemma 5.5. p. 19

The Schur complement of L in Q_r is $-\varphi_r$. Over $\Lambda = \mathbb{C}[F(u, v, w)]$, Q_r is associated with $I \oplus (-\varphi_r)$. For every ring $\mathcal{U} \supseteq E_1$: $\varphi_r^{-1} \in \mathcal{U} \iff Q_r^{-1} \in M(\mathcal{U})$.

↳ Used in: [Theorem 5.2](#) (p. 16)

Proof

L is triangular with letters (units of Λ) on the diagonal, and the inverse of the block of chain τ has $(1, r)$ entry $\ell_{\tau,1}^{-1} \cdots \ell_{\tau,r}^{-1}$; block elimination gives the first two claims. The block inverse of Q_r uses only L^{-1} (entries are products of inverse letters, in E_1), φ_r^{-1} and the entries of α, ω , which lie in A . $\square$

Evidence & scope

- **Program:** phi_3, phi_4 generated from the design equal the printed ones; Schur(Q_r) = -phi_r (supplement C08). scripts/auditoria_S12/s12_fr_simbolico.py

Lemma 5.6. (target profile) p. 20

Let $r \geq 3$ and $k \geq r + 1$, $\gamma = \gamma_k$. Each chain block of L^γ has exactly one part $\geq (r - 1)k + 1$. Hence $\nu_{(r-1)k+1}(L^\gamma) = 5$ and $\nu_{(r-1)k+1}(Q_r^\gamma) \geq 3$.

↳ Used in: [Theorem 5.2](#) (p. 16), [8.1](#) (p. 34), [8.2](#) (p. 35)

Proof

A chain block is equivalent to $I \oplus \ell_{\tau,r}^\gamma \cdots \ell_{\tau,1}^\gamma$ (the -1 's are unit pivots). For $r = 3$ the product is $U^3 = t^{k+1}U$, with parts $(k + 2, 2k + 1)$. For $r \geq 4$ write $U_i = t(\mathbf{a}_i \mathbf{b}_i - t^{k-1} E_{21})$ with $\mathbf{a}_i = (1, it)^\top$, $\mathbf{b}_i = (it, -1)$, so $\mathbf{b}_a \mathbf{a}_b = (a - b)t$. If the letters of the product have indices $i_1, \dots, i_r$ with $i_j \neq i_{j+1}$, its leading term is $t^r \cdot t^{r-1} \prod_j (i_j - i_{j+1}) \mathbf{a}_{i_1} \mathbf{b}_{i_r}$, whose $(1, 2)$ entry has valuation exactly $2r - 1$; corrections have valuation $\geq r + k - 1 \geq 2r$. So the first invariant factor is t^{2r-1} and, as $\det U_i = -t^{k+1}$, the second is $t^{r(k+1)-(2r-1)}$. Now $rk - r + 1 \geq (r - 1)k + 1 \iff k \geq r$ and $2r - 1 < (r - 1)k + 1$; for $r = 3$, $2k + 1 \geq 2k + 1 > k + 2$. Five chains give $\nu(L^\gamma) = 5$, and Lemma 3.2(3) with $m = 2$ gives ≥ 3 . $\square$

Lemma 5.7. (atoms) p. 20

1. For $n \geq 2$ and $p \geq 1$, $H_{n,p}$ is an atom of $\mathbb{C}\langle u_1, \dots, u_n \rangle$.
2. For $r \geq 3$, Q_r is an atom of $P = \mathbb{C}\langle u, v, w \rangle$.

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ChainPencilConditions.lean (SHA-256 of the compiled file: e266df18e48e9700...; a source page on this site may show it with comments edited, under its own hash).

Proof

Proof of (1). Let $R = [\mathbf{1} \mid I_{n-1}] \in M_{(n-1) \times n}(\mathbb{C})$, with columns $r_1 = \mathbf{1}$ and $r_i = e_{i-1}$ for $i \geq 2$, and let $U = \text{diag}(u_1, \dots, u_n)$. Then $H_{n,p} = R U^p R^\top$. The matrix $H_{n,p}$ is full, since $H_{n,p}(1, \dots, 1) = I + \mathbf{1}\mathbf{1}^\top$ has determinant n . It is not invertible, since it vanishes at 0 (§2.2(a)). Put $N = np - 1$.

The pencil. For $p = 1$ let $\mathbf{L} = H_{n,1} = R U R^\top$. For $p \geq 2$ take the vector $v = (v_0, v_1, \dots, v_{p-1})$ with $v_0 \in \mathbb{C}^{n-1}$ and $v_k \in \mathbb{C}^n$, and let $\mathbf{L}$ be the $N \times N$ block matrix with

$$\mathbf{L}_{01} = R U, \quad \mathbf{L}_{kk} = I_n \ (k \geq 1), \quad \mathbf{L}_{k,k+1} = -U \ (1 \leq k \leq p-2), \quad \mathbf{L}_{p-1,0} = -U R^\top,$$

and zero blocks elsewhere. The block T on the indices $1, \dots, p-1$ is $I - N$, where N is block strictly upper triangular with U on the block superdiagonal. So $T^{-1} = \sum_{m < p-1} N^m$ is polynomial, and $(T^{-1})_{1,p-1} = U^{p-2}$. The Schur complement of T is $-R U U^{p-2} (-U R^\top) = H_{n,p}$. By §2.2(b),(c) it suffices to prove that $\mathbf{L}$ is an atom.

Translation. Write $\mathbf{L} = \mathbf{L}_0 + \sum_i u_i \mathbf{L}_i$. Substitute $u_i = 1 + z_i$, an affine automorphism (§2.2(d)). The new pencil has constant term $K = \mathbf{L}(1, \dots, 1)$ and the same coefficients $\mathbf{L}_i$. Its determinant is $\det K = \det T(1) \cdot \det(R R^\top) = n \neq 0$.

Rank- p factorization of the coefficients. For $c = (c_1, \dots, c_p) \in \mathbb{C}^p$ let $\iota_i c$ be the vector with block 0 equal to $r_i c_1$, block k equal to $-e_i c_{k+1}$ ($1 \leq k \leq p-1$). Let

$$\kappa_i v = (e_i^\top v_1, \dots, e_i^\top v_{p-1}, r_i^\top v_0) \in \mathbb{C}^p.$$

Then $\mathbf{L}_i = \iota_i \kappa_i$. The map ι_i is injective, because its p components land in different blocks and $r_i, e_i \neq 0$. The map κ_i is onto, for the same reason. For $p = 1$, $\iota_i = r_i$ and $\kappa_i = r_i^\top$.

Monic. We have

$\sum_i \text{im } \mathbf{L}_i = \sum_i \text{im } \iota_i = (\sum_i \text{span}(r_i)) \oplus \bigoplus_{k \geq 1} (\sum_i \text{span}(e_i)) = \mathbb{C}^N$, because R has rank $n-1$. Next, $\bigcap_i \ker \mathbf{L}_i = \bigcap_i \ker \kappa_i$: this forces $v_k = 0$ for $k \geq 1$, and $R^\top v_0 = 0$, so $v_0 = 0$. Hence the row and the column of coefficients have rank N .

The transfer matrices. Let $\vartheta_i = K^{-1} \iota_i$, injective. To compute $v = \vartheta_i c$, solve $K v = \iota_i c$:

- blocks $1 \leq k \leq p-2$ give $v_{k+1} = v_k + e_i c_{k+1}$;
- block $p-1$ gives $v_{p-1} = R^\top v_0 - e_i c_p$;
- block 0 gives $R v_1 = r_i c_1$.

Hence $R R^\top v_0 = r_i \sigma$ with $\sigma = \mathbf{1}^\top c$, and

$$v_k = G e_i \sigma - e_i \sum_{m > k} c_m \quad (1 \leq k \leq p-1), \quad G = R^\top (R R^\top)^{-1} R.$$

The matrix G is the orthogonal projector onto $\text{im } R^\top = (\ker R)^\perp$. Since $\ker R = \langle \mathbf{n} \rangle$ with $\mathbf{n} = (1, -1, \dots, -1)$, we have $G = I - \mathbf{n}\mathbf{n}^\top/n$. Also $r_j^\top v_0 = e_j^\top G e_i \sigma$. Therefore

$$\kappa_j \vartheta_i = g_{ji} \mathbf{1}\mathbf{1}^\top - \delta_{ij} \mathbf{N}_p, \quad g_{ji} = \delta_{ij} - \frac{n_j n_i}{n}, \quad (5.2)$$

where $(\mathbf{N}_p)_{km} = 1$ for $m > k$ and 0 otherwise. In particular $g_{ji} = \pm 1/n \neq 0$ for $i \neq j$. (For $p = 1$, $\mathbf{N}_1 = 0$ and (5.2) reads $r_j^\top (RR^\top)^{-1} r_i = g_{ji}$.)

Cyclicity. Let $M = a\mathbf{1}\mathbf{1}^\top - \mathbf{N}_p$ with $a \in \mathbb{C}$. Then $\mathbf{1}$ is a cyclic vector for M , and $\mathbf{1}^\top$ is cyclic for M acting on row vectors from the right.

- Since $Mx = -\mathbf{N}_p x + a(\mathbf{1}^\top x)\mathbf{1}$, induction gives $\text{span}\{M^m \mathbf{1}\}_{m < q} = \text{span}\{\mathbf{N}_p^m \mathbf{1}\}_{m < q}$ for every q .
- The vector $\mathbf{N}_p^m \mathbf{1}$ vanishes below position $p - m$ and has entry 1 there. So these vectors form a triangular basis.
- The row case is the same: $\mathbf{1}^\top \mathbf{N}_p^m$ vanishes at positions $\leq m$ and equals 1 at position $m + 1$.

No invariant subspace. Let $A_i = K^{-1} \mathbf{L}_i = \vartheta_i \kappa_i$, and let $W \neq 0$ be invariant under all A_i . Put $M_i = \kappa_i \vartheta_i = g_{ii} \mathbf{1}\mathbf{1}^\top - \mathbf{N}_p$. Then $A_i^m \vartheta_i = \vartheta_i M_i^m$, and $A_j \vartheta_i = g_{ji} \vartheta_j \mathbf{1}\mathbf{1}^\top$ for $j \neq i$.

1. Pick $w \in W \setminus 0$. By the monic condition some $c = \kappa_i w \neq 0$, and $A_i w = \vartheta_i c \in W \setminus 0$.
2. Since $c \neq 0$ and $\mathbf{1}^\top$ is cyclic for M_i , there is m with $\beta = \mathbf{1}^\top M_i^m c \neq 0$. Fix $j \neq i$ (possible since $n \geq 2$). Then $W \ni A_j A_i^m \vartheta_i c = A_j \vartheta_i M_i^m c = g_{ji} \beta \vartheta_j \mathbf{1}$, so $\vartheta_j \mathbf{1} \in W$.
3. Then $A_j^m \vartheta_j \mathbf{1} = \vartheta_j M_j^m \mathbf{1} \in W$ for all m . Since $\mathbf{1}$ is cyclic for M_j , $\text{im } \vartheta_j \subseteq W$.
4. For every $l \neq j$, $\vartheta_j e_1 \in W$ by step 3, and $A_l \vartheta_j e_1 = g_{lj} \vartheta_l \mathbf{1}\mathbf{1}^\top e_1 = g_{lj} \vartheta_l \mathbf{1} \in W$. So $\vartheta_l \mathbf{1} \in W$, and by step 3 $\text{im } \vartheta_l \subseteq W$.

Hence $W \supseteq \sum_l \text{im } \vartheta_l = K^{-1} \sum_l \text{im } \iota_l = \mathbb{C}^N$. By Lemma 2.1 the translated pencil is an atom; so are $\mathbf{L}$ (§2.2(d)) and $H_{n,p}$ (§2.2(c)). $\square$

Evidence & scope

- **Program:** the eight rank conditions of the design, $r = 3..40$ (supplement C09). `scripts/smith_v1/verifica_T160.py`
- **Program:** $H_{n,p}$: epic and full coefficient algebra modulo two primes (supplement C10). `scripts/smith_v1/atomo_epic.py`

Proposition 5.8. p. 24

Let $N \geq n \geq 3$, $A = \mathbb{C}[F_N]$, $u_i = 1 + x_i$, and form E_1, V_1 over A . For $s_n = \sum_{i \leq n} u_i^{-2}$ we have $s_n^{-1} \in \mathcal{U}_2$ and $s_n^{-1} \notin E_1[V_1^{-1}]$.

