Inversion height two in the free field of the free group: strict hierarchies and a non-membership criterion

Associative rings · division rings 16K40 16S85 16S10 16S34 16E60

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At a glance

What the paper contributes

The free field D of $A = \mathbb{C}[F_3]$ is built in stages: E_1 inverts the nonzero elements of A, E_2 those of E_1, and so on. The stages never stabilise (Reutenauer; Herbera–Sánchez). The paper works **inside the single stage E_2**, where one restricts which elements are inverted.

- **A non-membership criterion** (Theorem A = Theorem 3.4). If H^{-1} has its entries in the ring U_Σ generated by the inverses of a family Σ of full matrices, each atomic factor of H is stably associated with an atomic factor of some $Q \in \Sigma$ (Lemma 3.1). To refute that, it suffices that one atomic factor of H be more singular than every Q at some evaluation, measured by an invariant that does not grow on factors: Smith counts along an arc, the corank on a curve of integral quaternions, the corank at a character. The allowed denominators may themselves be singular there.
- **Two hierarchies inside E_2, strict at every step** (Theorem B). The rings $\mathcal{U}_r$ generated by inverses of sums of products of at most r inverse binomials: $\mathcal{U}_{r-1} \subsetneq \mathcal{U}_r$, with witness $(u^{-2}+v^{-2}+w^{-2})^{-1}$ for r = 2; their union is a proper subring of E_2. The matrix-size chain $A_{d-1} \cap E_2 \subsetneq A_d \cap E_2$ for every $d \geq 2$. The two hierarchies are independent, and the size chain is already strict inside $\mathcal{U}_2$ (Corollary 7.10).
- **Infinitely many stopping steps of the Dubrovin–Gräter–Hanke series inside E_2** (Theorem C): for every $k \geq 3$ the stratum of sums of k elements of D_π, with explicit witnesses f_k. That there are infinitely many stopping steps follows from Herbera–Sánchez, but no infinite family had been located within a fixed stage. The rings reached at these steps form a strict chain whose union is $\mathcal{U}_2$.

Main tools: Cohn's theory of firs, atoms and stable association, and of rational closures; Smith normal forms over $\mathbb{C}[[t]]$ along an exact arc of 2×2 matrices; the free group generated by 1+2**i**, 1+2**j**, 1+2**k**; a grading argument that gives an atom for every k.

🚀 **Start here: Theorem A**: the non-membership criterion: one atomic factor against the allowed denominators.

Main results:

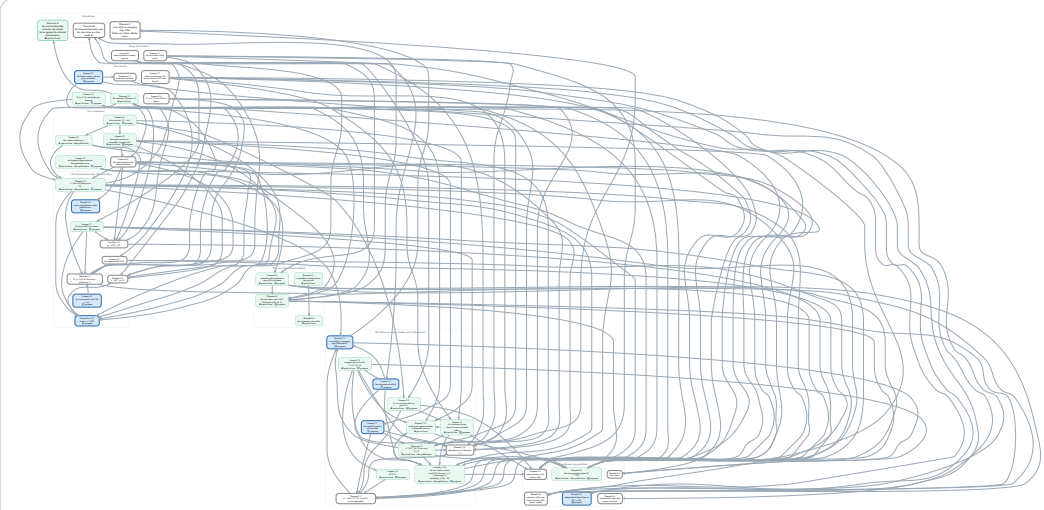
- **Theorem A**: the non-membership criterion: one atomic factor against the allowed denominators
- **Theorem B**: the binomial hierarchy and the size chain are strict inside E_2
- **Theorem C**: every $k[\pi]$ is a stopping step of the Dubrovin–Gräter–Hanke series

47 results | ✅ **0** entirely in Lean | ◐ **21** partly in Lean | 💻 **22** checked by a program

written proof only (a result can carry several tags)

🗺️ Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



📐 **Setting (copied from the paper, §2.1)**

$P = \mathbb{C}\langle x, y, z\rangle \subset A = \mathbb{C}[F_3] \subset D$, where D is the free field, the universal field of fractions of P and of A. We write $u = 1 + x$, $v = 1 + y$, $w = 1 + z$. $E_1 = A[s^{-1} : 0 \neq s \in A] \subset D$ and $E_2 = E_1[e^{-1} : 0 \neq e \in E_1]$. A *binomial* is $\ell = 1 - \beta h$, $h \in F_3 \setminus \{1\}$, $\beta \in \mathbb{C}^\times$. For $r \geq 0$, $\mathcal{F}_r$ is the span of the products $a_0 \ell_1^{-1} a_1 \cdots \ell_q^{-1} a_q$ with $q \leq r$, ℓ_i binomials and $a_i \in A$, and $\mathcal{U}_r = E_1[\mathcal{F}_r^{-1}]$, the subring of D generated by E_1 and the inverses of the nonzero elements of $\mathcal{F}_r$; $\mathcal{U}_\infty = \bigcup_r \mathcal{U}_r$. A_b is the subring of D generated by A and the entries of G^{-1} for all full $a \times a$ matrices G over A with $a \leq b$. A square matrix over a subring of D is *full* if it is invertible over D.

Abstract.

Let D be the free field of $A = \mathbb{C}[F_3]$ and E_2 the ring obtained from A by inverting twice. Two chains of subrings of E_2 are strict at every step: the rings generated over E_1 by inverses of sums of products of at most r inverse binomials (with coefficients in A), whose union is a proper subring of E_2, and the rings generated by A and the entries of the inverses of full matrices over A of size at most d, intersected with E_2. We also locate infinitely many stopping steps of the Dubrovin–Gräter–Hanke complexity series inside E_2, with explicit witnesses; that there are infinitely many stopping steps follows from the work of Herbera and Sánchez, but no infinite family had been located within a fixed stage. The tool is a non-membership criterion that compares one atomic factor with the allowed denominators through invariants that do not grow under passage to factors.

> **Companion site.** karlesmarin.github.io/blueprints/inversion-height-two: every statement with its dependencies and its checks (Lean, programs), the figures in motion and labs to try one's own values. What it holds is described in Appendix A.4.

1 Introduction

The rational closure of $A = \mathbb{C}[F_3]$ in its free field D is built in stages: E_1 is generated by A and the inverses of its nonzero elements, E_2 by E_1 and the inverses of the nonzero elements of E_1, and so on. These stages never stabilise [HS15, Thms 4.5, 4.6] (for infinitely many generators this is [Reu96, Thm 2.1]). Inside a single stage, where one restricts which elements are inverted, much less is known. How many nested inversions an element needs is a basic measure of its complexity, and the complexity series of Dubrovin, Gräter and Hanke [DGH03] records it step by step: at each inversion step (Case 3) it asks whether the inverses of one stratum already lie in the ring built so far [Jak19, Def. 1.1.4]. Answering that is a non-membership problem, and non-membership must exclude every formula, not exhibit one. We give a method that proves non-membership inside a single stage and use it to show that, already inside E_2, two natural hierarchies of subrings are strict at every step and infinitely many stopping steps of the series occur.

The criterion. For a family Σ of full matrices over $P = \mathbb{C}\langle x, y, z\rangle$ (full: invertible over D) let U_Σ be the subring of D generated by P and the entries of Q^{-1}, $Q \in \Sigma$; every subring of D generated by A and the entries of the inverses of a family of full matrices over A has this form (by polynomial lifts, §3.4), for a family containing x, y, z. A full matrix over the fir P has a complete factorization into atoms, unique up to the order of the factors and stable association [Coh06, Thm 3.2.7 and the remark after it]. If H^{-1} has its entries in U_Σ, each atomic factor of H is stably associated with an atomic factor of some $Q \in \Sigma$ (Lemma 3.1). To refute such an association one evaluates.

Theorem A. (Theorem 3.4) p. 2

> *Let Σ be a family of full matrices over P and H a full square matrix over P with an atomic factor G in a complete factorization. If for every $Q \in \Sigma$ there is a monotone invariant μ_Q (Definition 3.3) with $\mu_Q(G) > \mu_Q(Q)$, then H^{-1} does not have entries in U_Σ.*

Lean · ours, in part show

as for Theorem 3.4: with the conclusion of Lemma 3.1 as a hypothesis

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✅ Checked when this page was built: Criterion.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — not_inverseInU: propext, Classical.choice, Quot.sound.

Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean Criterion.lean` (SHA-256 of the compiled file: `b3acc608d6445b7e`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 1 result, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** part: as for Theorem 3.4: with the conclusion of Lemma 3.1 as a hypothesis · **Cite:** p. 2, doi:10.5281/zenodo.23481332

🆕 **Classical and new:** new, as far as the paper found: no use of Smith profiles along an arc, nor of coranks of atoms at a representation where the allowed denominators are singular, to prove non-membership (§8.8). It rests on Lemma 3.1, which the paper does not claim (Remark 8.6); kernel dimensions already separate stable association classes [SV26].

↳ Not used by another result of the paper.

💡 **In plain words (commentary, not part of the paper)**

1. To show that an element is *not* in a ring generated by inverting a family Σ of matrices, one has to exclude every formula. Lemma 3.1 reduces this to atoms: if H^{-1} has its entries in U_Σ, every atomic factor of H is stably associated with an atomic factor of a single $Q \in \Sigma$. The criterion then compares numbers: a monotone invariant (Definition 3.3) does not change under stable association and does not grow when one passes to a factor, so if one atomic factor G of H scores higher than Q, for every $Q \in \Sigma$, then H^{-1} is not in U_Σ. 2. The paper uses three invariants: the number ν_j of Smith invariant factors of order $\geq j$ along an arc on which u^2, v^2, w^2 become the same scalar (§4.1); the corank on a curve of quaternions on which every binomial is invertible (§4.2); and the corank at the character $\chi\colon x_j \mapsto -1$. The invariant may be chosen separately for each Q, and it need not vanish on the allowed denominators: it only has to be smaller there than on the target. That is what makes the criterion work inside E_2, where the allowed denominators are singular at the evaluations that detect the witnesses. 3. New and not new, as the paper says: the paper found no use of Smith profiles along an arc, nor of coranks of atoms at a representation where the allowed denominators are singular, to prove non-membership (§8.8). Lemma 3.1 is the analogue, for U_Σ and without factor-completeness, of results of Cohn, and the correspondence behind it is not claimed (Remark 8.6); kernel dimensions already separate stable association classes [SV26].

A monotone invariant is a number attached to square matrices that is unchanged by stable association and does not grow when passing to a factor. Write $u = 1 + x$, $v = 1 + y$, $w = 1 + z$, where x, y, z are the generators of F_3. We use three: the number ν_j of Smith invariant factors of order $\geq j$ of the

evaluation along an arc $P \to M_2(\mathbb{C}[[t]])$ on which u^2, v^2, w^2 become the same scalar (§4.1), the corank on a curve of quaternions, integral at $t = 2$, on which every binomial is invertible (§4.2), and the corank at the character $\chi\colon x_j \mapsto -1$, at which u, v, w vanish. The invariant need not vanish on the allowed denominators; it only has to be smaller there than on the target. So the denominators may be singular at the evaluation, a single curve bounds infinitely many of them at once, and an arc may be chosen separately for each denominator.

Results. A *binomial* is $\ell = 1 - \beta h$ with $h \in F_3 \setminus \{1\}$ and $\beta \in \mathbb{C}^\times$. Let $\mathcal{U}_r$ be the subring generated by E_1 and the inverses of the nonzero sums of products $a_0 \ell_1^{-1} a_1 \cdots \ell_q^{-1} a_q$ with $q \le r$ and $a_i \in A$, and A_b the subring generated by A and the entries of the inverses of full $a \times a$ matrices over A, $a \le b$ (the chain of Gerasimov and Malcolmson [Ger82, Mal84a]).