↳ Used in: 5.9 (p. 21)

Proof

The proof of Theorem 5.1 applies with three changes.

- The arc sends every letter to the exact arc, $u_i \mapsto U_{i-1}(t, t^k)$ for $i = 1, \dots, N$; Lemma 4.1, the trace law and Lemmas 4.3–4.4 hold verbatim for distinct indices. (Sending the extra letters to $-I$ does not work: $1 - [x_N, x_1]$ then evaluates to 0.)
- The target is $H_{n,2}$, with §5.1 read with u_2 in place of v . Its $2(n-1)$ parts all equal $k+1$, since $\det(\mathbf{1}\mathbf{1}^\top + I_{n-1}) = n$, against the bound 2 of Lemma 4.4.
- $H_{n,2}$ is an atom of $P_N = \mathbb{C}\langle x_1, \dots, x_N \rangle$. The translated pencil of the proof of Lemma 5.7(1) is a pencil over P_N whose coefficients at $x_{n+1}, \dots, x_N$ are zero. Zero coefficients change neither the ranks of the row and the column of coefficients nor the common invariant subspaces of the M_i . So Lemma 2.1 applies over P_N , and §2.2(b)–(d) give the claim as in Lemma 5.7(1). $\square$

Evidence & scope

- **Program:** N-axis arc, 80 random borders, margin $2(n-2)$, with controls (supplement C11). `scripts/auditoria_S27/s27_n_ejes.py`

Remark 5.9. (beyond $\mathcal{U}_1$) p. 24

In the setting of Proposition 5.8, with Γ'_a and D' formed over $A = \mathbb{C}[F_N]$ (§2.4, §7.2), $s_n^{-1} \notin \Gamma'_{n-2}$ for $3 \leq n \leq N$; in particular $s_4^{-1} \notin \Gamma'_2 \supseteq D'$ for $N \geq 4$. The proof is that of Theorem 7.2 (§7.5) with $H_{n,2}$ in place of F_ψ . If $s_n^{-1} \in \Gamma'_{n-2}$, Woodbury with $\Delta_n = \text{diag}(u_2^2, \dots, u_n^2)$ gives $H_{n,2}^{-1} = \Delta_n^{-1} - \Delta_n^{-1} \mathbf{1} s_n^{-1} \mathbf{1}^\top \Delta_n^{-1} \in M_{n-1}(U_{\Sigma'_{n-2}})$ (§5.1, and Lemma 3.6 with $a = n-2$, §7.2). $H_{n,2}$ is a full atom of P_N and its $2(n-1)$ parts at γ_k all equal $k+1$ (proof of Proposition 5.8). For $Q \in \Sigma'_{n-2}$ and $k > e(Q)$, $\nu_{k+1}(Q^{\gamma_k}) \leq 2(n-2)$ (≤ 2 for Q_s, x_j and the borders of $\mathcal{U}_1$ by Lemma 4.4 and the remark after it, $\leq 2(n-2)$ for $\tilde{B}_M$ by Lemma 7.7; their proofs use only Lemma 4.3, which holds on the arc with N axes), and Theorem 3.4 gives a contradiction. For $n = 3$ this is Proposition 5.8, since $\Gamma'_1 = \mathcal{U}_1$. Finally $D' \subseteq \Gamma'_2$ by Lemma 7.6, whose proof uses only that each element of $D_\pi = V_1 \cup D_{\omega+1} D_{\omega+1}$ (read as a set, as after Lemma 7.4) lies in V_1 or is pure. We do not claim here that $n[\pi]$ is a stopping step of the series over $\mathbb{C}[F_N]$ for $n \geq 4$; that would need Lemmas 2.2, 7.4 and 7.5 over F_N .

↳ Not used by another result of the paper.

6 The size chain (Theorem B(ii))

Theorem 6.1. p. 25

For every $d \geq 2$, $\rho_d \in A_d \cap E_2$ and $\rho_d \notin W_{d-1}$. Consequently

1. $\mathcal{U}_\infty = W_1 \cap E_2 \subsetneq E_2$; a witness is $\rho_2 = (p_1 - c p_2^{-1} c)^{-1}$ with $p_j = \xi^2 + (j+1)\eta^2 - (j+2)\zeta^2$, the diagonal entries of $\hat{F}_d$;
2. $A_{d-1} \cap E_2 \subsetneq A_d \cap E_2$ and $B_{d-1} \subsetneq B_d$ for every $d \geq 2$, and $\bigcup_b B_b = E_2$.

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InverseIdentities.lean (SHA-256 of the compiled file: 3e2df75becd9213c...; a source page on this site may show it with comments edited, under its own hash).

↳ Used in: Theorem 7.2 (p. 27), 8.2 (p. 35), 8.4 (p. 38)

Proof

Fix $d \geq 2$. $\hat{F}_d$ is a full $d \times d$ matrix over A (proof of (6.1)), so $\rho_d \in A_d$. By (6.2), $\sigma \in E_1$ and $\sigma \neq 0$, so $\rho_d = \sigma^{-1} \in E_2$.

Suppose $\rho_d \in W_{d-1}$. Since $E_1 \subseteq W_1 \subseteq W_{d-1}$ and T_d^{-1} has entries in E_1 , (6.2) gives $\hat{F}_d^{-1} \in M_d(W_{d-1}) = M_d(U_{\Sigma^{(d-1)}})$. Apply Theorem 3.4 with $G = H = \hat{F}_d$ (a full atom, Lemma 6.2(c)), $\Sigma = \Sigma^{(d-1)}$ and, for every Q , μ_Q the corank at Θ (Example (b) with $K = \mathbb{C}(t)$; the case $G = H$, $\mu = \text{corank}$ of Theorem 3.4):
 $\text{corank } \hat{F}_d^\Theta = 2d > 2(d-1) \geq \text{corank } Q^\Theta$ for every $Q \in \Sigma^{(d-1)}$ (Lemma 6.2(a),(b)).
 This is a contradiction, so $\rho_d \notin W_{d-1}$. (Figure 3(b) shows both sides of this comparison for $d \leq 6$.)

The consequences: $A_{d-1} \subseteq W_{d-1}$, so $\rho_d \in (A_d \cap E_2) \setminus A_{d-1}$ and $\rho_d \in B_d \setminus B_{d-1}$. For $d = 2$, $W_1 \cap E_2 = \mathcal{U}_\infty$ (§2.1), so $\rho_2 \in E_2 \setminus \mathcal{U}_\infty$. Finally $\bigcup_b B_b = E_2$: a nonzero $e \in E_1$ has a representation $e = \gamma - \mathbf{u}M^{-1}\mathbf{v}$ with M full over A [Coh06, Thm 7.1.2(d)], with Σ the full matrices over A , since the Σ -rational closure of A is a subfield of D containing A , hence all of D ; then $N = \begin{pmatrix} M & \mathbf{v} \\ \mathbf{u} & \gamma \end{pmatrix}$ is full over A of some order n and $e^{-1} = (N^{-1})_{nn} \in A_n \subseteq W_n$; E_2 is generated by E_1 and such inverses, and the W_b increase. $\square$

The strictness of the chain A_b alone does not need Lemma 4.5: with $\Sigma_A^{(b)} = \Sigma_0 \cup \{\text{lifts of full } a \times a \text{ matrices over } A, a \leq b\}$ one has $A_b = U_{\Sigma_A^{(b)}}$ (as in Lemma 3.5), and the lifts carry only the pivots x_j , which are invertible at Θ , so their corank is at most $2a$ directly. Lemma 4.5, whose proof amounts to the freeness of the group generated by $1 + 2\mathbf{i}, 1 + 2\mathbf{j}, 1 + 2\mathbf{k}$ modulo scalars, is needed only for W_b , that is, for $\mathcal{U}_\infty \subsetneq E_2$ and for $B_{d-1} \subsetneq B_d$. For the witnesses f_k the character χ already suffices (Remark 7.11); it does not detect ρ_d , since $\hat{F}_d^\chi = 8C_d$ is invertible.

Evidence & scope

- **Program:** the formula for ρ_d , the orientation of the cycle, F_d vanishes on Θ (supplement C12). `scripts/smith_V1/verifica_T162.py`

Lemma 6.2. p. 25

(a) For $Q \in \Sigma^{(b)}$, $\text{corank } Q^\Theta \leq 2b$. (b) $\hat{F}_d^\Theta = 0$, so $\text{corank } \hat{F}_d^\Theta = 2d$; likewise $(F^{(\lambda)})^\Theta = 0$ for every λ . (c) For $d \geq 2$, $\hat{F}_d$ is a full atom over P ; for $\lambda \in \mathcal{T} := \mathbb{C} \setminus \{-1, 0, 1, 2\}$, $F^{(\lambda)}$ is a full atom over P .

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↳ Used in: Theorem 6.1 (p. 21), 8.1 (p. 34), 8.4 (p. 38), 8.5 (p. 38)

Proof

(a) For Q_s ($s \in A$) the pivot block is triangular with $\Theta(x_j)$ on the diagonal, invertible, so $\text{corank } Q_s^\Theta = \text{corank } s^\Theta \leq 2$, also when $s^\Theta = 0$ (e.g. $s = c$). $\Theta(x_j)$ has corank 0. In $\tilde{B}_G^\Theta$ the triangular pivot evaluates to an invertible matrix ($\Theta(x_j)$ on its diagonal), and its Schur complement is B_G^Θ , since Θ is a ring homomorphism on A . In B_G^Θ the block L^Θ is block upper triangular with the 2×2 blocks ℓ^Θ on the diagonal, invertible by Lemma 4.5. Taking Schur complements does not change the corank, so $\text{corank } \tilde{B}_G^\Theta$ is the corank of a $2a \times 2a$ matrix over $\mathbb{C}(t)$, at most $2a \leq 2b$. (No element of R_{bin} is evaluated at Θ ; only the polynomial matrix $\tilde{B}_G$ is.)

1. $q_\lambda^\Theta = -t^2(1 + (\lambda - 1) - \lambda)I = 0$ and $c^\Theta = 0$.
2. We prove a slightly more general statement.

Claim. Let $C, D \in M_d(\mathbb{C})$ satisfy:

- (H1) C and D have no common invariant subspace other than 0 and $\mathbb{C}^d$;
- (H2) D is invertible;
- (H3) $E := D - I - C^2$ is invertible.

Then $F = (\xi^2 - \eta^2)I + (\eta^2 - \zeta^2)D + cC$, with $c = \xi\eta + \eta\xi$, is a full atom of $P = \mathbb{C}\langle \xi, \eta, \zeta \rangle$.

Proof. Let $\mathbf{L} = A_0 + \xi A_\xi + \eta A_\eta + \zeta A_\zeta$ be the $4d \times 4d$ pencil:

$$\mathbf{L} = \begin{pmatrix} 0 & \xi I + \eta C & \xi C + \eta(D - I) & -\zeta D \\ -\xi I & I & 0 & 0 \\ -\eta I & 0 & I & 0 \\ -\zeta I & 0 & 0 & I \end{pmatrix}.$$

Its lower right block is I_{3d} , and the Schur complement is

$(\xi I + \eta C)\xi + (\xi C + \eta(D - I))\eta - \zeta D\zeta = \xi^2 I + cC + \eta^2(D - I) - \zeta^2 D = F$. So $\mathbf{L}$ is stably associated with F (§2.2(b)). F is full, because $F(1, 0, 0) = I$, and not invertible, because $F(0) = 0$.

Monic. Write vectors as (v_0, v_1, v_2, v_3) in blocks of size d . For a row f :

$$fA_\xi = (-f_1, f_0, f_0 C, 0), \quad fA_\eta = (-f_2, f_0 C, f_0(D - I), 0), \quad fA_\zeta = (-f_3, 0, 0, -f_0)$$



If all three vanish, then $f = 0$. For a column v :

$$A_\xi v = (v_1 + Cv_2, -v_0, 0, 0), \quad A_\eta v = (Cv_1 + (D - I)v_2, 0, -v_0, 0), \quad A_\zeta v = (-Dv_3, 0, 0, v_0)$$



If all three vanish, then $v_0 = 0$, $v_1 = -Cv_2$, $Ev_2 = 0$ and $Dv_3 = 0$, so $v = 0$ by (H2), (H3).