Theorem B. p. 3

> *(i) $\mathcal{U}_{r-1} \subsetneq \mathcal{U}_r$ for every $r \ge 1$; for $r = 2$ a witness is $(u^{-2} + v^{-2} + w^{-2})^{-1}$ (Theorems 5.1, 5.2). (ii) $A_{d-1} \cap E_2 \subsetneq A_d \cap E_2$ for every $d \ge 2$, and $\bigcup_r \mathcal{U}_r \subsetneq E_2$ (Theorem 6.1).*

🆕 **Classical and new:** the rings $\mathcal{U}_r$ have not been studied, to the paper's knowledge; nor the strictness of the size chain inside E_2 for $d \ge 3$. The case $d = 2$, and the strictness of the chain A_b in E(b), follow from [HS15, Thm 4.6] by pivoting (§8.8). The separation of the whole stages $E(n) \subsetneq E(n+1)$ is [Reu96], [HS15].

↳ Not used by another result of the paper.

💡 **In plain words (commentary, not part of the paper)**

1. Inside the second stage E_2 of the rational closure of $A = \mathbb{C}[F_3]$ two natural ways of measuring complexity give hierarchies that are strict at every step. (i) Count the inverse binomials: $\mathcal{U}_r$ is generated by E_1 and the inverses of sums of products of at most r inverse binomials, with coefficients in A; then $\mathcal{U}_{r-1} \subsetneq \mathcal{U}_r$ for every $r \ge 1$, and for $r = 2$ the witness is $(u^{-2} + v^{-2} + w^{-2})^{-1}$. (ii) Count the size of the inverted matrices: $A_{d-1} \cap E_2 \subsetneq A_d \cap E_2$ for every $d \ge 2$, and the union of all $\mathcal{U}_r$ is still a proper subring of E_2. 2. Why: each step is an application of Theorem A. For (i) the exact arc makes the squares u^2, v^2, w^2 the same scalar $t^{k+1} I$, so a quadratic target has four Smith parts at level $k + 1$, while every allowed denominator, with k chosen after it, has at most two (Theorem 5.1; Theorem 5.2 and Lemma 5.6 for $r \ge 3$). For (ii) the quaternion curve makes every binomial invertible, so every allowed border of an $a \times a$ matrix has corank at most $2a$, while the target $\hat{F}_d$ vanishes there and has corank $2d$ (Theorem 6.1, Lemma 6.2). 3. New and not new, as the paper says: the rings $\mathcal{U}_r$, and the strictness of the size chain inside E_2 for $d \ge 3$, have not been studied to the paper's knowledge; the case $d = 2$ follows from [HS15, Thm 4.6] by pivoting (§8.8), and the separation of the whole stages $E(n) \subsetneq E(n+1)$ is [Reu96], [HS15]. The two hierarchies are independent (Remark 8.2, Remark 7.11).

The two hierarchies are independent. For $r, b \ge 2$, $\mathcal{U}_r$ is incomparable with $A_b \cap E_2$; the chain A_d is already strict inside $\mathcal{U}_1$, $A_{d-1} \cap \mathcal{U}_r \subsetneq A_d \cap \mathcal{U}_r$ for $d \ge 2$ and $r \ge 1$, by inverses of sums of $d + 1$ inverse binomials (Corollary 7.10(a)); the witnesses φ_r^{-1} of Theorem B(i) ($r \ge 3$, Theorem 5.2) all lie in A_5, while $f_k^{-1} \in \mathcal{U}_2$ needs matrices of size $k - 1$ (Remarks 7.11, 8.2).

The series of [DGH03] builds D as a union of subsets D_α, its *strata*, by adjoining sums, products and, once the current set is a ring, the inverses of the first stratum whose inverses are missing; such a stratum is a *stopping step* (§2.3). After A and the inverses of binomials, the first stratum that contains products of two inverse binomials is D_π.

Theorem C. p. 3

> *For every $k \ge 3$, the stratum of sums of k elements of D_π is a stopping step of the Dubrovin–Gräter–Hanke series of D for the monoid $\mathbb{C}^\times F_3$ of scalar multiples of group*

> *elements, in Jakobs' formulation [Jak19, Def. 1.1.4]. Witnesses are* $u^{-2}+v^{-2}+w^{-2}$ *for* $k=3$ *and* $f_k = u^{-2}+w^{-2}+\sum_{i=1}^{k-2} y^i u^{-2} y^{-i}$ *for* $k \geq 4$ *(Theorems 7.1, 7.3).*

🆕 **Classical and new:** infinitely many stopping steps follow from [HS15], but none had been located within a fixed stage; the first stopping step (sums of two monomials) is implicit in [Gra20, §4].

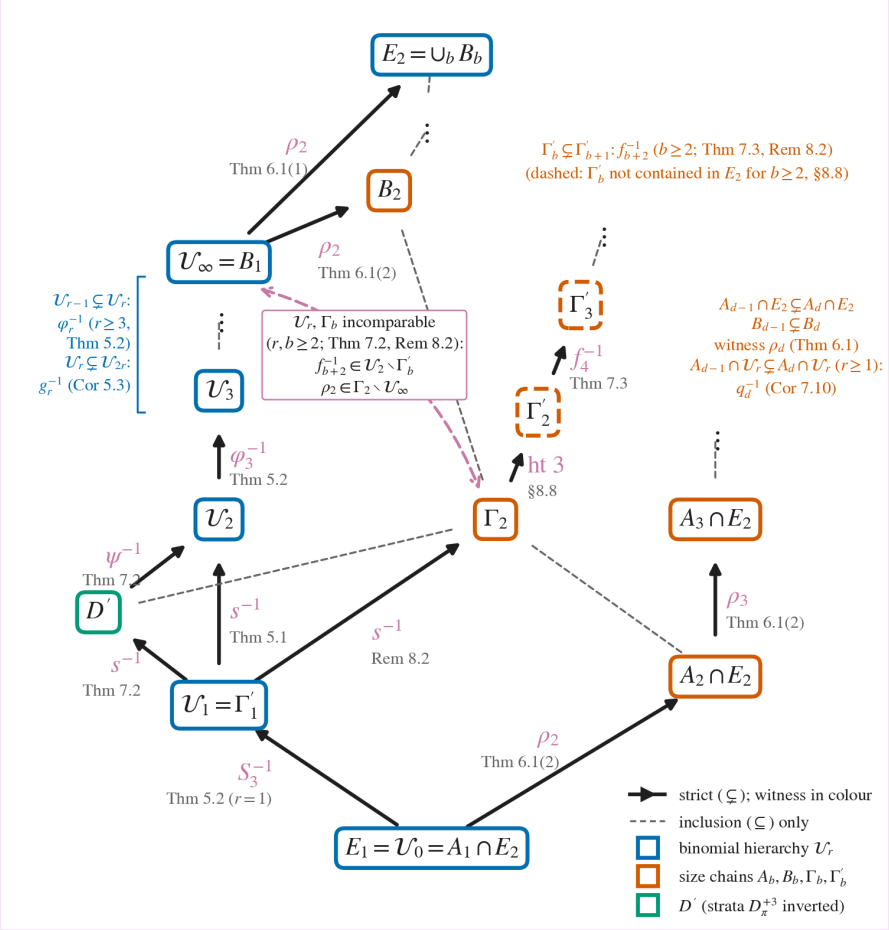
↳ Not used by another result of the paper.

💡 **In plain words (commentary, not part of the paper)**

1. The Dubrovin–Gräter–Hanke series builds the free field step by step: it adds sums, products and, when the current set is a ring, the inverses of the first stratum whose inverses are missing; such a stratum is a stopping step. After A and the inverse binomials, D_π is the first stratum with products of two inverse binomials. The theorem says that for every $k \geq 3$ the stratum of sums of k elements of D_π is a stopping step, and all of this happens while the ring built is still inside E_2. 2. Why: when the series reaches the stratum $k[\pi]$, the ring built so far lies in $\Gamma'_{k-2} \cap E_2$ (Lemma 7.5, using the compression of Lemma 7.4), so it suffices to find a sum of k elements of D_π whose inverse is not in Γ'_{k-2}: $s = u^{-2}+v^{-2}+w^{-2}$ for $k=3$ (Theorem 5.1) and $f_k = u^{-2}+w^{-2}+\sum_{i=1}^{k-2} y^i u^{-2} y^{-i}$ for $k \geq 4$ (Theorem 7.3). For every k an atom is needed; the paper gets it in the group algebra of the free group on u, y, w by a grading argument (Lemma 7.9) and applies Theorem A to that single atomic factor. 3. New and not new, as the paper says: that the series has infinitely many stopping steps in total already follows from [HS15, Thm 4.6]; what is added is that infinitely many occur inside the second stage, and which strata they are. The first stopping step (sums of two monomials) is implicit in [Gra20, §4]. The result is relative to the monoid $\mathbb{C}^\times F_3$, in Jakobs' formulation [Jak19, Def. 1.1.4], and it does not decide the intermediate strata $(k-1)[\pi]+x$ with $x \neq 0$, $k \geq 4$ (Remark 8.3).

The rings reached at these steps form a strict chain $\mathcal{U}_1 = \mathcal{D}_3 \subsetneq \mathcal{D}_4 \subsetneq \cdots$ whose union is $\mathcal{U}_2$ (Corollary 7.10(c)). The proof of Theorem C needs an atom for every k; we obtain it in the group algebra of the free group on u, y, w by a grading argument and apply Theorem A to that single atomic factor. Figure 1 summarises the inclusions, with their witnesses.

Figure 1



The subrings of D studied in this paper. A solid arrow is a strict inclusion, labelled by an element of the larger ring that is not in the smaller one and by the result that proves it; a dashed grey line is an inclusion not marked as strict; the dashed double arrow joins the incomparable rings $\mathcal{U}_\infty$ and Γ_2 (Theorems 6.1 and 7.2; more generally $\mathcal{U}_r$ and Γ_b are incomparable for $r, b \geq 2$, Remark 8.2). The inclusion $\Gamma_2 \subsetneq \Gamma'_2$ is witnessed by an element of inversion height 3 (§8.8). Dashed boxes: the rings Γ'_b, which for $b \geq 2$ are not contained in E_2 (§8.8). The rings Γ_b, Γ'_b and D' are defined in §2.4.

Relation to previous work. The known separations use witnesses of growing inversion height, so all but the first ones leave E_2. Reutenauer (for infinitely many generators) and Herbera–Sánchez separate the whole stages $E(n) \subsetneq E(n+1)$ [Reu96], [HS15], inverting every nonzero element. In a free algebra the chain A_b is strict by the elements of prescribed inversion height of [HS15, Thm 4.5]; over $\mathbb{C}[F_3]$, [HS15, Thm 4.6] yields elements of A_b of height $b-1$; a pivot at a unit entry turns them into entries of the inverse of a full $(b-1) \times (b-1)$ matrix over A, hence into witnesses for every step $A_{b-2} \subsetneq A_{b-1}$, and a second pivot gives $E_1 \subsetneq \mathcal{U}_1$; the case $b = 3$ also gives $d = 2$ in Theorem B(ii) (§8.8). Since the height is unbounded, they also give infinitely many stopping steps, none located in a fixed stage; for $b \geq 4$ these witnesses lie outside E_2. The first stopping step, the stratum of sums of two monomials, is implicit in [Gra20, §4]. Working inside E_2 is harder because the allowed denominators are singular at the evaluations that detect the witnesses; Theorem A is designed for that situation. It uses Cohn's theory of torsion modules over semifirs and of rational closures [Coh06, §3.2, Thm 7.1.2, Lemma 7.11.2]; Lemma 3.1 is the analogue, for U_Σ and without factor-completeness, of Cohn's results on localization at factor-complete sets [Coh06, Thm 7.11.9]. Over $\mathbb{C}$ the free-locus theorem [HKV22, Thm 2.12] gives an alternative route (Remark 3 after Lemma 3.1); kernel dimensions already separate stable association classes [SV26]. What the invariants of Definition 3.3 add is a certificate that an atom is absent from the allowed denominators when these are singular at the evaluation. A more detailed comparison is in §8.8.

Section 2 fixes the rings and the complexity series; §3 proves the criterion; §4 builds the two evaluations; §§5–7 prove Theorems B(i), B(ii) and C; §8 collects remarks and open problems. The appendix lists what is formally verified in Lean 4 and how the computations were checked. A companion site, karlesmarin.github.io/blueprints/inversion-height-two, shows every numbered result

with its dependencies and what checked it, has interactive labs for Lemmas 4.1 and 4.5 and Theorem 5.1, and draws Figures 2–4 live (Appendix A.4).

2 Rings and notation

2.1 Rings

$P = \mathbb{C}\langle x, y, z\rangle \subset A = \mathbb{C}[F_3] \subset D$, where D is the free field, the universal field of fractions of P and of A. For A: $A = P_\Sigma$ with Σ the diagonal matrices with entries in $\{x, y, z, 1\}$ [Coh06, Cor. 7.11.8], and a universal localization of a ring with universal field D that maps to some field has D as its universal field as well [Coh06, Ex. 7.5.11–12]. (Directly: an epic A-field K is an epic P-field, so D specializes to K. The domain of the specialization is a local ring containing P in which each x_j maps to a nonzero element, so it contains the x_j^{-1}, and the specialization is one of A-fields.) We work over $\mathbb{C}$ throughout; Remark 8.1 shows where characteristic 0 is used; in characteristic p the witness of Theorem 5.1 fails. We write $u = 1 + x$, $v = 1 + y$, $w = 1 + z$; the affine change $x \mapsto u$ is an automorphism of P, so $P = \mathbb{C}\langle u, v, w\rangle$.

- $E_1 = A[s^{-1} : 0 \neq s \in A] \subset D$ and $E_2 = E_1[e^{-1} : 0 \neq e \in E_1]$. Height is measured over A: the inverses x_j^{-1} do not count; this is the height $h_{\iota'}$ of [HS15, §3] for the embedding of A.
- A *binomial* is $\ell = 1 - \beta h$, $h \in F_3 \setminus \{1\}$, $\beta \in \mathbb{C}^\times$. Every element $\alpha g + \beta h$ of A with two terms equals $\alpha g(1 + \alpha^{-1}\beta g^{-1} h)$, a unit of A times a binomial, and $m\ell^{-1}m'$ is again a unit multiple of an inverse binomial for monomials m, m'.
- For $r \geq 0$,

$$\mathcal{F}_r = \mathrm{span}_{\mathbb{C}}\{a_0\ell_1^{-1}a_1 \cdots \ell_q^{-1}a_q : \ q \leq r,\ \ell_i \text{ binomials},\ a_i \in A\}, \qquad \mathcal{U}_r = E_1[\mathcal{F}_r^{-1}],$$

  the subring of D generated by E_1 and the inverses of the nonzero elements of $\mathcal{F}_r$. Thus $\mathcal{F}_0 = A$, $\mathcal{U}_0 = E_1$, $\mathcal{F}_1 = V_1 := A + \sum A\ell^{-1}A$ and $\mathcal{U}_1 = E_1[V_1^{-1}]$. Since $\mathcal{F}_r \subset E_1, \mathcal{U}_r \subseteq E_2$.
- **The coefficients a_i lie in A, not in E_1.** This is essential: with $a_0 = u^{-2r} \in E_1$ and $q = 0$ one would get $g_r \in \mathcal{F}_0$ and Corollary 5.3 would be false.
- Let $\chi : A \to \mathbb{C}$ be the character $x_j \mapsto -1$ (so $u, v, w \mapsto 0$). A binomial $1 - \beta h$ is *exceptional* if h is conjugate to x_j^n for some j and $n \neq 0$ and $\beta(-1)^n = 1$. Binomials with $\beta\chi(h) \neq 1$ are units at χ.

A square matrix over a subring of D is *full* if it is invertible over D. For a semifir with universal field D (here P, A and the group algebra Λ of §5.3) this agrees with Cohn's notion (inner rank n), since the embedding into D preserves the inner rank [Coh06, p. 291 and Thm 7.5.13, p. 450].

Let $R_{\mathrm{bin}} \subset E_1$ be the subring generated by A and the inverses of all binomials, and $\mathcal{U}_\infty = \bigcup_r \mathcal{U}_r$. Let A_b (resp. W_b) be the subring of D generated by A (resp. R_{bin}) and the entries of G^{-1} for all full $a \times a$ matrices G over A (resp. over R_{bin}) with $a \leq b$, and put $B_b = W_b \cap E_2$.

- $R_{\mathrm{bin}} = \bigcup_r \mathcal{F}_r$, since $\mathcal{F}_r\mathcal{F}_s \subseteq \mathcal{F}_{r+s}$. Hence $\mathcal{U}_\infty = E_1[(R_{\mathrm{bin}} \setminus 0)^{-1}] \subseteq E_2$.
- A full $a \times a$ matrix G with $a<b$ gives the full matrix $G \oplus I_{b-a}$, so A_b is also generated by A and the entries of inverses of full matrices of order exactly b. Since $A \subseteq R_{\mathrm{bin}}$, $A_b \subseteq W_b$. Inverses of nonzero elements of R_{bin} are entries of inverses of 1×1 full matrices, so $W_1 = \mathcal{U}_\infty$, $A_1 = E_1$, and $B_1 = \mathcal{U}_\infty$.
- **The chain A_b and Malcolmson's chain.** Malcolmson's R_b is the universal localization of A at the full $b \times b$ matrices; it embeds in the universal field [Mal84a, Thm 3], and its image is A_b: R_Σ is generated by R and the entries of the inverses of the matrices in Σ, and its image is generated by the images. Our W_b is the analogue over R_{bin}; it is not Malcolmson's ring ($W_1 = \mathcal{U}_\infty \neq E_1 = A_1$ by Theorem 5.2 with $r = 1$).

2.2 Matrices over P

Full matrices over P are as in §2.1. Two matrices are *stably associated* if $F_2 \oplus I = M(F_1 \oplus I)N$ with M, N invertible over P [Coh06, §0.5], [HKV22, §1]. A full matrix is an *atom* if it is not invertible and is not a product of two non-invertible full matrices; stable association preserves atoms [Coh06, Cor. 0.5.5]. For a family Σ of full matrices over P we write $U_\Sigma = P[\text{entries of } Q^{-1} : Q \in \Sigma] \subseteq D$.

Pencils and atoms. Let $k = \mathbb{C}$ and $P_g = k\langle x_1, \dots, x_g\rangle$. A *pencil* is $L = A_0 + \sum_{i=1}^{g} A_i x_i$ with $A_i \in M_N(k)$. Following Cohn, L is *right monic* if $[A_1 \ \cdots \ A_g]$ has a right inverse over k (rank N), *left monic* if the column $[A_1; \dots; A_g]$ has rank N, and *monic* if both hold [Coh06, §5.8, p. 312]. This is the *epic* condition of [HKV22, p. 5]. For a full $n \times n$ matrix F over P_g write $M(F) = P_g^{1\times n}/P_g^{1\times n}F$.

We use the following classical facts.

1. A non-full $n \times n$ matrix factors as AB with A of size $n \times (n-1)$, so it is singular at every point of k^g. Hence a matrix that is invertible at one scalar point is full; and a matrix that is singular at one scalar point is not invertible over P_g, since evaluation is a ring homomorphism.
2. If $F = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$ with T invertible over P_g, then
$$F = \begin{pmatrix} I & YT^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} I & 0 \\ T^{-1}Z & I \end{pmatrix}, \qquad S = X - YT^{-1}Z,$$
and $S \oplus T = (I \oplus T)(S \oplus I)$. So F is associated with $S \oplus I$, that is, stably associated with the Schur complement S.
3. Stably associated full matrices have isomorphic modules $M(\cdot)$. Indeed $M(F \oplus I) \cong M(F)$, and $M(UFV) \cong M(F)$ via $x \mapsto xV^{-1}$ for U, V invertible. Over a semifir a full matrix is an atom iff $M(F)$ is Tor-simple [Coh06, Thm 3.2.3 and the paragraph after it, p. 195]. Hence stable association preserves atoms [Coh06, Cor. 0.5.5, p. 30], [HKV22, p. 5].
4. A ring automorphism of P_g, for instance an affine change of variables $x_i \mapsto x_i + a_i$, preserves units, fullness and factorizations, hence atoms.

Lemma 2.1. (atom criterion) p. 6

> *Let $N \geq 1$ and let $L = A_0 + \sum_i A_i x_i$ be an $N \times N$ monic pencil with A_0 invertible. Then L is full and not invertible, and L is an atom iff the matrices $M_i = A_0^{-1}A_i$ ($1 \leq i \leq g$) have no common invariant subspace other than 0 and k^N.*

NEW **Classical and new:** classical: Cohn [Coh06, Thm 5.8.8], in the form of [HKV22, Rem. 2.6].

↳ Used in: Lemma 5.7 (p. 30), Proposition 5.8 (p. 34), Lemma 6.2 (p. 37), Lemma 7.8 (p. 45), 7.12 (p. 51), 8.4 (p. 56)

Proof

$L(0) = A_0$ is invertible, so L is full (§2.2(a)). A right monic matrix is not a unit [Coh06, Lemma 5.8.1, p. 312]. For $N = 1$, [Coh06, Thm 5.8.8, p. 317] says that L is an atom, and the subspace condition holds trivially. Let $N>1$. By [Coh06, Thm 5.8.8], L is not an atom iff there are $P, Q \in GL_N(k)$ and $r, s>0$ with $r + s = N$ such that every PA_jQ ($j = 0, \dots, g$) has a zero $r \times s$ block in the upper right corner.

- Given such P, Q, put $W = Q\,\mathrm{span}(e_{r+1}, \dots, e_N)$ and $W' = P^{-1}\mathrm{span}(e_{r+1}, \dots, e_N)$. Then $A_jW \subseteq W'$ for all j. Since A_0 is injective and $\dim W = \dim W'$, we get $A_0W = W'$, hence $M_iW \subseteq A_0^{-1}W' = W$, with $0<\dim W = s<N$.
- Conversely, let W be invariant with $0<s = \dim W<N$. Take Q whose last s columns span W, and $P = Q^{-1}A_0^{-1}$. Then $PA_0Q = I$ and $PA_iQ = Q^{-1}M_iQ$ maps $\mathrm{span}(e_{r+1}, \dots, e_N)$ into itself, so the upper right $r \times s$ block vanishes for all j. □

This is the reformulation [HKV22, Rem. 2.6, p. 7] of [Coh06, Thm 5.8.8]; we include the two-line proof because [Coh06, Cor. 5.8.9] cannot be applied verbatim with $A_0 = I$: there "monic" must be assumed, since $I + 0 \cdot x$ is a unit.

2.3 The Dubrovin–Gräter–Hanke series

Let $\mathcal{M} = \mathbb{C}^{\times} F_3$ (monomials with coefficients; written M in [Jak19], and D is written R there). We use Jakobs' formulation of the series (D_α) [Jak19, Defs. 1.1.4–1.1.5, pp. 3–4], after [DGH03]. The series is indexed by ordinals. In Cases 1 and 2 the stratum to be adjoined is chosen as the least element of the free commutative monoid $N(\alpha')$ on the ordinals $\leq \alpha'$, ordered lexicographically starting from the largest ordinal [Jak19, Thm 1.1.3, p. 3]. For an ordinal π we write $k[\pi]$ for the multiset with k copies of π and $k[\pi] + x$ for its sum with a multiset x; a stratum born by Case 1 is named by the multiset that selected it (so $D_{k[\pi]}$ is the stratum born when Case 1 selects $k[\pi]$, and $\mathrm{cp}(a) = k[\pi]$ means that a first appears there). Case 1 adjoins the minimal admissible sum, Case 2 products, only when the current set is additively closed, and Case 3 inverses, only when it is a ring $D_{\alpha'}$ (α' the current index), at the minimal ordinal α_1 with $D_{\alpha_1}^{-1} \not\subseteq D_{\alpha'}$; $\mathrm{cp}(a) = \min\{\alpha : a \in D_\alpha\}$ [Jak19, Def. 1.1.5]. A *stopping step* is an index α_1 selected by Case 3, i.e. a stratum whose inverses are adjoined at that point (named, for instance, $k[\pi]$; the Case 3 itself happens at the current index $\alpha' \geq \alpha_1$). Selected indices increase strictly, since once $D_{\alpha_1}^{-1}$ is adjoined it stays, and they are never limit ordinals, since an element of $D_\beta = \bigcup_{\gamma<\beta} D_\gamma$ already lies in some D_γ. For an index β, the *inversion pointer reaches* β at the first Case 3 whose selected index is $\geq \beta$. By the minimality of α_1, the current ring at that step contains D_γ^{-1} for every $\gamma<\beta$, whether or not γ was itself selected.

We read S^{-1}, for a stratum S, as the set of inverses of the nonzero elements of S (in D the units are the nonzero elements; in [Jak19, Def. 1.1.1] and [Gra20, §4] only units of the ambient ring are inverted). Note $D_1 = \mathcal{M} \cup \{0\}$, since $-1 \in \mathcal{M}$.