Normalization. Substitute $\xi = 1 + \xi'$ (§2.2(d)). The constant term becomes

$$K = \mathbf{L}(1, 0, 0) = \begin{pmatrix} 0 & I & C & 0 \\ -I & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad K^{-1}(y_0, y_1, y_2, y_3) = (y_0 - y_1 - Cy_2, y_0 \cdot$$



By Lemma 2.1 it suffices to show that $M_a = K^{-1}A_a$ ($a = \xi, \eta, \zeta$) have no common invariant subspace $W \neq 0$ other than $\mathbb{C}^{4d}$. Use the coordinates $z = (v_0, v_1 + Cv_2, v_2, v_3)$, an invertible change. A direct computation gives

$$\begin{aligned} M_\xi z &= (z_0 + z_1, z_1, 0, 0), & M_\zeta z &= (-Dz_3, -Dz_3, 0, -z_0), \\ M_\eta z &= (Cz_0 + Cz_1 + Ez_2, Cz_1 + Ez_2, -z_0, 0). \end{aligned}$$

The polynomials $P = 2M_\xi - M_\xi^2$ and $N = M_\xi^2 - M_\xi$ in M_ξ act by $Pz = (z_0, z_1, 0, 0)$ and $Nz = (z_1, 0, 0, 0)$. Let B_k be the k -th block subspace.

1. W meets B_0 . Let $0 \neq z \in W$.
 - If $z_1 \neq 0$, take Nz .
 - If $z_1 = 0 \neq z_0$, take Pz .
 - If $z_0 = z_1 = 0 \neq z_2$, take $NM_\eta z = (Ez_2, 0, 0, 0)$, nonzero by (H3).
 - Otherwise take $NM_\zeta z = (-Dz_3, 0, 0, 0)$, nonzero by (H2).
2. $B_0 \subseteq W$. Let $W_0 = \{a : (a, 0, 0, 0) \in W\} \neq 0$. On B_0 we have $PM_\eta(a, 0, 0, 0) = (Ca, 0, 0, 0)$ and $NM_\zeta^2(a, 0, 0, 0) = N(Da, Da, 0, 0) = (Da, 0, 0, 0)$. So W_0 is invariant under C and D , and $W_0 = \mathbb{C}^d$ by (H1).
3. *The other blocks.* Let $a \in \mathbb{C}^d$.
 - $M_\zeta(a, 0, 0, 0) = (0, 0, 0, -a)$ gives $B_3 \subseteq W$.
 - $M_\eta(a, 0, 0, 0) = (Ca, 0, -a, 0)$, minus $(Ca, 0, 0, 0) \in B_0$, gives $B_2 \subseteq W$.
 - $M_\eta(0, 0, a, 0) = (Ea, Ea, 0, 0)$, minus $(Ea, 0, 0, 0)$, gives $(0, Ea, 0, 0) \in W$. By (H3), $B_1 \subseteq W$.

So $W = \mathbb{C}^{4d}$. By Lemma 2.1 and §2.2(c),(d), F is an atom. $\square$

Evidence & scope

- **Program:** Schur complement of the 4d x 4d pencil and the criterion (supplement C13). `scripts/auditoria_S11/s11_a2_a4.py`
- **Program:** F_d atoms for $d = 2, 3, 4$: epic, algebra of dimension $(4d)^2$ (supplement C14). `scripts/auditoria_S16/s16_atomo_Fd.py`
- **Program:** coranks of borders at Theta: 0, 1, 2 (supplement C15). `scripts/auditoria_S10/s10_corango.py`

Remark 6.3. (a candidate second witness for part (1))  p. 27

An earlier route suggests a second witness. Let $\theta \in \mathbb{C}$ be transcendental, $\tau_i'^\theta = 1 + x_i + \theta x_{i+1}$ (indices mod 3) and $g_\theta = \sum_i (\tau_i'^\theta)^{-2} \in E_1$. We expect $g_\theta^{-1} \in E_2 \setminus \mathcal{U}_\infty$ but do not prove it; it is not used. The argument would be the same

corank comparison ($4 > 2$) at the point $X_i = \lambda_\theta I + T_i$, $\lambda_\theta = -1/(1 + \theta)$,

$T_i = (N_i - \theta N_{i+1} + \theta^2 N_{i+2})/(1 + \theta^3)$, $N_i = \begin{pmatrix} it^2 & -t \\ i^2 t^3 & -it^2 \end{pmatrix}$, where

$I + X_i + \theta X_{i+1} = N_i$ with $N_i^2 = 0$, so that the target $H_{3,2}(\tau'_0, \tau'_1, \tau'_2)$ vanishes; the target is an atom because $(x_0, x_1, x_2) \mapsto (\tau_0^\theta, \tau_1^\theta, \tau_2^\theta)$ is affine with linear part of determinant $1 + \theta^3 \neq 0$. The missing step is that no allowed binomial is identically singular on the curve, $\det(I - \beta h(X(t))) \neq 0$ for all $h \neq 1$ and $\beta \neq 0$; the route assumes θ transcendental, and for $\theta = 2, 1, \frac{1}{4}$ some binomials do become singular on the curve. We prefer the integer witness ρ_2 .

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↳ Not used by another result of the paper.

Remark 6.4. (the symmetric curve fails) ■ p. 28

With $\theta = 1$, i.e. $\tau'_i = 1 + x_i + x_{i+1}$, the analogous argument breaks: on the whole variety where $\tau'_0, \tau'_1, \tau'_2$ are nilpotent, the commutator $h = [xy, yx]$ (a word with exponent sum 0) is unipotent, so $1 - h$ degenerates. The reason is an identity in GL_2 : if $\text{tr } X = \text{tr } Y = -1$ and $Z = I + X + Y$ satisfies $Z^2 = 0$, then $(XY)(YX)(XY)^{-1}(YX)^{-1} = I + (\det X^{-1} - \det Y^{-1})Z$. If $\tau'_0, \tau'_1, \tau'_2$ evaluate to nilpotent 2×2 matrices, then $\text{tr}(I + X_i + X_{i+1}) = 0$ for $i = 0, 1, 2$, so every $\text{tr } X_i = -1$, and the identity applies with $X = X_0, Y = X_1$ (supplement C26).

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↳ Not used by another result of the paper.

7 The Dubrovin–Gräter–Hanke series (Theorem C)

Theorem 7.1. ■ p. 28

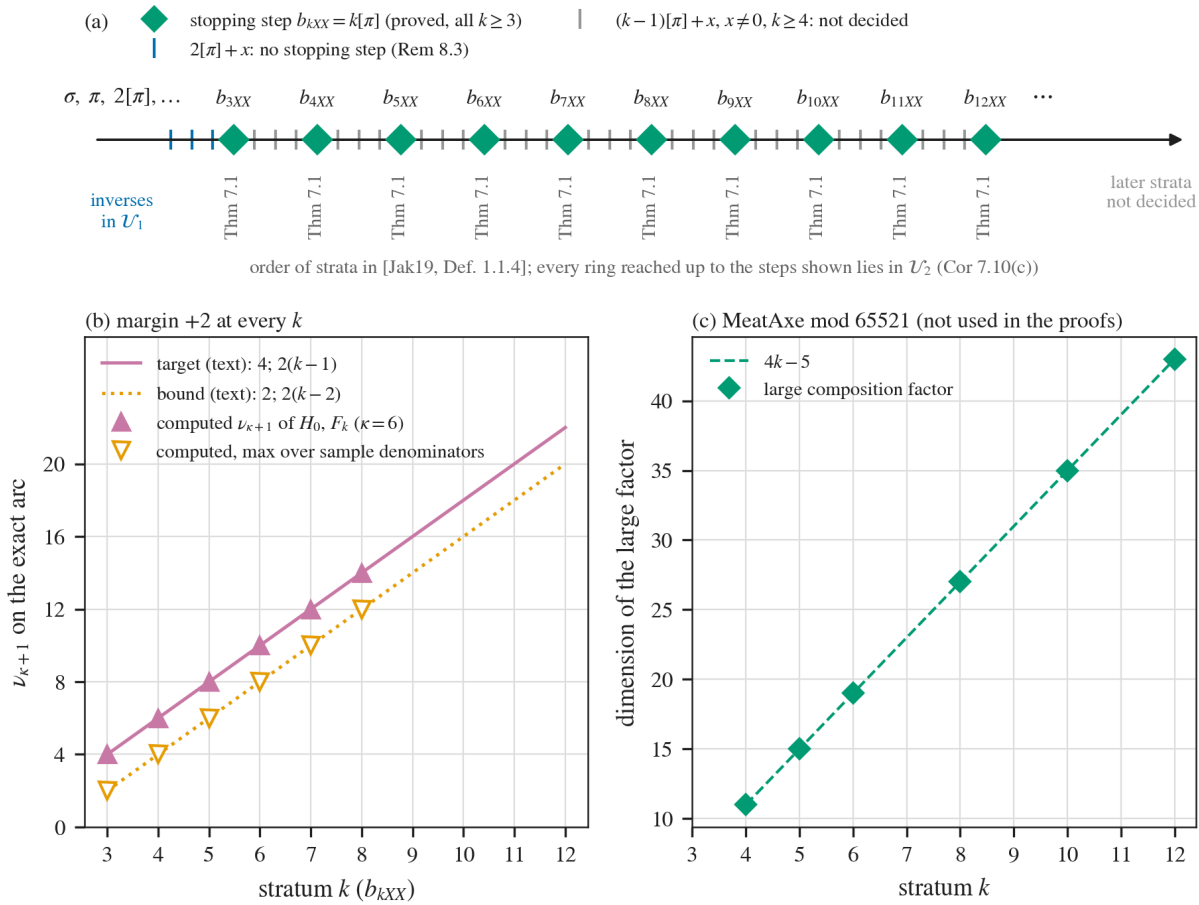
For every $k \geq 3$ the step $b_{kXX} = k[\pi]$ is a stopping step of the series of §2.3, for D and $\mathcal{M} = \mathbb{C}^\times F_3$ in Jakobs' formulation [Jak19, Def. 1.1.4]. When the inversion pointer reaches b_{kXX} , the ring built so far lies in $\Gamma'_{k-2} \cap E_2$; for $k = 3$ it is exactly $\mathcal{U}_1 = E_1[V_1^{-1}]$. A witness is $s = u^{-2} + v^{-2} + w^{-2}$ for $k = 3$, and $f_k = u^{-2} + w^{-2} + \sum_{i=1}^{k-2} y^i u^{-2} y^{-i}$ for $k \geq 4$ (ψ below is a second witness for $k = 4$). In particular infinitely many stopping steps occur while the ring built by the series is still inside E_2 .

↳ Used in: 8.3 (p. 37)

Proof

Let D_x be as in Lemma 7.5. For $k = 3$: $s \in D_\pi^{+3}$ and $s^{-1} \notin \mathcal{U}_1 = D_x$ (Theorem 5.1). For $k \geq 4$: $f_k \in D_\pi^{+k}$ and $f_k^{-1} \notin \Gamma'_{k-2} \supseteq D_x$ (Theorem 7.3); for $k = 4$ also $\psi \in D_\pi^{+4}$ and $\psi^{-1} \notin \Gamma'_2$ (Theorem 7.2). In each case the witness lies in no stratum born before b_{kXX} , since its inverse would then be in D_x ; so its complexity is $\text{cp} = b_{kXX}$, $D_{b_{kXX}}^{-1} \not\subseteq D_x$, and the Case 3 at which the pointer reaches b_{kXX} selects it. That Case 3 exists: otherwise every Case 3 would select an earlier stratum, and the union of the series, which is D [Jak19, Def. 1.1.4], would lie in Γ'_{k-2} , against $f_k^{-1} \notin \Gamma'_{k-2}$ (or $s^{-1} \notin \mathcal{U}_1$). The ring reached after the step, $D_x[(D_\pi^{+k} \setminus 0)^{-1}]$, still lies in E_2 , since $D_\pi^{+k} \subseteq E_1$. $\square$

 **Figure**



The Dubrovin–Gräter–Hanke series inside E_2 . (a) Order of the strata [Jak19, Def. 1.1.4]: every $b_{kXX} = k[\pi]$, $k \geq 3$, is a stopping step (Theorem 7.1); the strata $2[\pi] + x$ are not stopping steps, and the strata $(k-1)[\pi] + x$ with $x \neq 0$, $k \geq 4$, are not decided (Remark 8.3). (b) The margin behind each step: the target (H_0 for $k = 3$; F_k for $k \geq 4$, with the count of its atomic factor G_k) has $2(k-1)$ Smith parts at level $\kappa + 1$ and every allowed denominator at most $2(k-2)$. Lines: values proved in the text; markers: exact computations with $\kappa = 6$, the lower marker being the maximum over sample denominators. (c) For comparison only: the size of the large composition factor of the linear pencil of F_k , computed with MeatAxe modulo 65521, is $4k - 5$; this is not used in the proofs, and absolute irreducibility was confirmed only for $k = 6$ and 12.