Lemma 2.2. (the first strata) p. 7

1. *For $1 \leq n<\omega$, D_n is the set of sums of at most n elements of D_1, and $D_\omega = A$.*
2. *The first Case 3 occurs at $\alpha' = \omega$, with $\alpha_1 = 2$; $D_{\omega+1} = A \cup D_2^{-1}$, and the nonzero elements of D_2^{-1} are the monomials and the unit multiples $\ell^{-1}m$ of inverse binomials.*
3. *From $\omega + 1$ on, Case 1 applies up to the first index σ at which the set is additively closed, and $D_\sigma = V_1$.*
4. *If $D_{\omega+1}D_{\omega+1} \not\subseteq V_1$, Case 2 applies at σ with the multiset $2[\omega + 1]$, so $\pi = \sigma + 1$ and $D_\pi = V_1 \cup D_{\omega+1}D_{\omega+1}$.*
5. *Let $k \geq 2$ and suppose $D_\pi^{+j} \not\subseteq D_\pi^{+(j-1)} + V_1$ for $2 \leq j \leq k$. Then the strata born after π and before $k[\pi]$ are among the $j[\pi] + x$ with $1 \leq j<k$ and $x \in N(\sigma)$, each contained in $D_\pi^{+j} + V_1$, no Case 2 or Case 3 occurs in between, and $D_{k[\pi]} = D_\pi^{+k}$.*

↳ Used in: 5.9 (p. 35), Lemma 7.5 (p. 44), Corollary 7.10 (p. 48), 7.12 (p. 51)

Proof

(1) For $n<\omega$, D_n is not additively closed (it misses sums of $n + 1$ distinct monomials), so Case 1 applies. For $n<\omega$, every multiset smaller than $(n + 1)[1]$ involves only the ordinals 0 and 1, with at most n copies of 1, so its sum lies in D_n; and $(n + 1)[1]$ gives sums of $n + 1$ distinct monomials, which are not in D_n.

1. $D_\omega = A$ is a ring and $D_1^{-1} \subseteq D_1$. A sum $\alpha g + \beta h$ of two monomials with $g \neq h$ is not a unit of A [Jak19, Thm 1.3.10], so $D_2^{-1} \not\subseteq A$; and $(\alpha g + \beta h)^{-1} = \ell^{-1}(\alpha g)^{-1}$ with $\ell = 1 + \alpha^{-1}\beta g^{-1}h$.
2. Case 2 needs an additively closed set and Case 3 a ring. Sums of elements of $D_{\omega+1}$ lie in V_1, which is the additive closure of $D_{\omega+1}$ (§2.1: $m\ell^{-1}m'$ is a unit multiple of an inverse binomial). So the first additively closed set is V_1.

3. In $N(\sigma)$ the multisets below $2[\omega+1]$ are those over $\{0,\dots,\omega\}$ and the $[\omega+1]+x$ with x over $\{0,\dots,\omega\}$; any multiset containing an ordinal in $(\omega+1,\sigma]$ is larger. The corresponding products lie in $A\cdot(A\cup D_2^{-1})\cdot A\subseteq V_1$.
4. As long as Case 1 applies, the indices are taken in increasing order; every multiset containing an ordinal born after π exceeds every multiset of $N(\pi)$ [Jak19, p. 3], and $j[\pi]+x<(j+1)[\pi]$. The strata over $N(\sigma)$ lie in V_1, so $D_{j[\pi]+x}\subseteq D_\pi^{+j}+V_1$ (recall $V_1\subseteq D_\pi$). Before $k[\pi]$ the current set lies in $D_\pi^{+(k-1)}+V_1$, which by hypothesis does not contain D_π^{+k}; it is therefore not additively closed, so neither Case 2 nor Case 3 can occur, and $k[\pi]$ is eventually the least multiset with a new sum. □

Here D_π^{+m} is the set of sums of m elements of D_π; the equality $D_{k[\pi]}=D_\pi^{+k}$ holds because the current set lies in $D_\pi^{+(k-1)}+V_1\subseteq D_\pi^{+k}$ (as $V_1\subseteq D_\pi\ni 0$).

An element of $D_{\omega+1}D_{\omega+1}$ is $ab\in A$; $a\ell^{-1}m$ or $\ell^{-1}ma$, in V_1; or $\ell_1^{-1}m_1\ell_2^{-1}m_2=(\ell_2'\ell_1)^{-1}m_1m_2=:d^{-1}$ with $\ell_2'=m_1\ell_2m_1^{-1}$ and $d\in A$. We call the last type *pure*; so every element of D_π lies in V_1 or is pure. We write $b_{kXX}=k[\pi]$. The hypotheses of (4) and (5) hold for every k; this is shown in §7.2.

2.4 The rings D' and Γ_b

Let $D'=\mathcal{U}_1[(D_\pi^{+3}\setminus 0)^{-1}]$. Let Γ_2' be the subring generated by $\mathcal{U}_1$ and the entries of M^{-1} for full $a\times a$ matrices M over V_1, $a\le 2$, and $\Gamma_2\subseteq\Gamma_2'$ the same with M over A. For $a\ge 1$ let Γ_a' be defined as Γ_2' with $a'\times a'$ matrices, $a'\le a$, and Γ_a likewise with matrices over A; thus $\Gamma_b=\mathcal{U}_1[\text{entries of inverses of full } a\times a \text{ matrices over } A,\ a\le b]$.

3 The criterion

Question. How can one show that an element of D is not in a ring U_Σ? Membership is witnessed by a formula; non-membership must exclude every formula. Lemma 3.1 reduces it to finitely many atoms, and an invariant that cannot grow when passing to a factor turns the comparison of atoms into a comparison of numbers (Theorem 3.4).

3.1 Localization

Lemma 3.1. (matrix localization) p. 8

Let R be a semifir with universal field of fractions D, and Σ a set of full matrices over R, each with a complete factorization (this holds for every Σ if R is fully atomic, for instance a fir such as P or A). Let $U_\Sigma\subseteq D$ be the subring generated by R and the entries of Q^{-1}, $Q\in\Sigma$. If H is a full matrix over R and H^{-1} has entries in U_Σ, then every atomic factor of H is stably associated with an atomic factor of a complete factorization of a single member $Q\in\Sigma$. Conversely, if H has a complete factorization whose factors are all stably associated with atomic factors of members of Σ, then H^{-1} has entries in U_Σ.

🆕 **Classical and new:** not claimed: the analogue, for U_Σ and without factor-completeness, of Cohn's results on localization at factor-complete sets [Coh06, Thm 7.11.9] (Remark 8.6).

↳ Used in: Theorem 3.4 (p. 16), Lemma 7.9 (p. 47), 7.12 (p. 51), 8.1 (p. 52), 8.4 (p. 56), 8.5 (p. 56), 8.6 (p. 57)

Proof

1. By (T3) the torsion submodules of M form a sublattice $L(M)$ of the submodule lattice, so $L(M)$ is modular.

2. For $L \subseteq L'$ in $L(M)$, the quotient L'/L is torsion by (T2). The interval $[L, L']$ has no interior point exactly when L'/L is 0 or Tor-simple: the torsion submodules of L'/L are the L''/L with $L'' \in [L, L']$, by (T2).
3. By the Schreier refinement theorem [Coh06, App. Thm A.3, p. 521], applied in the modular lattice $L(M)$ of torsion submodules (not in the lattice of all submodules), the chains $0 \subseteq N \subseteq M$ and (M_j) have refinements in $L(M)$ whose intervals pair off into projective intervals.
4. Projective intervals are isomorphic as lattices and give isomorphic quotient modules [Coh06, p. 521]. A refinement of (M_j) only repeats terms, so its nontrivial intervals are the $[M_{j-1}, M_j]$.
5. Hence every nontrivial interval of the refinement of $0 \subseteq N \subseteq M$ has a Tor-simple quotient isomorphic to some M_j/M_{j-1}. The part between 0 and N is a $\mathcal{C}$-filtration of N.
6. The part between N and M is a chain $N = L_0 \subseteq \cdots \subseteq L_t = M$. The L_i/N are torsion submodules of M/N with $(L_i/N)/(L_{i-1}/N) \cong L_i/L_{i-1}$, which gives a $\mathcal{C}$ -filtration of M/N.
7. If M/N is Tor-simple, a $\mathcal{C}$-filtration of it with distinct terms is $0 \subset M/N$, so M/N is isomorphic to a member of $\mathcal{C}$. □

The ingredients are standard; we include the proof for completeness, because it uses neither a finiteness nor a factor-inverting hypothesis on Σ (compare [Coh06, §7.11] and [Coh85a, Ex. 7.11.13]). An *atomic factor* of an $N \times N$ matrix H is an atom F with $H = H_1 F H_2$ for some $N \times N$ matrices H_1, H_2 over R.

Conventions. Matrices act on columns. For a full $n \times n$ matrix G over R put $\operatorname{coker} G = R^n/GR^n$, a right R-module. Left multiplication by a full matrix is injective on R^n, since the matrix is invertible over $D \supseteq R$.

We use the following facts about the category Tor_R of (right) torsion modules over a semifir R [Coh06, §3.2]:

- (T1) A finitely presented module is torsion if and only if it is $\operatorname{coker} G$ for a full square G [Coh06, Prop. 3.2.1(i), p. 192].
- (T2) In a short exact sequence of modules, if two terms are torsion then so is the third. Kernels and cokernels of maps between torsion modules are torsion [Coh06, Prop. 3.2.1(ii),(iii), p. 192].
- (T3) Tor_R is abelian, and the torsion submodules of a torsion module are closed under sums and intersections [Coh06, Thm 3.2.3, p. 195].
- (T4) $\operatorname{coker} G$ is Tor-simple if and only if G is a matrix atom [Coh06, p. 195].
- (T5) A factorization $G = G_1 \cdots G_k$ into full square matrices gives the chain $R^n \supseteq G_1 R^n \supseteq \cdots \supseteq G_1 \cdots G_k R^n = GR^n$. Left multiplication by $G_1 \cdots G_{j-1}$ identifies its j-th quotient with $\operatorname{coker} G_j$ [Coh06, p. 194].
- (T6) Full matrices over a semifir are regular [Coh06, p. 193]. For regular matrices, isomorphic cokernels are equivalent to stable association [Coh06, Cor. 0.5.5, p. 30].

Let $\mathcal{C}$ be the set of isomorphism classes of the modules $\operatorname{coker} G$, where G runs over the atomic factors of complete factorizations of members of Σ. A torsion module M is *$\mathcal{C}$-filtered* if it has a chain $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ of torsion submodules whose quotients M_j/M_{j-1} are Tor-simple and isomorphic to members of $\mathcal{C}$.

Claim A (Jordan–Hölder in Tor_R). *Let M be $\mathcal{C}$-filtered and $N \subseteq M$ a torsion submodule. Then N and M/N are $\mathcal{C}$-filtered. In particular, a Tor-simple torsion quotient of M is isomorphic to a member of $\mathcal{C}$.*

Claim B. *Let $\bar{\Sigma}$ be the set of block upper triangular matrices whose diagonal blocks lie in $\Sigma \cup \{1\}$, with any blocks above the diagonal. Then $\bar{\Sigma}$ is upper multiplicative in the sense of [Coh06, p. 411–412], its members are full, and* $\operatorname{coker} Q$ *is $\mathcal{C}$-filtered for every $Q \in \bar{\Sigma}$.*

Proof.

1. Closure under $\left(\begin{smallmatrix} Q_1 & Z \\ 0 & Q_2 \end{smallmatrix}\right)$ is clear. Members are invertible over D, being block triangular with invertible diagonal blocks.
2. Let $Q = \left(\begin{smallmatrix} Q_1 & Z \\ 0 & Q_2 \end{smallmatrix}\right)$ with Q_1, Q_2 full. The maps $x \mapsto (x, 0)$ and $(x, y) \mapsto y$ give an exact sequence $0 \to \operatorname{coker} Q_1 \to \operatorname{coker} Q \to \operatorname{coker} Q_2 \to 0$:
   - The second map is well defined because $Q(u, v) = (Q_1 u + Zv,\ Q_2 v)$.
   - Its kernel is the image of the first map, because $(x, Q_2 v) \equiv (x - Zv, 0)$.
   - The first map is injective, because $(x, 0) = Q(u, v)$ forces $Q_2 v = 0$, so $v = 0$ and $x = Q_1 u$.
3. Take $\mathcal{C}$-filtrations of $\operatorname{coker} Q_1$ and $\operatorname{coker} Q_2$. Their images and preimages in $\operatorname{coker} Q$ are torsion by (T2), so they splice into a $\mathcal{C}$-filtration of $\operatorname{coker} Q$.
4. By induction on the number of diagonal blocks, it remains to treat $Q \in \Sigma \cup \{1\}$.
   - For the 1×1 identity the cokernel is 0.
   - For $Q \in \Sigma$, take a complete factorization $Q = G_1 \cdots G_k$. By (T5) and (T4) the chain of (T5) has Tor-simple quotients $\operatorname{coker} G_j$, which lie in $\mathcal{C}$ by definition.
   - Its terms are torsion: $G_1 \cdots G_j R^n / Q R^n \cong \operatorname{coker}(G_{j+1} \cdots G_k)$. $\square$

Proof of Lemma 3.1. Let H be $N \times N$ and F an atomic factor, $H = H_1 F H_2$.

Step 1 (the factor is inverted in U_Σ). H is invertible over the field D, so the square matrices H_1 and H_2 are as well. Hence $F^{-1} = H_2 H^{-1} H_1$ has entries in U_Σ.

Step 2 (one denominator).

1. The inclusion $R \to D$ is $\bar{\Sigma}$-inverting (Claim B).
2. By [Coh06, Thm 7.1.2, p. 412–413], the $\bar{\Sigma}$-rational closure $R_{\bar{\Sigma}}(D)$ is a subring of D containing R. It contains the entries of Q^{-1} for $Q \in \Sigma \subseteq \bar{\Sigma}$, so it contains U_Σ.
3. By (d) of the same theorem, each entry of F^{-1} has the form $(F^{-1})_{ij} = b_{ij} A_{ij}^{-1} c_{ij}$ with $A_{ij} \in \bar{\Sigma}$, b_{ij} a row and c_{ij} a column over R.
4. Put $Q = \bigoplus_{i,j} A_{ij} \in \bar{\Sigma}$, of order K, say. Let $B \in {}^N R^K$ have b_{ij} in row i and block column (i, j), and $C \in {}^K R^N$ have c_{ij} in block row (i, j) and column j, with zeros elsewhere. Then

$$F^{-1} = B Q^{-1} C, \qquad \text{so} \qquad I_N = (FB)\, Q^{-1}\, C \quad \text{over } D.$$

Step 3 (Cohn's lemma). R is a semifir, hence a Sylvester domain [Coh06, p. 291], and Q is full. Apply [Coh06, Lemma 7.11.2, p. 502] (= [Coh85a, Lemma 7.10.2, p. 455], stated there for semifirs) to the relation $I_N = (FB) Q^{-1} C$. It gives $P', Q' \in R_K$, $B' \in {}^N R^K$ and $C' \in {}^K R^N$ with

$$Q = Q'P', \qquad FB = B'P', \qquad C = Q'C', \qquad I_N = B'C'.$$

$Q'P' = Q$ is invertible over D, so the square matrices P' and Q' are invertible over D, i.e. full.

Step 4 ($\operatorname{coker} F$ *is a quotient of* $\operatorname{coker} P'$*).* Define

$$\varphi : \operatorname{coker} P' \to \operatorname{coker} F, \qquad x + P' R^K \mapsto B' x + F R^N.$$

- φ is well defined, since $B'P'y = FBy \in F R^N$.
- φ is right R-linear.
- φ is surjective, since $y = B'(C'y)$ for every $y \in R^N$.

(This is Cohn's description of homomorphisms between torsion modules by pairs of matrices [Coh06, p. 194].) Both modules are torsion by (T1), so $\ker \varphi$ is a torsion submodule by (T2), and $\operatorname{coker} F \cong \operatorname{coker} P' / \ker \varphi$.

Step 5 ($\operatorname{coker} P'$ *is a torsion submodule of* $\operatorname{coker} Q$*).* Left multiplication by Q' is injective and maps $P' R^K$ onto $Q R^K$, so $\operatorname{coker} P' \cong Q' R^K / Q R^K \subseteq \operatorname{coker} Q$; it is a torsion submodule, being isomorphic to the torsion module $\operatorname{coker} P'$.

Step 6 (conclusion).

1. By Claim B, coker Q is $\mathcal{C}$-filtered. By Claim A (Step 5), so is coker P'.
2. F is an atom, so coker F is Tor-simple (T4). By Claim A (Step 4), it is isomorphic to coker G for an atomic factor G of a complete factorization of one $Q_0 \in \Sigma$.
3. By (T6), F and G are stably associated.

Converse. If $Q = G_1 F G_2$ with $Q \in \Sigma$ and G_1, G_2 square, then $F^{-1} = G_2 Q^{-1} G_1$ has entries in U_Σ. If $F' \oplus I = S(F \oplus I)T$ with S, T invertible over R, then $F'^{-1} \oplus I = T^{-1}(F^{-1} \oplus I)S^{-1}$. Finally $H = F_1 \cdots F_k$ gives $H^{-1} = F_k^{-1} \cdots F_1^{-1}$. ■

Remarks on Lemma 3.1.

1. *Where the hypotheses are used.*
   - That R is a semifir: (T1)–(T6) and [Coh06, Lemma 7.11.2].
   - That D is the universal field: [Coh06, Lemma 7.11.2], through the rank-preserving embedding $R \to D$ [Coh06, Cor. 5.4.10, p. 287].
   - That members of Σ have complete factorizations: only in Claim B.
   - The base field and its characteristic play no role.
   - Without complete factorizations the conclusion cannot even be phrased as stated: in a commutative valuation ring R with value group $\mathbb{Q} \times \mathbb{Z}$ (lexicographic), $\Sigma = \{q\}$ with $v(q) = (1, 0)$ and $H = p$ with $v(p) = (0, 1)$, we have $p^{-1} = (q/p)\, q^{-1} \in U_\Sigma$ while q has no complete factorization (although p divides q). We do not know whether the weaker conclusion (F is stably associated with an atom dividing some $Q \in \Sigma$) holds for every semifir.
2. *Inverting in D versus in R_Σ.* In D a square factor of an inverted matrix is inverted (Step 1). In the universal localization R_Σ this fails in general. Take $R = k\langle x, y\rangle$, $\Sigma = \{xy\}$: R_Σ contains the idempotent $yux \neq 0, 1$ [Coh06, p. 500], so x is not invertible in R_Σ (if it were, with $u = (xy)^{-1}$ we would get $yux = 1$). This is why Cohn assumes Σ factor-inverting or factor-complete in [Coh06, §7.11]. We never pass through R_Σ, so no such assumption is needed.
3. *Sketches of other proofs.* For $R = P$ over $\mathbb{C}$ there is an analytic proof: [KVV12] gives the inclusion of free loci, and [HKV22, Thm 2.12] together with Claim A gives the conclusion.

3.2 Smith profiles

Let M be a matrix over $\mathbb{C}[[t]]$ with invariant factors $t^{\lambda_1}, t^{\lambda_2}, \ldots$, $\lambda_i \in \mathbb{N} \cup \{\infty\}$ (zero invariant factors count as ∞). For $j \geq 1$ put

$$\nu_j(M) = \#\{i : \lambda_i \geq j\} = \mu\big(t^{j-1} \operatorname{coker} M\big),$$

the minimal number of generators. By *the parts* of M we mean the λ_i. Unimodular equivalence over $\mathbb{C}[[t]]$ and padding with an identity block do not change ν_j for $j \geq 1$.

For a ring homomorphism $\gamma : A \to M_m(\mathbb{C}[[t]])$ (an *arc*) and a matrix F over A we write F^γ for the entrywise evaluation, an $m \cdot \#\text{rows}$ by $m \cdot \#\text{columns}$ matrix over $\mathbb{C}[[t]]$. An arc on A restricts to a ring homomorphism $P \to M_m(\mathbb{C}[[t]])$, which we also call an arc. Theorem 3.4 needs only these restrictions, because every matrix it evaluates is over P. The arcs of §4.1 are defined on A: the images of the x_j are invertible. Section 4.1 uses this to evaluate matrices over A and the triangular pivots of the lifts; Theorem 3.4 does not.

Let γ be an arc.

Lemma 3.2. (cf. [Tho79]; we give a direct proof) p. 11

Let $j \geq 1$.

1. *(Factors.) If $Y = G_1 F G_2$ with all matrices square of the same size, then $\nu_j(F^\gamma) \leq \nu_j(Y^\gamma)$, whether or not G_1^γ, G_2^γ are singular.*

2. *(Upper interlacing.) If* $Y = \begin{pmatrix} L & \boldsymbol{\omega} \\ \boldsymbol{\alpha} & \delta \end{pmatrix}$ *with a bottom block of size* m *after evaluation, then* $\nu_j(Y) \le \nu_j(L) + m$.
3. *(Lower interlacing.) In the same situation,* $\nu_j(L) - m \le \nu_j(Y)$.
4. *(Chains.) If* L *is block upper triangular with diagonal blocks* $\Delta_1, \dots, \Delta_q$*, each with* $\det \Delta_i \not\equiv 0$ *and largest part* e_i*, then every part of* L *is at most* $e_1 + \cdots + e_q$*. If* L *is block diagonal, its largest part is the largest over the blocks.*

🆕 **Classical and new:** cf. [Tho79]; the paper gives a direct proof.

↳ Used in: 3.3 (p. 15), Lemma 4.4 (p. 23), Lemma 5.6 (p. 29)

Proof

(1) $\operatorname{im}(G_1 F G_2) \subseteq \operatorname{im}(G_1 F)$, so $\operatorname{coker}(G_1 F)$ is a quotient of $\operatorname{coker} Y$; likewise $\operatorname{coker} F^{\mathsf{T}}$ is a quotient of $\operatorname{coker}(F^{\mathsf{T}} G_1^{\mathsf{T}})$, and transposition does not change Smith forms; a quotient does not increase $\mu(t^{j-1}\cdot)$. (2) $\operatorname{coker} Y$ maps onto $\operatorname{coker}[L\ \boldsymbol{\omega}]$ with kernel generated by m elements, and $\operatorname{coker}[L\ \boldsymbol{\omega}]$ is a quotient of $\operatorname{coker} L$; over a discrete valuation ring a submodule of an m-generated module is m-generated. (3) $\operatorname{coker}[L\ \boldsymbol{\omega}] = \operatorname{coker} L / \operatorname{im} \boldsymbol{\omega}$ and $t^{j-1}(\mathcal{X}/\mathcal{X}') \cong t^{j-1}\mathcal{X}/(t^{j-1}\mathcal{X} \cap \mathcal{X}')$ with $t^{j-1}\mathcal{X} \cap \mathcal{X}'$ at most m-generated, so $\mu(t^{j-1}\cdot)$ drops by at most m; then $\operatorname{coker} Y \twoheadrightarrow \operatorname{coker}[L\ \boldsymbol{\omega}]$. (4) Since each Δ_i is injective, $\operatorname{coker} L$ is an iterated extension of the $\operatorname{coker} \Delta_i$, and $t^{e_1+\cdots+e_q}$ kills it. □

Evidence & scope

- **Program:** nu_j does not increase on factors: 480 random pairs, with a control (supplement C01). `scripts/auditoria_S23/s23_monotonia.py`

(All matrices here are evaluated at γ; we drop the superscript.)

3.3 Monotone invariants and the criterion

Definition 3.3. p. 11

A *monotone invariant* is a function μ from square matrices over P to $\mathbb{N} \cup \{\infty\}$ such that

1. $\mu(F \oplus I) = \mu(F)$ for identity matrices I of every size, and $\mu(MFN) = \mu(F)$ for M, N invertible over P of the size of F;
2. $\mu(F) \le \mu(G_1 F G_2)$ whenever F, G_1, G_2 are square of the same size.

Examples.

- 1. For an arc (a unital ring homomorphism) $\gamma : P \to M_m(\mathbb{C}[[t]])$ and $j \ge 1$, $\mu(F) = \nu_j(F^\gamma)$. Condition 1 holds because M^γ, N^γ are invertible over $\mathbb{C}[[t]]$ and padding does not change ν_j; condition 2 is Lemma 3.2(1).
- 1. For a unital ring homomorphism $\Theta : P \to M_m(K)$ into matrices over a field (not necessarily commutative), $\mu(F) = \operatorname{corank} F^\Theta$. Condition 1 holds because invertible matrices go to invertible matrices; condition 2 holds because $\operatorname{rank}(XYZ) \le \operatorname{rank} Y$.

Neither example needs the arc or Θ to extend to D, or even to A. Example (b) is m times the nullity of the Sylvester rank function $F \mapsto \operatorname{rank} F^\Theta / m$, and every Sylvester rank function with values in $\frac{1}{m}\mathbb{Z}$ gives a monotone invariant in the same way [Mal80], [Sch85, Ch. 7]. Example (a) is a difference of two instances of (b): if $\Theta_j : P \to M_{mj}(\mathbb{C})$ is the reduction of γ modulo t^j followed by the regular representation of $\mathbb{C}[t]/t^j$, then $\nu_j(F^\gamma) = \operatorname{corank} F^{\Theta_j} - \operatorname{corank} F^{\Theta_{j-1}}$, because multiplication by t^λ on $\mathbb{C}[t]/t^j$ has kernel of dimension $\min(\lambda, j)$, so $\operatorname{corank} F^{\Theta_j} = \sum_i \min(\lambda_i, j)$. Its monotonicity is Lemma 3.2(1); it does not follow from that of (b).