Evidence & scope

- **Program:** lexicographic order of $N(O)$, 8 checks with a control (supplement C19).
scripts/auditoria_S15/orden_N.py

Theorem 7.2. p. 28

Let $\psi = u^{-2} + v^{-2} + w^{-2} + yu^{-2}y^{-1}$. Then $\psi^{-1} \in \mathcal{U}_2$ and $\psi^{-1} \notin \Gamma'_2$. Since $D' \subseteq \Gamma_2 \subseteq \Gamma'_2$, this gives $\mathcal{U}_1 \subsetneq D' \subsetneq \mathcal{U}_2$. Moreover $\mathcal{U}_\infty$ and Γ_2 are incomparable.

↳ Used in: [Theorem 7.1](#) (p. 25), [7.12](#) (p. 34)

Proof

$u^{-2} = u^{-1} \cdot 1 \cdot u^{-1}$ and $yu^{-2}y^{-1}$ has outer coefficients $y, y^{-1} \in A$, so $\psi \in \mathcal{F}_2$. By Lemma 7.8, F_ψ is full, so $H_\psi = \Delta_\psi(I + \Delta_\psi^{-1}\mathbf{1}u^2\mathbf{1}^\top)$ is full, and since $I + \mathbf{a}\mathbf{b}^\top$ is invertible iff $1 + \mathbf{b}^\top\mathbf{a}$ is, $u^2\psi \neq 0$; hence $\psi^{-1} \in \mathcal{U}_2$. Also $yu^{-2}y^{-1} = (yuy^{-1})^{-1}(yuy^{-1})^{-1}$ with $yuy^{-1} = 1 + yxy^{-1}$ a binomial, so $\psi \in D_\pi^{+4}$.

Suppose $\psi^{-1} \in \Gamma'_2$. Since $H_\psi = \Delta_\psi + \mathbf{1}u^2\mathbf{1}^\top$ and $\psi = u^{-2} + \mathbf{1}^\top\Delta_\psi^{-1}\mathbf{1}$, Woodbury gives $H_\psi^{-1} = \Delta_\psi^{-1} - \Delta_\psi^{-1}\mathbf{1}\psi^{-1}\mathbf{1}^\top\Delta_\psi^{-1}$ with $\Delta_\psi^{-1} \in M_3(E_1)$, and $F_\psi^{-1} = \text{diag}(1, 1, y^{-1})H_\psi^{-1} \in M_3(\Gamma'_2) = M_3(U_{\Sigma'_2})$ (Lemma 3.6). Apply Theorem 3.4 with $G = H = F_\psi$ and $\mu_Q = \nu_{k+1}$ along γ_k (the case $G = H, \mu = \nu_j$ of Theorem 3.4). Given $Q \in \Sigma'_2$, choose k larger than $e(Q)$: then $\nu_{k+1}(Q^{\gamma_k}) \leq 2$ for $Q = Q_s$ or x_j and for the borders of $\mathcal{U}_1$ (Lemma 4.4 with $q = 1$), and ≤ 4 for $Q = \tilde{B}_M$ (Lemma 7.7). Since $\nu_{k+1}(F_\psi^{\gamma_k}) = 6$ for every k (Lemma 7.8), this contradicts Theorem 3.4. Hence $\psi^{-1} \notin \Gamma'_2$.

$D' \subseteq \Gamma_2 \subseteq \Gamma'_2$ by Lemma 7.6, so $\psi^{-1} \in \mathcal{U}_2 \setminus D'$. Each element of D_π lies in $\mathcal{F}_2$, so $D' \subseteq \mathcal{U}_2$; and $s = u^{-2} + v^{-2} + w^{-2} \in D_\pi^{+3}$ gives $s^{-1} \in D' \setminus \mathcal{U}_1$ by Theorem 5.1. Finally $\psi^{-1} \in \mathcal{U}_2 \subseteq \mathcal{U}_\infty$ and $\psi^{-1} \notin \Gamma_2$, while $\rho_2 \in \Gamma_2$ ($\hat{F}_2$ is a full 2×2 matrix over A) and $\rho_2 \notin \mathcal{U}_\infty$ (Theorem 6.1). Both rings lie in E_2 : $\mathcal{U}_\infty \subseteq E_2$ (§2.1), and for a full 2×2 matrix over A a pivot (after a permutation if needed) leaves a Schur complement in E_1 , so the entries of its inverse lie in E_2 . $\square$

Theorem 7.3. p. 28

For every $k \geq 4$, $f_k = u^{-2} + w^{-2} + \sum_{i=1}^{k-2} y^i u^{-2} y^{-i}$ is a sum of k elements of D_π , and $f_k^{-1} \notin \Gamma'_{k-2}$.

↳ Used in: [Theorem 7.1](#) (p. 25), [Corollary 7.10](#) (p. 32), [7.12](#) (p. 34), [8.2](#) (p. 35)

Proof

Each summand of f_k is the inverse of the square of w or of a binomial $y^i u y^{-i} = 1 + y^i x y^{-i}$, so $f_k \in D_\pi^{+k} \cap \mathcal{F}_2$, and $f_k \neq 0$ (Lemma 7.9(a)). Suppose $f_k^{-1} \in \Gamma'_{k-2}$. Woodbury gives $F_k^{-1} = \text{diag}(y^{-b})(\Delta_k^{-1} - \Delta_k^{-1}\mathbf{1}f_k^{-1}\mathbf{1}^\top\Delta_k^{-1})$ with Δ_k^{-1} over E_1 , so F_k^{-1} has entries in $\Gamma'_{k-2} = U_{\Sigma'_{k-2}}$ (Lemma 3.6 for $a = k - 2$). Given $Q \in \Sigma'_{k-2}$, choose $\kappa > e(Q)$; then $\nu_{\kappa+1}(Q^{\gamma_\kappa}) \leq 2(k - 2)$ (Lemma 7.7, Lemma 4.4), while $\nu_{\kappa+1}(F_k^{\gamma_\kappa}) = 2(k - 1)$ (Lemma 7.9(d)). Theorem 3.4, with $G = G_k, H = F_k$ and $\mu_Q = \nu_{\kappa+1}$ along γ_κ , gives a contradiction. $\square$

Lemma 7.4. (compression at level k) p. 29

Let $k \geq 3$ and let $0 \neq f \in D_\pi^{+n} + V_1$ with $n \leq k - 1$. Then $f^{-1} \in \Gamma'_{k-2}$.

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Proof

Write $f = d_1^{-1} + \dots + d_{n'}^{-1} + r$ with $n' \leq n$ pure summands and $r \in V_1$ (§2.3; V_1 is closed under sums). If $n' \leq 2$, then $f = d_1^{-1} + d_2^{-1} + r$, $d_1^{-1} + r$ or r , and

$$f^{-1} = d_2(d_1 r d_2 + d_1 + d_2)^{-1} d_1, \quad (1 + d_1 r)^{-1} d_1 \quad \text{or} \quad r^{-1},$$

with $d_1 r d_2 + d_1 + d_2$, $1 + d_1 r \in V_1$, so $f^{-1} \in \mathcal{U}_1 \subseteq \Gamma'_{k-2}$. If $n' \geq 3$, then $k \geq 4$. Put $Y = d_1 r d_2 + d_1 + d_2 \in V_1$ and

$$M = \begin{pmatrix} Y & d_1 \mathbf{1}_{n'-2}^\top \\ -\mathbf{1}_{n'-2} d_2 & \text{diag}(d_3, \dots, d_{n'}) \end{pmatrix} \in M_{n'-1}(V_1).$$

The Schur complement of the diagonal block is $Y + d_1 (\sum_{i \geq 3} d_i^{-1}) d_2 = d_1 f d_2 \neq 0$, so M is full and $f^{-1} = d_2 (M^{-1})_{11} d_1$, with $n' - 1 \leq k - 2$. $\square$

Evidence & scope

- **Program:** `f^-1 = d_2 (M^-1)_11 d_1` for $m = 2..7$, with controls (supplement C20).
`scripts/auditoria_S17/s17_p2_compresion.py`

Lemma 7.5. (the ring reached at $k[\pi]$) p. 29

Let $k \geq 3$, and let D_x be the ring built by the series when the inversion pointer reaches b_{kXX} . Then D_x is generated by $\mathcal{U}_1$ and the inverses of the nonzero elements of $D_\pi^{+n} + V_1$, $n \leq k - 1$; $\mathcal{U}_1 \subseteq D_x \subseteq \Gamma'_{k-2} \cap E_2$; and for $k = 3$, $D_x = \mathcal{U}_1$.

Proof

By Lemma 2.2, every stratum born before $k[\pi]$ is contained in A , in V_1 , or in $D_\pi^{+n} + V_1$ with $n \leq k - 1$. By §2.3, at the Case 3 where the pointer reaches b_{kXX} the current ring contains D_γ^{-1} for every earlier stratum γ , whether γ was a stopping step (its inverses were adjoined there) or not (the pointer passed it at a Case 3 where its inverses already lay in the ring); and between two Case 3 steps Cases 1 and 2 only add sums and products. So D_x is generated by A and the inverses of the nonzero elements of all earlier strata. Conversely, for $n \leq k - 1$ the elements of $D_\pi^{+n} + V_1$ are the sums indexed by the multiset $n[\pi] + [\sigma] < k[\pi]$; since Case 1 selects the least multiset with a new sum (Lemma 2.2(5)), they already lie in the set built before b_{kXX} , so the inverses of its nonzero elements lie in D_x . These inverses lie in E_1 , in $\mathcal{U}_1$, or in Γ'_{k-2} by Lemma 7.4; and D_x contains A^{-1} and V_1^{-1} , since $D_\omega = A$ and $D_\sigma = V_1$ are earlier strata. This gives the first two claims, and the third because $\Gamma'_1 = \mathcal{U}_1$. Finally $\mathcal{U}_1 \subseteq E_2$ and $D_\pi^{+n} + V_1 \subseteq E_1$, so the inverses of its nonzero elements lie in E_2 . $\square$

Evidence & scope

- **Program:** order of the strata: 6480 multisets, $k = 4..8$, with a control (supplement C21).
scripts/auditoria_S17/s17_p4_orden.py

Lemma 7.6. ($D' \subseteq \Gamma_2$) ■ p. 29

Let $e \in D_\pi^{+3}$ be nonzero.

1. If $e = d_1^{-1} + d_2^{-1} + d_3^{-1}$ is a sum of three pure elements, then

$$H_3 = \begin{pmatrix} d_1 + d_2 & d_1 \\ d_1 & d_1 + d_3 \end{pmatrix} \in M_2(A) \text{ is full and}$$

$$e^{-1} = d_1 - (d_1 \ d_1) H_3^{-1} \begin{pmatrix} d_1 \\ d_1 \end{pmatrix} \in \Gamma_2.$$

2. Otherwise $e^{-1} \in \mathcal{U}_1$.

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↳ Used in: [Theorem 7.2](#) (p. 27)

Proof

(1) $H_3 = \Delta + \mathbf{1}d_1\mathbf{1}^\top$ with $\Delta = \text{diag}(d_2, d_3)$, and $e = d_1^{-1} + \mathbf{1}^\top \Delta^{-1} \mathbf{1}$; Woodbury gives the formula, and H_3 is full iff $1 + d_1(d_2^{-1} + d_3^{-1}) = d_1 e \neq 0$. (2) With two pure summands and one in V_1 , $e = p'^{-1} + q'^{-1} + r$ and $e^{-1} = q'(p'rq' + p' + q')^{-1}p'$ with $p'rq' + p' + q' \in V_1$ (Lemma 7.4); with one pure summand, $(d^{-1} + r)^{-1} = (1 + dr)^{-1}d$ with $1 + dr \in V_1$; with none, $e \in V_1$. $\square$

Evidence & scope

- **Program:** three-term inverse, Woodbury, intermediate strata, with controls (supplement C16). scripts/auditoria_S16/s16_prop51_lemas6.py

Lemma 7.7. (the bound 2a) ■ p. 29

For every $j \geq 1$, $\nu_j(B_M^{\gamma_k}) \leq 2a + \sum_s \nu_j(\ell_s^{\gamma_k})$. In particular $\nu_{k+1}(\tilde{B}_M^{\gamma_k}) \leq 2a$ for $k > e(M)$.

↳ Used in: [Theorem 7.2](#) (p. 27), [Theorem 7.3](#) (p. 27), [7.12](#) (p. 34)

Proof

Over $R_j = \mathbb{C}[[t]]/t^j$, a square matrix N of size n satisfies $\nu_j(N) = n - \mu(\text{im } N)$, with μ the minimal number of generators. Let $n = 2a + 2m$ be the size of B_M^γ and pr the projection onto the last $2m$ coordinates. Then $\text{pr}(\text{im } B_M) = \text{im}[-C \ \Delta] \supseteq \text{im } \Delta$. An image of a module needs no more generators, and neither does a submodule, viewing R_j -modules as $\mathbb{C}[[t]]$ -modules with the same μ ($R_j = \mathbb{C}[[t]]/t^j$ is not a principal ideal domain, but $\mathbb{C}[[t]]$ is), so $\mu(\text{im } B_M) \geq \mu(\text{im } \Delta)$ and $\nu_j(B_M) \leq 2a + 2m - \mu(\text{im } \Delta) = 2a + \nu_j(\Delta) = 2a + \sum_s \nu_j(\ell_s)$. For $k > e(M)$, every $\ell_s^{\gamma_k}$ has largest part at most k (Lemma 4.3: units, exceptional binomials with parts $(1, k)$, the others with parts $< k$), so $\nu_{k+1}(\ell_s^{\gamma_k}) = 0$. The pivots of the lift are units on the arc. $\square$

Evidence & scope

- **Program:** the bound 2a on 588 presentations, $k = 4..10$, with controls (supplement C17). `scripts/auditoria_S14/s14_falsa.py`

Lemma 7.8. p. 30

(a) F_ψ is a full atom over P . (b) For every $k \geq 1$, $F_\psi^{\gamma_k} = t^{k+1}((I_3 + J_3) \otimes I_2) \text{diag}(I, I, Y)$ with $Y = y^{\gamma_k}$, $\det Y = 1 - t^{k+1}$; hence all six parts of $F_\psi^{\gamma_k}$ equal $k + 1$ and $\nu_{k+1}(F_\psi^{\gamma_k}) = 6$.

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↳ Used in: Theorem 7.2 (p. 27), Lemma 7.9 (p. 31), 8.1 (p. 34)

Proof

(b) $u^2, v^2, w^2 \mapsto t^{k+1}I$ (Lemma 4.1), central; $\det(I_3 + J_3) = 4$; and $\det(U - I) = 1 - \text{tr } U + \det U = 1 - t^{k+1}$ for $U = v^{\gamma_k}$, a unit of $\mathbb{C}[[t]]$.

- Let $Q = Q_0 + uQ_u + vQ_v + wQ_w$ be the 9×9 pencil with nonzero entries (row, column, coefficient)
 - Q_0 : $(i, i, 1)$ for $i = 4, \dots, 9$;
 - Q_u : $(1, 4, 1), (2, 4, 1), (3, 4, 1), (3, 7, -1), (4, 1, -1), (4, 2, -1), (4, 3, 1), (4, 9, 1), (7, 3, -1), (8, 7, 1)$;
 - Q_v : $(1, 5, 1), (3, 8, -1), (5, 1, -1), (9, 3, 1)$;
 - Q_w : $(2, 6, 1), (6, 2, -1)$.

Its block Q_{22} (rows and columns 4–9) is $I + N$ with $N^2 = 0$, so $Q_{22}^{-1} = I - N$ is polynomial, the Schur complement of Q_{22} is exactly $+F_\psi$, and $Q \sim F_\psi \oplus I_6$ over P . The pencil is epic (the stacked coefficient matrices have rank 9 both ways). At $u = v = w = 2$, $\det Q(2, 2, 2) = 256$, and $K_i = 16 Q(2, 2, 2)^{-1} Q_i$ ($i = u, v, w$) are integer matrices. With $\mathbf{r} = (0, 0, -1, 0, 0, 0, -2, 4, 1)^\top$ and $\ell = (2, 0, -2, 0, 2, 0, 0, 3, 0)$,

$$K_v^2(K_v^2 - 12K_v + 48I) = 2048 \mathbf{r}\ell, \quad (7.1)$$

and, writing $K_\omega = K_{\omega_1} K_{\omega_2} \cdots$ for a word ω (left to right),

$$\det[K_\omega \mathbf{r}]_{\omega \in \mathcal{R}} = 5 \cdot 2^{56}, \quad \det[\ell K_\omega]_{\omega \in \mathcal{O}} = 2^{49}, \quad (7.2)$$

with $\mathcal{R} = (\emptyset, u, uu, vu, wu, uuu, vuu, uvu, vvu)$ and

$\mathcal{O} = (\emptyset, u, w, uu, uv, wu, ww, uuu, uuv)$. By (7.1) the rank-one matrix $\mathbf{r}\ell$ is a polynomial without constant term in K_v , so it lies in the algebra generated by the K_i ; by (7.2) the vectors $K_\omega \mathbf{r}$ and the covectors $\ell K_{\omega'}$ form bases, so the products $K_\omega(\mathbf{r}\ell)K_{\omega'}$ span M_9 . The translated pencil is monic with invertible constant term

$Q(2, 2, 2)$, and its normalized coefficients generate M_9 , so it is an atom by Lemma 2.1; the shift is an automorphism (§2.2(d)) and $Q \sim F_\psi \oplus I_6$ (§2.2(b),(c)), so F_ψ is an atom. $Q(2, 2, 2)$ is invertible, so Q , and with it F_ψ , is full (§2.2(a),(b)). $\square$

Evidence & scope

- **Program:** the 9 x 9 certificate: Schur = +F, det 256, (7.1), (7.2), in exact arithmetic (supplement C18). `scripts/auditoria_S16/s16_cert9.py`

Lemma 7.9. p. 31

(a) f_k is an atom of Λ_y . (b) F_k is full, and an atom over Λ_y . (c) No atomic factor of F_k over P is stably associated with u or with w . (d) In a complete factorization of F_k over P exactly one atomic factor G_k is not stably associated with y , and $\nu_{\kappa+1}(G_k^{\gamma_\kappa}) = 2(k-1)$ for every $\kappa \geq 1$.

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↳ Used in: Theorem 7.3 (p. 27), Corollary 7.10 (p. 32), 7.11 (p. 33), 8.1 (p. 34)

Proof

(a) Grade Λ_y by the exponent of w and put $T = \sum_{i=0}^m y^i u^{-2} y^{-i}$, so that $f' = w f_k w = 1 + w T w$ has components in degrees 0 and 2 only. Let $f' = gh$. The extreme components multiply, and the lowest ones give $g_a h_c = 1$, so g_a is a unit, i.e. a scalar times a group element; replacing g, h by $g g_a^{-1}, g_a h$ we may assume $g_a = h_c = 1$ in degree 0. The widths of g and h add up to 2. If one of them is 0, that factor is 1. Otherwise $g = 1 + g_1, h = 1 + h_1$ with $g_1 + h_1 = 0$ and $g_1 h_1 = w T w$, so $-g_1^2 = w T w$. For a bi-invariant order of the free group the leading word of a product is the product of the leading words, so the leading word of $w T w$, which is one of the distinct reduced words $W_i = w y^i u^{-2} y^{-i} w$, would be a square. But W_i is cyclically reduced, so a square root of it would be cyclically reduced and $W_i = h h$ as words; the first letter of W_i is w and the letter in position $i+3$ is u^{-1} . Hence f' , which is not a unit, is an atom, and so is $f_k = w^{-1} f' w^{-1}$.

1. Over Λ_y , Δ_k is invertible and $H_k = \Delta_k(I + \mathbf{a}\mathbf{b}^\top)$ with $\mathbf{a} = \Delta_k^{-1}\mathbf{1}$, $\mathbf{b}^\top = u^2\mathbf{1}^\top$; $I + \mathbf{a}\mathbf{b}^\top$ is stably associated with $1 + \mathbf{b}^\top\mathbf{a} = u^2 f_k$. Since $\text{diag}(1, y, \dots, y^m)$ and u^2 are units of Λ_y , F_k is stably associated with $f_k \neq 0$, hence full, and stable association preserves atoms [Coh06, Cor. 0.5.5, Thm 3.2.3 and p. 195].
2. If an atomic factor were stably associated with u , then at any matrix point X where $u(X)$ is singular, that factor, and hence $F_k(X)$, would be singular (evaluate the stable association and the factorization). Take $n = m+1$, $u \mapsto U = \text{diag}(0, 1, \dots, 1)$, $w \mapsto I$ and $y \mapsto Y$ the cyclic permutation matrix, so that the evaluations of u^2 and of $y^i u^2 y^{-i}$ are $I - e_i e_i^\top$ ($i = 0, \dots, m$). A vector $\mathbf{v}$ in the kernel of H_k gives, with $s = (I - e_0 e_0^\top) \sum \mathbf{v}$, $\mathbf{v}_w = -s$, $\mathbf{v}_i = -s + c_i e_i$ with $s \perp e_0, \dots, e_m$ and $(m+2)s = \sum c_i e_i$; since $e_0, \dots, e_m$ is a basis, $s = 0$, so $\sum c_i e_i = 0$, $c_i = 0$ and $\mathbf{v} = 0$ (no division is needed). So F_k is invertible at a

point where u is singular. For w a scalar point suffices: at $u = y = 1, w = 0$, $F_k = H_k = \text{diag}(0, 1, \dots, 1) + J_{k-1}$, and subtracting its first row from the others leaves the rows $e_1^\top, \dots, e_m^\top$, so $\det F_k = 1$.

3. Let $F_k = G_1 \cdots G_s$ be a complete factorization over P . Over a weakly finite ring a product of square matrices is invertible only if every factor is: if XY is invertible, X has the right inverse $Y(XY)^{-1}$, so X is invertible, and then so is Y . By (b), F_k is not invertible over Λ_y , so some G_{i_0} is not. If another G_{i_1} were not invertible over Λ_y , say $i_1 > i_0$, then $F_k = XY$ with $X = G_1 \cdots G_{i_0}$ and $Y = G_{i_0+1} \cdots G_s$ full and both non-invertible over Λ_y , against (b). So each G_i , $i \neq i_0$, is invertible over $\Lambda_y = U_{\{u,y,w\}}$. By Lemma 3.1 with $\Sigma = \{u, y, w\}$ (atoms of P , being of degree one), it is stably associated with u, y or w , and by (c) with y . On the other hand, G_{i_0} is not stably associated with y , since it would then be invertible over Λ_y (converse part of Lemma 3.1). On the arc, $\det y^{\gamma_\kappa} = 1 - t^{\kappa+1}$ (as in Lemma 7.8(b)), so y^{γ_κ} , and by stable association every $G_i^{\gamma_\kappa}$ with $i \neq i_0$, is invertible over $\mathbb{C}[[t]]$. Hence $\nu_{\kappa+1}(G_{i_0}^{\gamma_\kappa}) = \nu_{\kappa+1}(F_k^{\gamma_\kappa})$. Each $y^i u^2 y^{-i}$ evaluates to $t^{\kappa+1} I$ (Lemma 4.1: $u^2 \mapsto t^{\kappa+1} I$ is central), so $F_k^{\gamma_\kappa} = t^{\kappa+1} ((I_{k-1} + J_{k-1}) \otimes I_2) \text{diag}(Y^b)$ with $\det(I + J) = k \neq 0$, and all $2(k-1)$ parts equal $\kappa + 1$. $\square$

Evidence & scope

- **Program:** words W_i are not squares, root lemma, F_k stably associated with f_k (supplement C22). `scripts/auditoria_S18/s18_comprobaciones.py`
- **Program:** F_k invertible where u (or w) is singular, $k = 3..10$ (supplement C23). `scripts/kXX_todo_k/paso_C_puntos.py`

Corollary 7.10. p. 32

(a) For every $d \geq 2$ and $r \geq 1$, $A_{d-1} \cap \mathcal{U}_r \subsetneq A_d \cap \mathcal{U}_r$; a witness is $e_d = q_d^{-1}$ with $q_d = w^{-1} + \sum_{i=0}^{d-1} y^i u^{-1} y^{-i}$, and $e_d \in (A_d \cap \mathcal{U}_1) \setminus A_{d-1}$. The range $r \geq 1$ is optimal, since $A_d \cap \mathcal{U}_0 = E_1$ for every d . (b) The bound of Lemma 7.4 is optimal: for $n \geq 3$ there is a sum f of n pure elements with $f^{-1} \notin \Gamma'_{n-2}$. (c) Let $\mathcal{D}_k$ be the ring D_x of Lemma 7.5, reached when the inversion pointer reaches b_{kXX} (it does, §7.7). Then $\mathcal{U}_1 = \mathcal{D}_3 \subsetneq \mathcal{D}_4 \subsetneq \mathcal{D}_5 \subsetneq \cdots$ and $\bigcup_k \mathcal{D}_k = \mathcal{U}_2$.