↳ Used in: 8.5 (p. 56)

Theorem 3.4. p. 11

> *Let Σ be a family of full matrices over P and H a full square matrix over P with an atomic factor G in a complete factorization. Suppose that for every $Q \in \Sigma$ there is a monotone invariant μ_Q with $\mu_Q(G) > \mu_Q(Q)$. Then H^{-1} does not have entries in U_Σ.*

Lean · ours, in part show

with the conclusion of Lemma 3.1 as a hypothesis

- `ThreeAxes.Criterion.not_inverseInU`

✅ Checked when this page was built: Criterion.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — not_inverseInU: propext, Classical.choice, Quot.sound.

Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean Criterion.lean` (SHA-256 of the compiled file: `b3acc608d6445b7e`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 1 result, standard axioms only

Proof: in this note · **Covers:** not formalized · **Cite:** p. 3, doi:10.5281/zenodo.23481332

↳ Used in: Theorem A (p. 4), Theorem 5.1 (p. 25), Theorem 5.2 (p. 26), Corollary 5.3 (p. 27), 5.9 (p. 35), Theorem 6.1 (p. 35), Theorem 7.2 (p. 41), Theorem 7.3 (p. 42), Corollary 7.10 (p. 48), 7.11 (p. 50), 7.12 (p. 51), 8.1 (p. 52), 8.2 (p. 53), 8.3 (p. 55), 8.4 (p. 56)

Proof

> Suppose that H^{-1} has entries in U_Σ. By Lemma 3.1 there are $Q \in \Sigma$ and an atomic factor of Q stably associated with G. In a complete factorization, $Q = G_1 \mathcal{A} G_2$ with $\mathcal{A}, G_1, G_2$ square of the size of Q and $\mathcal{A}$ an atom stably associated with that factor, hence with G [Coh06, Cor. 0.5.5]. Then $\mu_Q(G) = \mu_Q(\mathcal{A})$ by condition 1, and $\mu_Q(\mathcal{A}) \le \mu_Q(Q)$ by condition 2, a contradiction. □

The applications use three cases: $G = H$ with $\mu_Q = \nu_{j_Q}$ along an arc chosen for Q (Theorems 5.1, 5.2 with $r \ge 3$, Corollary 5.3, Theorem 7.2); $G = H$ with μ_Q the corank at one fixed representation for all Q (at χ for Theorem 5.2 with $r = 1$, at Θ for Theorem 6.1); and G an atomic factor that may be proper, with $\mu_Q = \nu_{j_Q}$ (Theorem 7.3) or the corank at χ (Corollary 7.10(a), Remark 7.11). In Remark 8.2 the invariant depends on the kind of Q: the corank at χ for some, ν_{j_Q} for the others. The invariant may depend on Q. In the applications the arc parameter k is chosen after Q, and the same μ_Q is applied to G and Q. Table 1 lists, for the main applications, the witness, the invariant and the two values that are compared.

Result	Witness	In	Not in	Invariant (Def. 3.3)	Target
Thm 5.2, $r = 1$	S_3^{-1}, $S_3 = u^{-1} + v^{-1} + w^{-1}$	$\mathcal{U}_1$	$E_1 = \mathcal{U}_0$	corank at χ (target $H_{3,1}$)	2
Thm 5.1	s^{-1}, $s = u^{-2} + v^{-2} + w^{-2}$	$\mathcal{U}_2$ (also D', Γ_2)	$\mathcal{U}_1$	ν_{k+1} on γ_k (target H_0)	4
Thm 5.2, $r \ge 3$	φ_r^{-1}, (5.1)	$\mathcal{U}_r$	$\mathcal{U}_{r-1}$	$\nu_{(r-1)k+1}$ on γ_k (target Q_r)	≥ 3

Cor 5.3	g_r^{-1}, $g_r = u^{-2r} + v^{-2r} + w^{-2r}$	$\mathcal{U}_{2r}$	$\mathcal{U}_r$	ν_{rk+1} on γ_k (target $H_{3,2r}$)	4
Thm 6.1	$\rho_d = (\hat{F}_d^{-1})_{11}$	$A_d \cap E_2$, B_d	W_{d-1} ($\supseteq A_{d-1}$, B_{d-1}; for $d=2$: $\mathcal{U}_\infty$)	corank at Θ over $\mathbb{C}(t)$ (target $\hat{F}_d$)	$2d$
Thm 7.2	ψ^{-1}, $\psi = u^{-2} + v^{-2} + w^{-2} + yu^{-2}y^{-1}$	$\mathcal{U}_2$	Γ_2' ($\supseteq \Gamma_2 \supseteq D'$)	ν_{k+1} on γ_k (target F_ψ)	6
Thm 7.3, $k \geq 4$	f_k^{-1}	$\mathcal{U}_2 \cap \Gamma'_{k-1}$, $\mathcal{U}_2 \cap A_{k-1}$ (Rem 7.11)	Γ'_{k-2}	$\nu_{\kappa+1}$ on γ_κ (atomic factor G_k of F_k)	$2(k-1)$
Cor 7.10(a), $d \geq 2$	$e_d = q_d^{-1}$, $q_d = w^{-1} + \sum_{i<d} y^i u^{-1} y^{-i}$	$\mathcal{U}_1 \cap A_d$	A_{d-1}	corank at χ (atomic factor G of L)	d
Rem 7.11, $k \geq 4$	f_k^{-1}	$\mathcal{U}_2 \cap A_{k-1}$	A_{k-2}	corank at χ (atomic factor G_k of F_k)	$k-1$
Rem 8.2, $r \geq 3$	φ_r^{-1}	$\mathcal{U}_r \cap A_5 \cap \Gamma'_4$	Γ_4 ($\supseteq A_4$)	corank at χ (lifts over A); $\nu_{(r-1)k+1}$ on γ_k (borders of $\mathcal{U}_1$); target M_r	5; ≥ 3

Table 1. Witnesses of the strict inclusions. In each row Theorem 3.4 is applied with the stated invariant: the target (an atom, or an atomic factor) beats every allowed denominator. The margin compares the target with the bound for the same μ_Q; in the rows with ν the arc parameter k (or κ) is chosen after Q, with $k > e(Q)$ ($e(Q)$ is defined in §4.1; and $k > r$ in the row of Theorem 5.2, $r \geq 3$).

Colour: blue shade = the number's size within its column.

3.4 Presentations of the rings

For $0 \neq s \in A$ let $Q_s = \begin{pmatrix} \tilde{s} & * \\ & T \end{pmatrix}$ be a polynomial matrix with T upper triangular, the x_j on its diagonal and polynomials above it, whose Schur complement is s; then T^{-1} has entries in A and the entries of Q_s^{-1} lie in E_1. (Several inverses in one monomial need a triangular, not diagonal, T: the $(1,2)$ entry of $\begin{pmatrix} x_1 & -x_2 \\ 0 & x_1 \end{pmatrix}^{-1}$ is $x_1^{-1}x_2x_1^{-1}$.) For instance, write s as a sum of products $a_0 x_{j_1}^{-1} a_1 \cdots x_{j_q}^{-1} a_q$ with $a_i \in P$, and take for Q_s the border B_{-s} defined below, built from these chains with the x_{j_i} in place of the binomials, with its rows and columns permuted; its Schur complement is s and its pivot is upper triangular with the x_j on the diagonal. The polynomial lift of a matrix over A is built in the same way, entry by entry, as for B_G below. With $\Sigma_0 = \{Q_s : 0 \neq s \in A\} \cup \{x, y, z\}$ we get $E_1 = U_{\Sigma_0}$.

For $0 \neq f \in \mathcal{F}_r$ fix one presentation $f = a_* + \sum_\tau a_{\tau,0}\ell_{\tau,1}^{-1}a_{\tau,1}\cdots\ell_{\tau,q_\tau}^{-1}a_{\tau,q_\tau}$ ($a_* \in A$, $q_\tau \leq r$ the number of inverses in the term τ) and form the *border*

$$B_f = \begin{pmatrix} L & \boldsymbol{\omega} \\ \boldsymbol{\alpha} & -a_* \end{pmatrix}, \qquad L = \bigoplus_\tau C_\tau,$$

where the *chain* C_τ is the $q_\tau \times q_\tau$ upper bidiagonal matrix with $\ell_{\tau,1}, \ldots, \ell_{\tau,q_\tau}$ on the diagonal and $-a_{\tau,1}, \ldots, -a_{\tau,q_\tau-1}$ above it, the row $\boldsymbol{\alpha}$ carries $a_{\tau,0}$ at the first position of chain τ and the column $\boldsymbol{\omega}$ carries a_{τ,q_τ} at its last position. Since $(C_\tau^{-1})_{1,q_\tau} = \ell_{\tau,1}^{-1}a_{\tau,1}\ell_{\tau,2}^{-1}\cdots\ell_{\tau,q_\tau}^{-1}$, the Schur complement of L in B_f is $-f$. B_f has entries in A; let $\tilde{B}_f$ be its polynomial lift, of which B_f is the Schur complement with respect to a triangular pivot with the x_j on the diagonal, as for Q_s.

Lemma 3.5. p. 13

> $\mathcal{U}_r = U_{\Sigma_r}$ *with* $\Sigma_r = \Sigma_0 \cup \{\tilde{B}_f : 0 \neq f \in \mathcal{F}_r\}$, *one border per* f.

Lean · ours, in part show

the chain inverse

- `ThreeAxes.InverseIdentities.chain_inv_corner`

✅ Checked when this page was built: InverseIdentities.lean compiles with Lean 4.35.0, no errors, no sorry.
Axioms — chain_inv_corner: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean InverseIdentities.lean` (SHA-256 of the compiled file: `3e2df75becd9213c`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 1 result, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** part: the chain inverse · **Cite:** Lemma 3.5, p. 13, doi:10.5281/zenodo.23481332

↳ Used in: Theorem 5.1 (p. 25), Corollary 5.3 (p. 27), Theorem 6.1 (p. 35), 8.2 (p. 53)

Proof

> $\mathcal{U}_r \subseteq U_{\Sigma_r}$: U_{Σ_r} is a ring containing E_1 and f^{-1}, since $-f^{-1}$ is a corner entry of $\tilde{B}_f^{-1}$.
> $U_{\Sigma_r} \subseteq \mathcal{U}_r$: the entries of $\tilde{B}_f^{-1}$ are polynomials in the entries of L^{-1} (whose entries lie in E_1, because $\ell^{-1} \in E_1$ and $a_i \in A$), in f^{-1} and in the entries of the inverse pivot, which lie in A. □

Evidence & scope

- **Program:** chain inverse and Schur(B_f) = -f with random non-commuting blocks, with controls (supplement C02). `scripts/auditoria_S12/s12_cadena_schur.py`

Only one $Q \in \Sigma_r$ at a time enters the argument below: sums and products of borders never appear, because Lemma 3.1 returns a single Q.

Presentations of W_b. For a full $G \in M_a(R_{\mathrm{bin}})$ write each entry as above and form the border

$$B_G = \begin{pmatrix} L & \Omega \\ \mathcal{A} & -G_* \end{pmatrix},$$

with L the direct sum of the chains of all entries, Ω (a columns) and $\mathcal{A}$ (a rows) carrying the boundary coefficients and $G_* \in M_a(A)$ the polynomial parts, so that the Schur complement of L is $-G$; let $\tilde{B}_G$ be its polynomial lift with pivots x_j. As in Lemma 3.5, $W_b = U_{\Sigma^{(b)}}$ with $\Sigma^{(b)} = \Sigma_0 \cup \{\tilde{B}_G : G \text{ full over } R_{\mathrm{bin}},\ a \leq b\}$, one border per G.

Presentations over V_1. Let $M \in M_a(V_1)$ be full, with entries $M_{ij} = a_{ij} + \sum_{s \in S_{ij}} b_s \ell_s^{-1} c_s$ ($a_{ij}, b_s, c_s \in A$, binomials ℓ_s, repetitions allowed). Put $m = \sum |S_{ij}|$, $\Delta = \operatorname{diag}(\ell_1, \dots, \ell_m)$ and

$$B_M = \begin{pmatrix} M_0 & R \\ -C & \Delta \end{pmatrix}, \qquad M_0 = (a_{ij}),\ R_{i,s} = b_s\ (s \in S_{i\cdot}),\ C_{s,j} = c_s\ (s \in S_{\cdot j}),$$

so that $M = M_0 + R\Delta^{-1}C$ is the Schur complement of Δ and M^{-1} is the top-left $a \times a$ block of B_M^{-1}. Let $\tilde{B}_M$ be the polynomial lift with pivots x_j.

Lemma 3.6. p. 13

> $\Gamma_2' = U_{\Sigma_2'}$ *with* $\Sigma_2' = \Sigma_1 \cup \{\tilde{B}_M : M \text{ full over } V_1,\ a \le 2\}$, *one lift per* M.

↳ Used in: 5.9 (p. 35), Theorem 7.2 (p. 41), Theorem 7.3 (p. 42)

Proof

> $\tilde{B}_M$ is full, because M is full and Δ is invertible over D. $\Gamma_2' \subseteq U_{\Sigma_2'}$: $U_{\Sigma_2'} \supseteq U_{\Sigma_1} = \mathcal{U}_1$; the block of $\tilde{B}_M^{-1}$ complementary to the triangular pivot is B_M^{-1}, and the $(1,1)$ block of B_M^{-1} is M^{-1} (Schur complement of Δ). $U_{\Sigma_2'} \subseteq \Gamma_2'$: the entries of $\tilde{B}_M^{-1}$ are polynomials in those of B_M^{-1}, of the inverse pivot (in A) and of elements of A, and
>
> $$B_M^{-1} = \begin{pmatrix} M^{-1} & -M^{-1}R\Delta^{-1} \\ \Delta^{-1}CM^{-1} & \Delta^{-1} - \Delta^{-1}CM^{-1}R\Delta^{-1} \end{pmatrix}$$
>
> with $\Delta^{-1} \in M_m(E_1) \subseteq M_m(\mathcal{U}_1)$ and R, C over A. □

Here and below R, C denote the coefficient blocks of B_M.

Neither "one border per f" nor the other conditions of Lemmas 3.5 and 4.4 enter Theorem 3.4: they are needed only to produce Σ and to bound $\mu_Q(Q)$.

4 Two evaluations

Question. Theorem 3.4 needs, for each allowed denominator Q, an invariant that the target exceeds. Which evaluations make the targets large and every allowed denominator small? The targets are built from squares, the denominators from binomials: we want the squares to degenerate together and the binomials not to. Section 4.1 makes all squares the same central scalar along an arc; Section 4.2 gives a point where no binomial degenerates at all.

4.1 The exact arc

The arc. For $i \in \{0,1,2\}$, t and ε put

$$U_i(t,\varepsilon) = \begin{pmatrix} it^2 & -t \\ i^2t^3 - \varepsilon & -it^2 \end{pmatrix} \in M_2(\mathbb{C}[t,\varepsilon]).$$

For $k \ge 1$ the *exact arc* γ_k sends u, v, w to U_0, U_1, U_2 at $\varepsilon = t^k$; equivalently $x \mapsto U_0 - I$, $y \mapsto U_1 - I$, $z \mapsto U_2 - I$. In Lemma 4.3 and in the proof of Lemma 5.6 we index the generators from 0: $x_0 = x$, $x_1 = y$, $x_2 = z$, and $u_j = 1 + x_j$; these indices, and those of U_i, N_i and X_i below (Remarks 6.3, 6.4), are taken mod 3; elsewhere (as in $H_{n,p}$ below) $u_1 = u$, $u_2 = v$, $u_3 = w$. At $t = 0$ every x_j goes to $-I$, which is invertible, so γ_k extends to $A \to M_2(\mathbb{C}[[t]])$: Laurent polynomials

evaluate integrally. At $\varepsilon = 0$, $U_i(t,0) = I - K^i J K^{-i}$ with $J = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $K = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, i.e. $x_i \mapsto \rho_t(x_i) = -K^i J K^{-i}$, the parabolic family. This holds for every ε:

$$U_i(t,\varepsilon) = K^i\, U_0(t,\varepsilon)\, K^{-i}, \qquad U_0(t,\varepsilon) = -(tE_{12} + \varepsilon E_{21}),$$

with E_{ab} the matrix units. So the exact arc deforms the nilpotent $-tE_{12} = I - J$ of the parabolic family into $-(tE_{12} + \varepsilon E_{21})$ on the three axes at once; since K commutes with E_{21}, $U_i = -K^i(tE_{12})K^{-i} - \varepsilon E_{21}$, the form used in the proofs of Lemmas 4.3 and 5.6.

Lemma 4.1. (exact arc) p. 14

> $\operatorname{tr} U_i = 0$, $\det U_i = -t\varepsilon$ *and* $U_i^2 = t\varepsilon\, I$. *In particular* $U_i(t,t^k)^2 = t^{k+1} I$, *and* $U_i(t,t^k)$ *has Smith parts* $(1,k)$.

Lean · ours, in part **show**

trace, determinant and square (not the Smith parts)

- `ThreeAxes.ExactArc.exactArc_trace`
- `ThreeAxes.ExactArc.exactArc_det`
- `ThreeAxes.ExactArc.exactArc_sq`
- `ThreeAxes.ExactArc.exactArc_sq_pow`
- `ThreeAxes.ParabolicFamily.exactArc_eq_conj`

✅ Checked when this page was built: ExactArc.lean, ParabolicFamily.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — exactArc_trace, exactArc_det, exactArc_sq, exactArc_sq_pow, exactArc_eq_conj: propext, Classical.choice, Quot.sound.

Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean ExactArc.lean ParabolicFamily.lean` (SHA-256 of the compiled file: `56b3b2ae3774446a…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 5 results, standard axioms only

Proof: in this note · **Covers:** not formalized · **Cite:** Lemma 3.6, p. 13, doi:10.5281/zenodo.23481332

↳ Used in: Lemma 4.2 (p. 21), Lemma 4.3 (p. 22), Proposition 5.8 (p. 34), Lemma 7.8 (p. 45), Lemma 7.9 (p. 47)

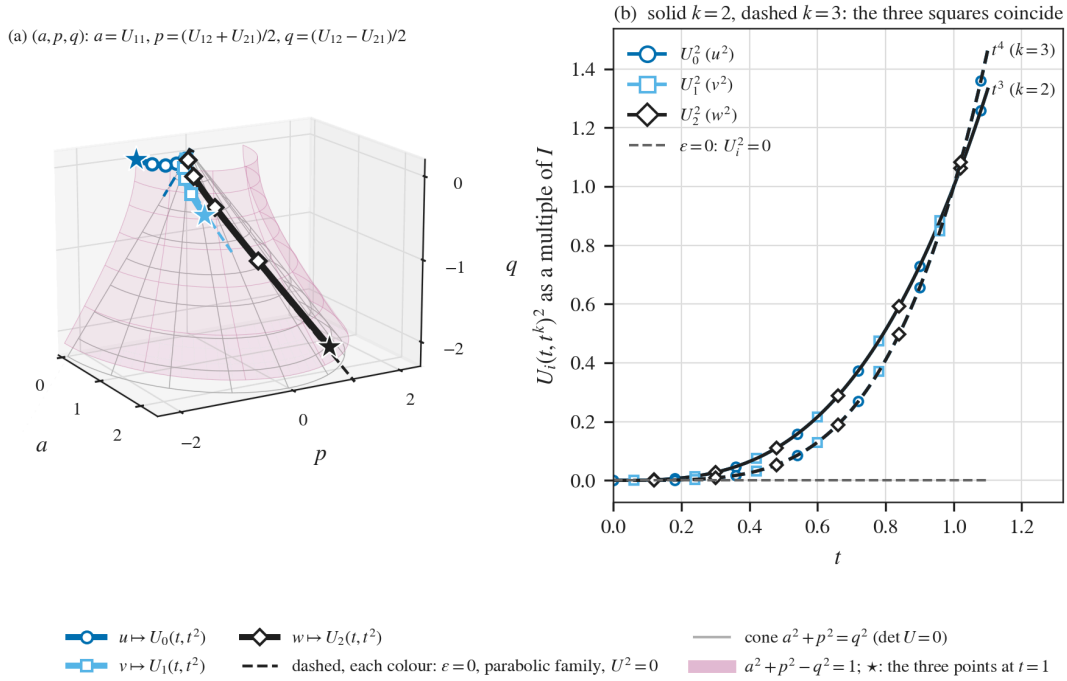
Proof

> U_0 has zero trace, determinant $-t\varepsilon$ and $U_0^2 = t\varepsilon I$, and conjugation by K^i preserves all three. The parts: every entry is divisible by t (since $k \geq 1$), the $(1,2)$ entry is $-t$, and $\det = -t^{k+1}$. □

Evidence & scope

- **Program:** U_i(t,0) = I - K^i J K^-i, trace, determinant and U_i^2, exactly (supplement C03). `scripts/smith_V1/t158_arco_exacto.py`

Figure 2



The exact arc. A traceless 2×2 matrix U is a point (a,p,q) with $a=U_{11}$, $p=(U_{12}+U_{21})/2$, $q=(U_{12}-U_{21})/2$, and $U^2=(a^2+p^2-q^2)I$. (a) The images of u,v,w for $\varepsilon=t^2$ (solid) leave the cone $a^2+p^2=q^2$ of nilpotent matrices, on which the parabolic family $\varepsilon=0$ lies (dashed); at each t the three points lie on the same quadric $a^2+p^2-q^2=t^3$, shown at $t=1$ (stars). (b) The scalar λ with $U_i(t,t^k)^2=\lambda I$ for $k=2,3$: the three curves coincide, $\lambda=t^{k+1}$ (Lemma 4.1). Live version, in 3D and for any k, in the blueprint.

Scalar evaluation. Let $\mathcal{J}=(u^2,v^2,w^2)$, the two-sided ideal of A, and for $n\geq 2$, $p\geq 1$ let

$$H_{n,p}=u_1^p\,\mathbf{1}\mathbf{1}^{\mathsf{T}}+\operatorname{diag}(u_2^p,\dots,u_n^p)\qquad((n-1)\times(n-1)),$$

so that $H_{3,2r}=\begin{pmatrix}u^{2r}+v^{2r} & u^{2r}\\ u^{2r} & u^{2r}+w^{2r}\end{pmatrix}$ and $H_0:=H_{3,2}$.

Lemma 4.2. (scalar evaluation) p. 14

> *Let $\gamma=\gamma_k$.*
>
> 1. *If F is a $d\times d$ matrix with entries in $\mathcal{J}^r$, every entry of F^γ is divisible by $t^{r(k+1)}$; hence $\nu_{r(k+1)}(F^\gamma)=2d$.*
> 2. $H^\gamma_{3,2r}=t^{r(k+1)}\left(\begin{pmatrix}2&1\\1&2\end{pmatrix}\otimes I_2\right)$. *Its four parts all equal $r(k+1)$.*

Lean · ours, in part show

added to the Lean repository after publication (21-oct-2026): part (2) for every r: H_{3,2r} = s^r • [[2,1],[1,2]] when u², v², w² equal s • 1 (the case r general was not stated in Lean)

- `ThreeAxes.ExactArc.H_eval_pow`
- `ThreeAxes.ExactArc.H0_eval`
- `ThreeAxes.ExactArc.prod_scalar_interleave`

✅ Checked when this page was built: ExactArc.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — H_eval_pow, H0_eval: propext, Classical.choice, Quot.sound; prod_scalar_interleave: propext, Quot.sound. Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean ExactArc.lean` (SHA-256 of the compiled file: `56b3b2ae3774446a`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 3 results, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** part: added to the Lean repository after publication (21-oct-2026): part (2) for every r: H_{3,2r} = s^r • [[2,1],[1,2]] when u², v², w² equal s • 1 (the case r general was not stated in Lean) · **Cite:** Lemma 4.2, p. 14, doi:10.5281/zenodo.23481332

↳ Used in: Theorem 5.1 (p. 25), Corollary 5.3 (p. 27), 8.1 (p. 52)

Proof

(1) A monomial of $\mathcal{J}^r$ is $a_0\,u_{i_1}^2 a_1 \cdots u_{i_r}^2 a_r$ with $a_i \in A$; each u_i^2 evaluates to the central scalar $t^{k+1}I$ (Lemma 4.1) and each a_i integrally. (2) Apply the same to U_i^r, whose square is $t^{r(k+1)}I$; $\det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3 \neq 0$. □

Binomials on the arc. For a binomial ℓ let $e_\ell \in \mathbb{N} \cup \{\infty\}$ be the largest Smith part of ℓ^{ρ_t}, where ρ_t is the parabolic family ($\varepsilon = 0$: $x_j \mapsto -K^j J K^{-j}$); $e_\ell = 0$ if ℓ^{ρ_t} is invertible.

Lemma 4.3. (binomials) p. 14

Let $\ell = 1 - \beta h$ be a binomial.