✓ Checked when this page was built: `BinomialSizeFamily.lean`, `PhiCompression.lean` compiles with Lean 4.35.0, no errors, no sorry. Axioms — `rankOne_mul_inv`, `rankOneInv_mul`, `inv_q_eq`, `conj_add_mul_pow`, `det_scalar_point`, `idemBlock_isUnit`, `inv_mul_conj`: `propext`, `Classical.choice`, `Quot.sound`.
 Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean BinomialSizeFamily.lean PhiCompression.lean` (SHA-256 of the compiled file: 01db4f029c94cedf...; a source page on this site may show it with comments edited, under its own hash).

NEW **Classical and new:** the refinement of the size chain inside $\mathcal{U}_1$ has not been studied, to the paper's knowledge (§8.8).

↳ Used in: 7.11 (p. 33), 8.1 (p. 34), 8.2 (p. 35)

Proof

(a) Let $d \geq 2$, $\Delta = \text{diag}(w, ywy^{-1}, \dots, y^{d-1}wy^{-(d-1)})$, $H = \Delta + uJ_d$ and $L = H \text{diag}(1, y, \dots, y^{d-1}) \in M_d(P)$, whose entries are $uy^b + \delta y^b u$ (w in place of $y^0 u$ for the first index). This is the construction of §7.6 with u, w in place of u^2, w^2 , and Lemma 7.9(a)–(c) and the first part of (d) hold for q_d and L with the same proofs. In (a), $wq_d = 1 + wT$ with $T = \sum_{i=0}^{d-1} y^i u^{-1} y^{-i}$ has components in degrees 0 and 1 only, so one of two factors has width 0, hence is a unit, and the argument with squares is not needed. In (b), $1 + u\mathbf{1}^\top \Delta^{-1} \mathbf{1} = uq_d$. In (c), with $n = d$, the matrix U is idempotent, so u and u^2 have the same evaluation, and the two points give the same matrices as there. Each summand of q_d is an inverse binomial ($w^{-1} = (1 + z)^{-1}$, $y^i u^{-1} y^{-i} = (1 + y^i x y^{-i})^{-1}$), so $q_d \in V_1$, $q_d \neq 0$ by (a), and $e_d \in \mathcal{U}_1 \subseteq \mathcal{U}_r$. Woodbury gives $e_d = u - u\mathbf{1}^\top H^{-1} \mathbf{1} u \in A_d$, since H is a full $d \times d$ matrix over A . If $e_d \in A_{d-1}$, then $H^{-1} = \Delta^{-1} - \Delta^{-1} \mathbf{1} e_d \mathbf{1}^\top \Delta^{-1}$, with Δ^{-1} over $E_1 = A_1$, and $L^{-1} = \text{diag}(y^{-b}) H^{-1}$ have entries in A_{d-1} . Every entry of L vanishes at χ , and the atomic factors of L other than the exceptional one, G , are stably associated with y , which is invertible at χ ; so $\text{corank } G^\chi = d$, while the matrices of $\Sigma_A^{(d-1)}$ have corank at most $d - 1$ at χ (Remark 7.11(i)). Theorem 3.4, with $G, H = L$ and μ the corank at χ , gives a contradiction. Finally $A_d \cap \mathcal{U}_0 = A_d \cap E_1 = E_1$, since $E_1 = A_1 \subseteq A_d$. (b) s for $n = 3$ (Theorem 5.1, $\Gamma'_1 = \mathcal{U}_1$) and f_n for $n \geq 4$ (Theorem 7.3). (c) Expanding the coefficients into monomials, a term of an element of $\mathcal{F}_2$ with two inverse binomials is $m_0 \ell_1^{-1} m_1 \ell_2^{-1} m_2 = (\ell_1^{-1} m_0 m_1)(\ell_2^{-1} m_2)$ with $\ell'_1 = m_0 \ell_1 m_0^{-1}$, a product of two elements of $D_{\omega+1}$ (Lemma 2.2(2)); terms with fewer inverses lie in $V_1 \subseteq D_\pi$. With $D_\pi \subseteq \mathcal{F}_2$ this gives $\mathcal{F}_2 = \bigcup_n D_\pi^{+n}$. By Lemma 7.5, $\mathcal{D}_k \subseteq \mathcal{D}_{k+1}$ and $\mathcal{D}_k \subseteq \mathcal{U}_2$, and $\bigcup_k \mathcal{D}_k$ contains E_1 and the inverse of every nonzero element of $\mathcal{F}_2$, so it is $\mathcal{U}_2$. Finally $s \in D_\pi^{+3}$ gives $s^{-1} \in \mathcal{D}_4 \setminus \mathcal{D}_3$ (Theorem 5.1), and for $k \geq 4$, $f_k \in D_\pi^{+k}$ gives $f_k^{-1} \in \mathcal{D}_{k+1} \setminus \mathcal{D}_k$, since $\mathcal{D}_k \subseteq \Gamma'_{k-2}$ (Lemma 7.5 and Theorem 7.3). $\square$

Evidence & scope

- **Program:** Woodbury for e_d and H^{-1} , polynomial entries of L , $\det$ of L at the idempotent and scalar points ($d = 2..10$), L vanishes at χ , with controls. `scripts/V168/verifica_qd.py`
- **Program:** the two Woodbury identities in exact rational arithmetic, $d = 2..6$, with a control. `scripts/V168/verifica_qd_exacta.py`

Remark 7.11. (matrix size and number of binomial inverses) p. 33

Let $\text{sz}(e)$ be the least b with $e \in A_b$.

1. For the chain A_b alone the character χ suffices. By §6.2, $A_b = U_{\Sigma_A^{(b)}}$ with $\Sigma_A^{(b)} = \Sigma_0 \cup \{\tilde{G} : G \text{ full over } A \text{ of size } a \leq b\}$, $\tilde{G}$ the polynomial lift with pivots x_j , and $\text{corank } \tilde{G}^\chi = \text{corank } G^\chi \leq a$ since $x_j^\chi = -1$; also $\text{corank } Q_s^\chi \leq 1$. Every entry of F_k vanishes at χ ($u^\chi = w^\chi = 0$), and the atomic factors of F_k other than G_k are stably associated with y , which is invertible at χ (Lemma 7.9(d)); so $\text{corank } G_k^\chi = k - 1$. For $k \geq 4$, if $f_k^{-1} \in A_{k-2}$, then $F_k^{-1} \in M(A_{k-2})$ by Woodbury (as in §7.6, with Δ_k^{-1} over $E_1 = A_1$), and Theorem 3.4 with μ the corank at χ gives a contradiction. Likewise $H_0^\chi = 0$, with H_0 an atom (Lemma 5.7(1)), gives $s^{-1} \notin A_1$, and $H_{3,2r}^\chi = 0$, with $H_{3,2r}$ an atom, gives

$g_r^{-1} \notin A_1$. So the exclusions $f_k^{-1} \notin A_{k-2}$, $s^{-1} \notin A_1$ and $g_r^{-1} \notin A_1$ do not need the arc either; the proof of Corollary 7.10(a) uses this argument.

2. Let $\text{bin}(e)$ be the least r with $e \in \mathcal{U}_r$, for $e \in \mathcal{U}_\infty$ (and $\text{bin}(e) = \infty$ otherwise). By (i), §5.1 and Woodbury: $\text{sz}(s^{-1}) = \text{sz}(g_r^{-1}) = 2$, $\text{sz}(f_k^{-1}) = k - 1$ ($k \geq 4$), $\text{sz}(e_d) = d$ with $\text{bin}(e_d) = 1$ ($d \geq 2$, Corollary 7.10(a)), $\text{sz}(\rho_d) = d$ ($d \geq 2$) and $\text{sz}(\varphi_r^{-1}) = 5$ for every $r \geq 3$ (Remark 8.2). The two complexities are independent: $\varphi_r^{-1} \in \mathcal{U}_r \setminus \mathcal{U}_{r-1}$ has size 5 for every $r \geq 3$, while $e_d \in \mathcal{U}_1 \setminus \mathcal{U}_0$ has size d and $f_k^{-1} \in \mathcal{U}_2 \setminus \mathcal{U}_1$ has size $k - 1$: neither of sz , bin is bounded in terms of the other, and the size is unbounded already for $\text{bin} = 1$.
3. For $n \geq 2$ and $d_1, \dots, d_n \in A \setminus 0$ with $f = \sum_i d_i^{-1} \neq 0$, $f^{-1} \in A_{n-1}$: the compression of Lemma 7.4 with $r = 0$ is a matrix over A . The size $n - 1$ is attained for every $n \geq 3$ already by a sum of n inverse binomials, q_{n-1} (Corollary 7.10(a)), and also by s for $n = 3$ and by f_n for $n \geq 4$, although f itself lies in $A_1 = E_1$. For $n = 2$, $f^{-1} = d_2(d_1 + d_2)^{-1}d_1$ is the parallel sum of Anderson and Duffin [AD69]; we did not find the compression for general n , or its optimality, stated. When the d_i are non-units, pairwise totally coprime and contained in a proper ideal $\mathfrak{c}$ with $A/\mathfrak{c}$ weakly finite, f has depth n [Coh06, Prop. 7.7.8] (without the ideal this fails: $(x + 1)^{-1} - (x + 2)^{-1} = ((x + 1)(x + 2))^{-1}$), a statement about f rather than about the size of the matrices needed for f^{-1} .

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 Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, lake env lean
 PhiCompression.lean (SHA-256 of the compiled file: f5da776cad749a60...; a source page on this site may show it with comments edited, under its own hash).

NEW Classical and new: for $n = 2$ the compression is the parallel sum of Anderson and Duffin [AD69]; the paper did not find the compression for general n , or its optimality, stated.

↳ Used in: 8.2 (p. 35)

Remark 7.12. (dependencies of Theorem 7.1) ■ p. 33

The theorem rests on: the reading of [Jak19, Defs. 1.1.4–1.1.5] in §2.3 and Lemma 2.2, including the order of the index monoid; Lemmas 7.4 and 7.5; Theorems 5.1 and 7.3 (Theorem 7.2 only for the second witness ψ), and through them the criterion (Theorem 3.4), the localization lemma (Lemma 3.1), the binomials on the arc (Lemma 4.3), the chain bound (Lemma 4.4), the bound $2a$ (Lemma 7.7) and the atom criteria (Lemma 2.1, Lemma 7.9). Like the series, it is relative to $\mathcal{M} = \mathbb{C}^\times F_3$ [Jak19, Rem. 1.1.6]. It does not decide whether the intermediate strata $(k - 1)[\pi] + x$ with $x \neq 0$, $k \geq 4$, are stopping steps; for $k = 3$ they are not (Remark 8.3).

Figure 5 places these stopping steps in the order of the strata and shows the margin behind each of them.

↳ Not used by another result of the paper.

8 Remarks and open problems

Remark 8.1. (characteristic p) ■ p. 35

Over a field of characteristic p , $(1+x)^p = 1+x^p$, so $u^{-2} = u^{p-2}(1+x^p)^{-1} \in V_1$, since $1+x^p$ is a binomial. Hence $s \in V_1$ and $s^{-1} \in E_1[V_1^{-1}]$: the witness of Theorem 5.1 fails in characteristic p . Whether $\mathcal{U}_1 \subsetneq \mathcal{U}_2$ holds in characteristic p is not studied here. For this witness the first step that breaks is Lemma 4.3(2): the binomial $1+x^p = (1+x)^p$ is exceptional but $1+X^p$ has a multiple root, so its parts on the arc exceed k (indeed $U^p = t^{(k+1)(p-1)/2}U$ for odd p) and the chain bound of Lemma 4.4 is lost. Characteristic 0, or the field $\mathbb{C}$, is used at least at the following places:

- Lemma 4.2(2): $\det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3$ (the consequence $\nu = 4$ survives by Lemma 4.2(1));
- Lemma 4.3(2) (simple roots of $1 - \beta X^n$) and Lemma 4.3(3) (the factors $\prod n_i$ and $\prod (j_{i+1} - j_i)$ of the trace law; in characteristic 2 the exact arc also identifies u and w);
- Lemma 5.6: the differences $i_j - i_{j+1} = \pm 1, \pm 2$;
- Lemma 5.7(1): $g_{ji} = \pm 1/n$ and $\det K = n$; Lemma 5.7(2): $\varphi_r(1) = 5$;
- Lemma 4.5: real quaternions at $t = 2$ and $\sqrt{-1}$;
- Lemma 6.2(c): the distinct eigenvalues $3, \dots, d+2$ and the Neumann series;
- Lemma 7.8: $\det Q(2, 2, 2) = 256$, the factor 16, (7.1), (7.2) and $\det(I_3 + J_3) = 4$;
- Lemma 7.9(d): $\det(I_{k-1} + J_{k-1}) = k$, for all parts to equal $\kappa + 1$ (if $p \mid k$, two parts are infinite and the count $\nu_{\kappa+1} = 2(k-1)$ survives, since ν_j counts infinite parts; what is lost in characteristic p is the bound on the allowed denominators, Lemmas 4.3–4.4).

By contrast, Lemma 3.1 holds over any field by its algebraic proof (§3.1), and the proofs of Lemma 7.9(a)–(c) and of the first part of (d) (exactly one atomic factor of F_k not stably associated with y) use no division by an integer and no property of $\mathbb{C}$, the free group algebra Λ_y having trivial units over any field. Computations modulo $p = 2, 3, 5, 7, 11$ for $k = 4, 5, 6$ agree; they are not used, and the tool that produced them miscounts the one-dimensional factors. The same holds for the proof of Corollary 7.10(a), whose other inputs are Theorem 3.4 and the presentation of A_b in §6.2; we have not restated §§2–3 over a general field.

‡ Not used by another result of the paper.

Remark 8.2. (two parameters) p. 35

The hierarchy $\mathcal{U}_r$ (number of inverse binomials) and the rings $\Gamma_b = \mathcal{U}_1[\text{entries of inverses of full } a \times a \text{ matrices over } A, a \leq b]$ (matrix size) are not nested: $\psi^{-1} \in \mathcal{U}_2 \setminus \Gamma_2$ and $\rho_2 \in \Gamma_2 \setminus \mathcal{U}_\infty$, and more generally, for all $r, b \geq 2$, $\mathcal{U}_r$ is incomparable with Γ_b , Γ'_b and $A_b \cap E_2$ ($f_{b+2}^{-1} \in \mathcal{U}_2 \setminus \Gamma'_b$ by Theorem 7.3, and $\rho_2 \in (A_2 \cap E_2) \setminus \mathcal{U}_\infty$ by Theorem 6.1, with $A_b \subseteq \Gamma_b \subseteq \Gamma'_b$); while $s^{-1} \in (\mathcal{U}_2 \cap \Gamma_2) \setminus \mathcal{U}_1$ by the identity $s^{-1} = v^2 - v^2(H_0^{-1})_{11}v^2$ of §5.1. In the plane of the rings $\mathcal{U}_r \cap \Gamma'_b$ (§2.4) the rows $r \geq 2$ are strict (with Γ_b as well, by the same witnesses). Note $\Gamma'_1 = \mathcal{U}_1$. For every $r \geq 2$, $s^{-1} \in (\mathcal{U}_r \cap \Gamma'_2) \setminus \Gamma'_1$ (Theorem 5.1). For $b \geq 2$ and $r \geq 2$, $f_{b+2}^{-1} \in (\mathcal{U}_r \cap \Gamma'_{b+1}) \setminus \Gamma'_b$ by Theorem 7.3, because $f_{b+2} \in \mathcal{F}_2$ and $f_{b+2}^{-1} = u^2 - u^2 \mathbf{1}^\top H_{b+2}^{-1} \mathbf{1} u^2$ with H_{b+2} a full $(b+1) \times (b+1)$ matrix over A . So

$\mathcal{U}_r \cap \Gamma'_1 \subsetneq \mathcal{U}_r \cap \Gamma'_2 \subsetneq \dots$, with union $\mathcal{U}_r$, since $\bigcup_b \Gamma'_b \supseteq \bigcup_b A_b = D$ [Mal84a, Thm 3]. The row $r = 1$ is constant, since $\Gamma'_1 = \mathcal{U}_1$; the chain $A_d \cap \mathcal{U}_1$, which is strict (Corollary 7.10(a)), is not visible in this plane, since every Γ'_b contains $\mathcal{U}_1$. In the columns (b fixed) the inclusions are as follows. Since $\mathcal{U}_1 \subseteq \Gamma'_b$, $\mathcal{U}_r \cap \Gamma'_1 = \mathcal{U}_1$ for every $r \geq 1$, and $\mathcal{U}_0 \cap \Gamma'_b = E_1 \subsetneq \mathcal{U}_1 = \mathcal{U}_1 \cap \Gamma'_b$ for every b (Theorem 5.2, $r = 1$). For $b \geq 2$, s^{-1} separates $r = 1$ from $r = 2$, and $g_r^{-1} \in (\mathcal{U}_{2r} \cap \Gamma'_2) \setminus \mathcal{U}_r$ for every $r \geq 1$ by Corollary 5.3 and the identity of §5.1 for $H_{3,2r}$; so no column $b \geq 2$ is eventually constant. For $r \geq 3$ let $d_\tau = \ell_{\tau,r} \cdots \ell_{\tau,1} \in P$, so that $\varphi_r = \sum_\tau \alpha_\tau d_\tau^{-1} \omega_\tau$, and let $T = \{0, 1, 3, 4\}$, $D_T = \text{diag}(d_\tau)_{\tau \in T}$, ω_T the column of the ω_τ and α_T the row of the α_τ , $\tau \in T$. Since $\omega_2 = 1$ and $\alpha_2 = w$ (§5.3), the matrix

$$M_r = \begin{pmatrix} D_T & -\omega_T d_2 \\ \alpha_T & w \end{pmatrix} \in M_5(P)$$

has Schur complement $w + \sum_{\tau \in T} \alpha_\tau d_\tau^{-1} \omega_\tau d_2 = \varphi_r d_2$ with respect to D_T , so it is full and $\varphi_r^{-1} = d_2 (M_r^{-1})_{55} \in A_5$. Pivoting on w instead leaves $S_r = D_T + \omega_T d_2 w^{-1} \alpha_T \in M_4(V_1)$, which is full, and $\varphi_r^{-1} = d_2 w^{-1} - d_2 w^{-1} \alpha_T S_r^{-1} \omega_T d_2 w^{-1} \in \Gamma'_4$. By Theorem 5.2, $\varphi_r^{-1} \in (\mathcal{U}_r \cap \Gamma'_4) \setminus \mathcal{U}_{r-1}$, so every column $b \geq 4$ is strict at every step (and every column $b \geq 5$ with Γ_b in place of Γ'_b). Whether $\mathcal{U}_{r-1} \cap \Gamma'_b \subsetneq \mathcal{U}_r \cap \Gamma'_b$ for $r \geq 3$ and $b \in \{2, 3\}$ is open.

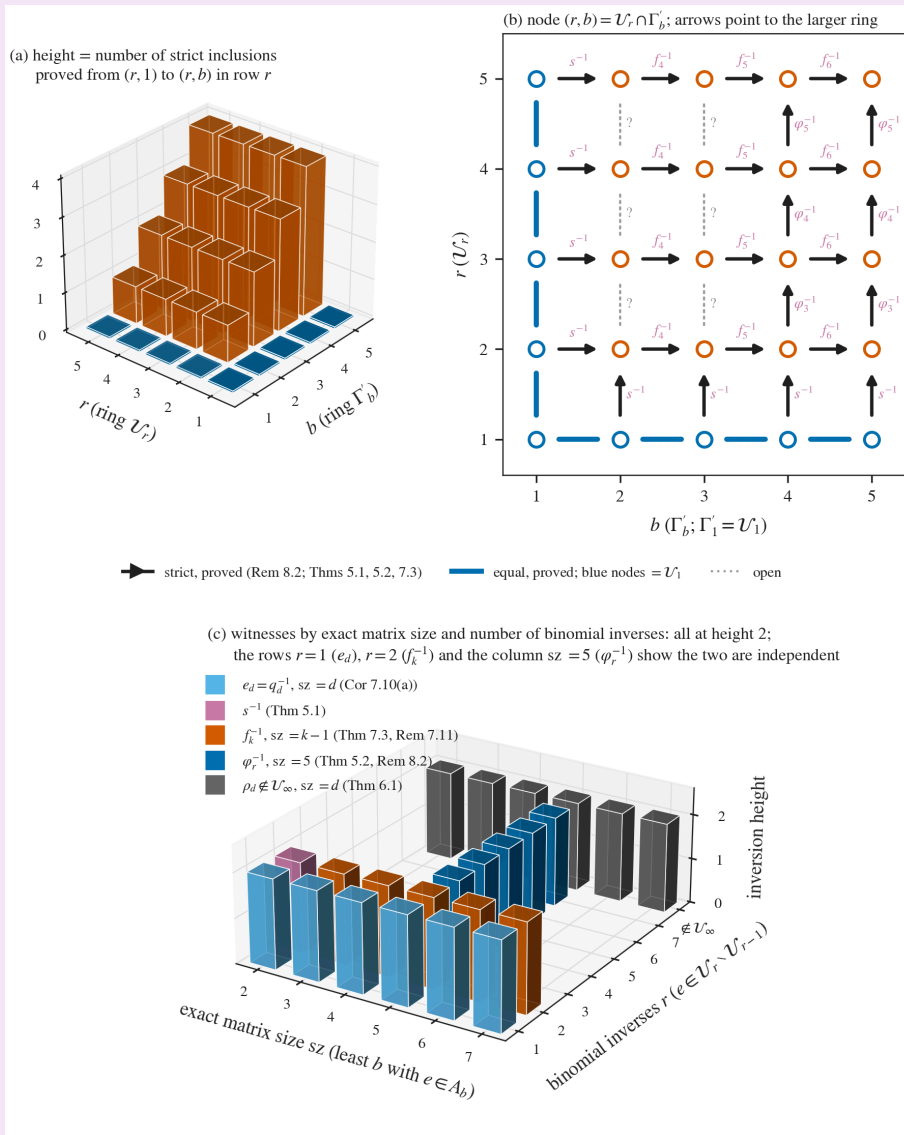
The same witnesses separate sizes inside E_2 . Up to permutations of rows and columns, M_r is the Schur complement in Q_r of the square block on the rows and columns of the $5(r-1)$ entries -1 of the chains and of the entry $\omega_2 = 1$; that block is triangular with diagonal ± 1 , hence invertible over P , so M_r is stably associated with Q_r and is an atom (Lemma 5.7(2)). Every entry of M_r is a product involving a letter, so $M_r^\chi = 0$ and $\text{corank } M_r^\chi = 5$. Now $\Gamma_4 = U_\Sigma$ with $\Sigma = \Sigma_1 \cup \Sigma_A^{(4)}$ (Lemma 3.5, §6.2), and if $\varphi_r^{-1} \in \Gamma_4$ then $M_r^{-1} \in M_5(\Gamma_4)$ (block inverse with respect to D_T , whose inverse is over E_1). Apply Theorem 3.4 with target M_r and μ_Q depending on Q : for Q_s , x_j and the lifts in $\Sigma_A^{(4)}$, the corank at χ , at most 4 (the pivots x_j evaluate to -1) against 5; for the borders of $\mathcal{U}_1$, $\nu_{(r-1)k+1}$ on γ_k with $k > \max(e(Q), r)$, at most 2 (Lemma 4.4) against at least 3 (Lemma 5.6; stable association preserves Smith profiles). Hence $\varphi_r^{-1} \in (A_5 \cap \Gamma'_4) \setminus \Gamma_4$ for $r \geq 3$, and in particular $\varphi_r^{-1} \notin A_4$. So $\Gamma_4 \subsetneq \Gamma'_4$ by elements of inversion height two. Figure 6 shows the plane for $r, b \leq 5$ and places the witnesses by exact matrix size and number of binomial inverses (Remark 7.11).

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Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, lake env lean ArrowMatrix.lean PhiCompression.lean (SHA-256 of the compiled file: f5da776cad749a60...; a source page on this site may show it with comments edited, under its own hash).

↳ Not used by another result of the paper.

Figure



The two-parameter plane of Remark 8.2; node (r, b) is the ring $\mathcal{U}_r \cap \Gamma'_b$. (a) Height: the number of strict inclusions proved along row r , starting from $b = 1$. (b) Status of each step for $r, b \leq 5$: rows $r \geq 2$ are strict (witnesses s^{-1} and f_{b+2}^{-1}) and row $r = 1$ is constant; column $b = 1$ is constant and the step $r = 1 \rightarrow 2$ is strict for $b \geq 2$ (witness s^{-1}); the column steps $r-1 \rightarrow r$ with $r \geq 3$ are open for $b \in \{2, 3\}$ (individually; in each such column one of the steps $2 \rightarrow 3, 3 \rightarrow 4$ is strict, since $g_2^{-1} \in (\mathcal{U}_4 \cap \Gamma_2) \setminus \mathcal{U}_2$) and strict for $b \geq 4$ (witness φ_r^{-1}). The plane starts at $r = 1$; the step from $\mathcal{U}_0 = E_1$ to $\mathcal{U}_1$ is strict in every column. Blue nodes are rings equal to $\mathcal{U}_1$. (c) Each witness placed by its exact matrix size sz (Remark 7.11) and by the least r with the witness in $\mathcal{U}_r$; all have inversion height 2. The rows $r = 1$ (e_d , size d , Corollary 7.10(a)) and $r = 2$ (f_k^{-1} , size $k-1$) and the column of size 5 (φ_r^{-1} , every $r \geq 3$) show that the two complexities are independent; ρ_d lies outside $\mathcal{U}_\infty$. The witnesses g_m^{-1} are not drawn: their least level is only known to lie between $m+1$ and $2m$.

Evidence & scope

- Program:** the 6 x 6 arrow matrix with Schur complement -phi_r, $r = 3..8$, with controls (supplement C24). `scripts/auditoria_S26/s26_comprobaciones.py`

Remark 8.3. (later strata) p. 36

Theorem 7.1 settles the steps $b_{kXX} = k[\pi]$. The band between $2[\pi]$ and $3[\pi]$ contains no stopping step: an element of a stratum $2[\pi] + x$ is $d_1^{-1} + d_2^{-1} + r$ with $r \in V_1$ (or has fewer pure summands), and if a Case 3 selected that stratum, the current ring would already contain V_1^{-1} (as $D_\sigma = V_1$ is an earlier stratum) and hence $(d_1^{-1} + d_2^{-1} + r)^{-1} = d_2(d_1 r d_2 + d_1 + d_2)^{-1} d_1$ (Lemma 7.4). With one pure summand, $(d_1^{-1} + r)^{-1} = (1 + d_1 r)^{-1} d_1$ with $1 + d_1 r \in V_1$, so π and the strata $[\pi] + x$ are not stopping steps either. The strata $(k - 1)[\pi] + x$ with $x \neq 0$ and $k \geq 4$ are not decided, nor any step involving ordinals born after π (products of three or more binomial inverses, or the inverses adjoined at the stopping steps themselves); nor do we decide which strata before π (sums of $n \geq 3$ monomials, sums of elements of $D_{\omega+1}$) are stopping steps. On the free algebra P the harmonic matrices need not be atoms. For a related family, not used here (axes u^2, v^2, w^2 and conjugates by single letters with pairwise distinct signed outer letters), we certified atomicity for $k \leq 9$ by computation. The $k = 10$ extensions we tested use a conjugator of length two that repeats an outer letter; exactly over $\mathbb{Q}$, their linear pencils have a one-dimensional composition factor of letter type, and so do the targets f_k of §7.6 already for $k = 4, 5$. Theorem 3.4, applied to one atomic factor, makes such factors harmless, because letters evaluate to units on the arc.

↳ Not used by another result of the paper.

Remark 8.4. (strictness of the size chain in general) ■ p. 36

Theorem 6.1 is specific to $A = \mathbb{C}[F_3]$ and uses the quaternion curve. In general, let R be a fully atomic semifir (for instance a fir) with universal field D , and $R_b \subseteq D$ the subring generated by R and the entries of the inverses of full matrices of order $\leq b$. Then $R_{b-1} \subsetneq R_b$ exactly when some $b \times b$ atom is not stably associated with a matrix of smaller size. The direction “ $\Leftarrow$ ” is Lemma 3.1, since the atomic factors of a full $n \times n$ matrix are square of order $\leq n$ [Coh06, Thms 3.2.3, 3.2.7]. Conversely, a full $b \times b$ matrix is a product of $b \times b$ matrices each stably associated with an atom of order $\leq b$ (R is fully atomic); if every $b \times b$ atom is stably associated with a matrix of smaller size, each factor has the entries of its inverse in R_{b-1} (stable association does not change this: converse part of Lemma 3.1), so $R_b = R_{b-1}$. A matrix with all its entries in a proper ideal $\mathfrak{a}$ with $R/\mathfrak{a}$ weakly finite is not stably associated with a matrix of smaller size [Coh85a, Lemma 3.4.11(i)], [Coh06, Lemma 3.3.11(i)]. This gives a second proof that the chain is strict at every b in $\mathbb{C}\langle x, y, z \rangle$; the first follows from [HS15, Thm 4.5] (§1). There $P_b = xI + y \operatorname{diag}(1, \dots, b) + z C_b$ is an atom: after $x \mapsto x + 1$ it is a monic pencil with invertible constant term whose coefficients generate $M_b(\mathbb{C})$ ($\operatorname{diag}(1, \dots, b)$ has distinct eigenvalues and C_b permutes its eigenlines cyclically, as in the proof of Lemma 6.2(c)), so Lemma 2.1 applies. All its entries lie in (x, y, z) . In an Ore domain, by contrast, $R_1 = D$. When $R/\mathfrak{a}$ is a field this criterion is the analogue over R of Theorem 3.4 with μ the corank at $R \rightarrow R/\mathfrak{a}$. It does not locate witnesses inside E_2 ; Theorem 6.1 does. The characterization by atoms is a short corollary of published lemmas.

↳ Not used by another result of the paper.

Remark 8.5. (independent adjunctions in $[\mathcal{U}_\infty, B_2]$) ■ p. 37

For $\lambda \in \mathcal{T} = \mathbb{C} \setminus \{-1, 0, 1, 2\}$ let $\rho^{(\lambda)} = ((F^{(\lambda)})^{-1})_{11} = (q_\lambda - c q_{\lambda+1}^{-1} c)^{-1}$, and for $J \subseteq \mathcal{T}$ let $\mathcal{U}_J$ be the subring generated by $\mathcal{U}_\infty$ and $\{\rho^{(\lambda)} : \lambda \in J\}$. Then $\mathcal{U}_\infty \subseteq \mathcal{U}_J \subseteq B_2$, and $J \mapsto \mathcal{U}_J$ is an order embedding $\mathcal{P}(\mathcal{T}) \hookrightarrow [\mathcal{U}_\infty, B_2]$: $\mathcal{U}_J \subseteq \mathcal{U}_K$ if and only if $J \subseteq K$. It preserves finite joins. Restricted to $J \subseteq \{3, 4, 5, \dots\}$ it gives $2^{\aleph_0}$ distinct rings generated over $\mathcal{U}_\infty$ by elements with integer coefficients; on all of $\mathcal{T}$ it gives $2^{\aleph_0}$. $\mathcal{T}$ is the set where the atom criterion of Lemma 6.2(c) applies; it is sufficient, not maximal. Intersections $\mathcal{U}_J \cap \mathcal{U}_K$ are not determined.

Proof. $\rho^{(\lambda)} \in B_2$ because $F^{(\lambda)}$ is a full 2×2 matrix over A , whose inverse has entries in E_2 (pivot argument of §7.5) and in W_2 . $J \subseteq K$ gives $\mathcal{U}_J \subseteq \mathcal{U}_K$, and $\mathcal{U}_{J \cup K}$ is generated by $\mathcal{U}_J \cup \mathcal{U}_K$, by definition; so it suffices to show $\rho^{(\lambda')} \in \mathcal{U}_J \Rightarrow \lambda' \in J$. Since $q_{\lambda+1}^{-1} \in E_1$, the entries of $(F^{(\lambda)})^{-1}$ lie in the ring generated by E_1 and $\rho^{(\lambda)}$ (block inverse, as in (6.2)); hence $\mathcal{U}_J = U_\Sigma$ with $\Sigma = \Sigma^{(1)} \cup \{F^{(\lambda)} : \lambda \in J\}$, and $\rho^{(\lambda')} \in \mathcal{U}_J$ gives $(F^{(\lambda')})^{-1} \in M_2(U_\Sigma)$. By Lemma 3.1 the atom $F^{(\lambda')}$ (Lemma 6.2(c)) is stably associated with an atomic factor of some $Q \in \Sigma$. It is not a factor of a member of $\Sigma^{(1)}$: at Θ every such Q has corank at most 2 (Lemma 6.2(a)), hence so do its atomic factors (Definition 3.3, Example (b)), while $F^{(\lambda')}$ has corank 4 (Lemma 6.2(b)). So $F^{(\lambda')}$ is stably associated with $F^{(\lambda)}$ for some $\lambda \in J$ (an atom, so its only atomic factor). Stable association over P survives abelianization, and the units of the commutative polynomial ring are the scalars, so $\det F^{(\lambda')} = a \det F^{(\lambda)}$ for some $a \in \mathbb{C}^\times$. At $(\xi, \eta, \zeta) = (\xi, 0, 1)$, $c = 0$ and $\det F^{(\lambda)} = (\lambda - \xi^2)(\lambda + 1 - \xi^2)$, so $\lambda' = \lambda \in J$. $\square$

↳ Not used by another result of the paper.

Evidence & scope

- **Program:** the excluded values of $F^{(\lambda)}$ (supplement C13).
scripts/auditoria_S11/s11_a2_a4.py

Remark 8.6. (intermediate rings and classes of atoms)  p. 38

The proof of Lemma 3.1 shows that, when H^{-1} has entries in U_Σ , the cokernel of every atomic factor of H is a composition factor, in the category of torsion modules, of the cokernel of a matrix built from Σ (Claims A and B). That universal localizations of a hereditary ring correspond to classes of modules is known in general: Schofield gives a bijection between universal localizations and well-placed subcategories of finitely presented modules, and between the localizations into which the ring embeds and Hom-perpendicular sets of finitely presented bound modules [Sch07, Thms 2.3, 2.5]; Krause and Šťovíček give, for a hereditary ring, a bijection between extension-closed abelian subcategories of finitely presented modules and universal localizations [KS10, Thm B]; and for a factor-complete set Σ over a fir, Cohn shows that every matrix atom either becomes a unit or remains an atom in R_Σ , and describes the divisor group of R_Σ by the Σ -relevant primes [Coh06, Thm 7.11.9, Prop. 7.11.10, Thm 7.11.11]. The rings P and A are firs, hence hereditary. We do not claim this correspondence; Lemma 3.1 uses only the instance above. Describing the intermediate rings of this paper ($\mathcal{U}_r, A_b \cap E_2, \Gamma'_b, D'$) by the classes of atoms they invert is a programme, not a result of this paper.

 NEW

Classical and new: the correspondence is known [KS10], [Sch07], [Coh06]; not claimed.

↳ Not used by another result of the paper.

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