1. *If $\beta\chi(h) \neq 1$, then ℓ^{γ_k} is invertible over $\mathbb{C}[[t]]$: no positive parts.*
2. *If ℓ is exceptional, ℓ^{γ_k} has parts $(1, k)$ for every $k \geq 1$.*
3. *Otherwise $e_\ell < \infty$, and for $k > e_\ell$ the parts of ℓ^{γ_k} equal those at $\varepsilon = 0$; in particular they are $< k$.*

Lean · ours, in part **show**

the trace of products of the N_j (core of (3))

- `ThreeAxes.ParabolicFamily.trace_prod_parabolic`

✅ Checked when this page was built: ParabolicFamily.lean compiles with Lean 4.35.0, no errors, no sorry.
Axioms — trace_prod_parabolic: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean ParabolicFamily.lean` (SHA-256 of the compiled file: `70adc1a90a531fd1…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 1 result, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** part: the trace of products of the N_j (core of (3)) · **Cite:** Lemma 4.3, p. 14, doi:10.5281/zenodo.23481332

↳ Used in: Lemma 4.4 (p. 23), Proposition 5.8 (p. 34), 5.9 (p. 35), Lemma 7.7 (p. 45), 7.12 (p. 51), 8.1 (p. 52)

Proof

(1) At $t = 0$, $\ell \mapsto (1 - \beta\chi(h))I$. (2) Write $\ell = g(1 - \beta x_j^n)g^{-1}$ with $\beta(-1)^n = 1$. For $n<0$ use $1 - \beta x_j^n = -\beta x_j^n(1 - \beta^{-1}x_j^{-n})$, a unit times a binomial of the same kind; so let $n>0$. The roots of $1 - \beta X^n$ are simple, so $1 - \beta X^n = (1 + X)c(X)$ with $c(-1) \neq 0$; since $x_j^\gamma \to -I$, $c(x_j)^\gamma$ and g^γ are units, and the parts are those of $u_j^\gamma = U_j$, namely $(1, k)$ (Lemma 4.1). (3) After conjugation, which changes ℓ^γ by a unit, write $h = x_{j_1}^{n_1} \cdots x_{j_q}^{n_q}$ cyclically reduced. If $q = 1$, then $\beta\chi(h) = 1$ makes ℓ exceptional, so $q \geq 2$ and $j_{i+1} \neq j_i$ for all i, indices mod q. Write $N = tE_{12}$ and $N_j = K^j N K^{-j} = t\,c_j r_j$ with $c_j = (1, jt)^{\mathsf{T}}$ and $r_j = (-jt, 1)$. Then $r_a c_b = (b - a)t$, $N_j^2 = 0$, and $\rho_t(x_j)^n = (-1)^n(I + nN_j)$ for $n \in \mathbb{Z}$. Expanding the product,

$$\operatorname{tr} \rho_t(h) = \chi(h) \sum_S \Big(\prod_{i \in S} n_i \Big) \operatorname{tr} \prod_{i \in S} N_{j_i},$$

over the subsets $S=\{i_1<\cdots<i_m\}$ of $\{1,\dots,q\}$, with the factors in increasing order. For $m\ge 1$,

$$\operatorname{tr}\bigl(N_{j_{i_1}}\cdots N_{j_{i_m}}\bigr)=t^m\prod_{l=1}^{m}r_{j_{i_l}}c_{j_{i_{l+1}}}=t^{2m}\prod_{l=1}^{m}\bigl(j_{i_{l+1}}-j_{i_l}\bigr)\qquad(i_{m+1}=i_1),$$

which vanishes for $m=1$. Hence $\operatorname{tr}\rho_t(h)$ is a polynomial in t^2 of degree q with leading coefficient $\chi(h)\prod_i n_i\prod_i(j_{i+1}-j_i)\neq 0$ (*trace law*; compare [LU69] for two parabolic generators). Since $\det\rho_t(x_j)=1$, $\det(I-\beta\rho_t(h))=1-\beta\operatorname{tr}\rho_t(h)+\beta^2$ is a nonconstant polynomial, hence $\not\equiv 0$, and $e_\ell<\infty$. Finally $\ell^{\gamma_k}=\ell^{\gamma}|_{\varepsilon=0}+O(t^k)$, and a perturbation $O(t^k)$ of a matrix whose parts are all $<k$ does not change them ($D+O(t^k)=D(I+O(t))$). □

Evidence & scope

- **Program:** leading coefficient of tr rho_t(h) for 60 words, both signs (supplement C04). `scripts/auditoria_S12/s12_traza.py`

For a border B_f let $e(f)$ be the maximum of e_ℓ over the non-exceptional binomials ℓ with $\beta\chi(h)=1$ occurring in it. For a presentation B_M over V_1 (§3.4) let $e(M)$ be the maximum of e_ℓ over the non-exceptional ℓ_s with $\beta_s\chi(h_s)=1$. For $Q\in\Sigma$ write $e(Q)=e(f)$ if $Q=\tilde B_f$, $e(Q)=e(M)$ if $Q=\tilde B_M$, and $e(Q)=0$ otherwise; a maximum over no binomials is 0. The hypothesis $k>e(f)$ cannot be dropped from the bound on the parts of L in Lemma 4.4: with $k=5$, borders containing $1-[[x,y],z]$ ($e=7$) already have a part $>2k$ in a chain of length 2; this does not contradict anything, because k is chosen after Q.

Lemma 4.4. (chain bound) p. 16

Let $0\neq f\in\mathcal{F}_q$ with border B_f and $k>e(f)$. Then every part of L^{γ_k} is at most qk, and $\nu_{qk+1}(\tilde B_f^{\gamma_k})\le 2$.

↳ Used in: Theorem 5.1 (p. 25), Theorem 5.2 (p. 26), Corollary 5.3 (p. 27), 5.4 (p. 28), Proposition 5.8 (p. 34), 5.9 (p. 35), Theorem 7.2 (p. 41), 7.12 (p. 51), 8.1 (p. 52), 8.2 (p. 53)

Proof

Each chain C_τ has at most q binomials on its diagonal, each with $\det\not\equiv 0$ on the arc and largest part $\le k$ (Lemma 4.3). By Lemma 3.2(4) the parts of L^γ are at most qk, so $\nu_{qk+1}(L^\gamma)=0$. The pivots x_j of the lift evaluate to units, so $\tilde B_f^\gamma$ and B_f^γ have the same parts up to zeros. The border is one row and one column, i.e. $m=2$ after evaluation, and Lemma 3.2(2) gives $\nu_{qk+1}\le 0+2$. □

The remaining members of Σ_0 are harmless at every level $j\ge 1$: Q_s^γ has the parts of the 2×2 matrix s^γ (its pivot evaluates to a unit), so $\nu_j(Q_s^\gamma)\le 2$, and x_j^γ is a unit, so $\nu_j=0$.

4.2 The quaternion curve

Write $\xi=x-1$, $\eta=y-1$, $\zeta=z-1$ and $c=\xi\eta+\eta\xi$. Let $\mathbf{i}=\operatorname{diag}(\sqrt{-1},-\sqrt{-1})$, $\mathbf{j}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $\mathbf{k}=\mathbf{ij}$, and

$$\Theta:\ x\mapsto I+t\mathbf{i},\quad y\mapsto I+t\mathbf{j},\quad z\mapsto I+t\mathbf{k}\ \in M_2(\mathbb{C}[t]).$$

Since $\det(I+t\mathbf{q})=1+t^2$ for $\mathbf{q}\in\{\mathbf{i},\mathbf{j},\mathbf{k}\}$, Θ extends to a ring homomorphism $A\to M_2(\mathbb{C}(t))$, with $\Theta(x)^{-1}=(I-t\mathbf{i})/(1+t^2)$, regular at $t=0$ and $t=2$. We do **not** extend Θ to R_{bin} or E_1: the argument evaluates only matrices over A and their polynomial lifts over P. Under Θ, $\xi^2=\eta^2=\zeta^2=-t^2I$ and $c=t^2(\mathbf{ij}+\mathbf{ji})=0$.

Lemma 4.5. (no binomial degenerates on Θ) p. 16

> *For every $h \in F_3 \setminus \{1\}$ and $\beta \in \mathbb{C}^\times$, $\det\left(I - \beta\, h^{\Theta}\right) \neq 0$ in $\mathbb{C}(t)$. Hence every binomial, and every element $\alpha g + \beta h$ of A with two terms, is invertible at Θ.*

Lean · ours, in part show

the finite part (residues modulo 5); added to the Lean repository after publication (21-oct-2026): the 2 × 2 representation Θ of §4.2: det(I + tq) = 1 + t², (I + tq)(I − tq) = (1 + t²) I, ξ² = −t² I and c ↦ 0

- `ThreeAxes.QuaternionResidues.res_mul_res_eq_zero_iff`
- `ThreeAxes.QuaternionResidues.prod_res_ne_zero`
- `ThreeAxes.QuaternionTheta.det_one_add_smul_mi`
- `ThreeAxes.QuaternionTheta.det_one_add_smul_mj`
- `ThreeAxes.QuaternionTheta.det_one_add_smul_mk`
- `ThreeAxes.QuaternionTheta.one_add_smul_mul_one_sub_smul`
- `ThreeAxes.QuaternionTheta.theta_c_eq_zero`
- `ThreeAxes.QuaternionTheta.smul_mul_smul_self`

✅ Checked when this page was built: QuaternionResidues.lean, QuaternionTheta.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — res_mul_res_eq_zero_iff, prod_res_ne_zero, det_one_add_smul_mi, det_one_add_smul_mj, det_one_add_smul_mk, one_add_smul_mul_one_sub_smul, theta_c_eq_zero, smul_mul_smul_self: propext, Classical.choice, Quot.sound.

Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean QuaternionResidues.lean QuaternionTheta.lean` (SHA-256 of the compiled file: `69a51a05a38ebe68…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 8 results, standard axioms only

Proof: in this note · **Covers:** not formalized · **Cite:** Lemma 4.4, p. 16, doi:10.5281/zenodo.23481332

🆕 **Classical and new:** the freeness of the group generated by 1+2i, 1+2j, 1+2k is classical [LPS88].

↳ Used in: Theorem 6.1 (p. 35), Lemma 6.2 (p. 37), 8.1 (p. 52)

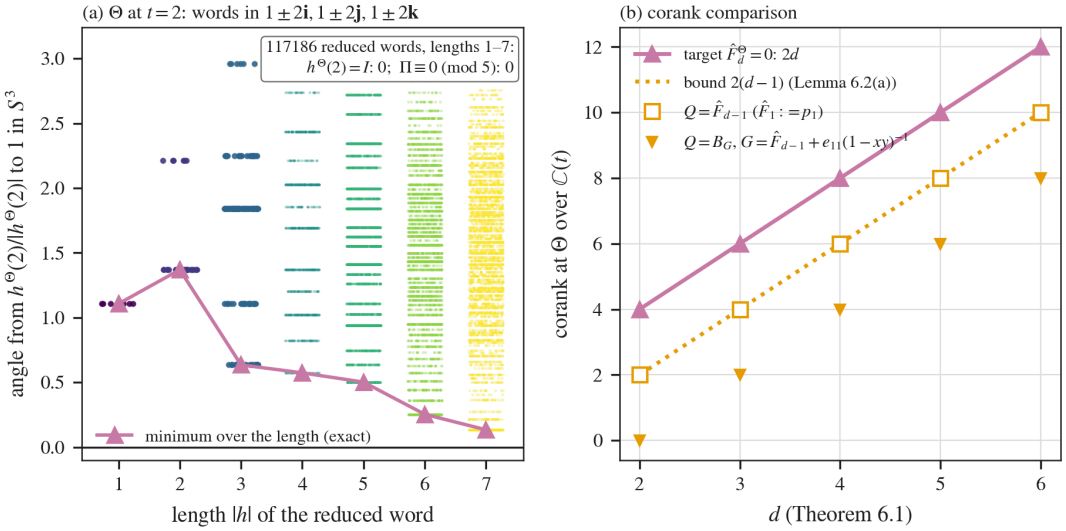
Proof

Entries of h^{Θ} are regular at $t = 0$ and $t = 2$, so evaluation there is a ring homomorphism. If $\beta \neq 1$: at $t = 0$, $h^{\Theta} = I$ and the determinant is $(1-\beta)^2 \neq 0$. If $\beta = 1$: at $t = 2$, $\Theta(x_j)$ is the Lipschitz quaternion $1 + 2\mathbf{q}_j$ of norm 5, and $5(1+2\mathbf{q}_j)^{-1} = 1 - 2\mathbf{q}_j$. For a reduced word h of length $|h|$ with n_- negative letters, $h^{\Theta}(2) = 5^{-n_-}\Pi$ with Π a product of $|h|$ elements of $\{1 \pm 2\mathbf{i}, 1 \pm 2\mathbf{j}, 1 \pm 2\mathbf{k}\}$, so $\operatorname{Norm}\Pi = 5^{|h|}$. Reduce modulo 5 through the ring homomorphism from the Lipschitz order to $M_2(\mathbb{F}_5)$ with $\mathbf{i} \mapsto \operatorname{diag}(2,3)$, $\mathbf{j} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The six residues have rank one, $M_a = u_a v_a^{\mathsf{T}}$, and $M_a M_b = (v_a^{\mathsf{T}} u_b)\, u_a v_b^{\mathsf{T}}$ vanishes exactly when b is the residue of the conjugate $1 \mp 2\mathbf{q}$ of $a = 1 \pm 2\mathbf{q}$ (a check of the 36 products; Appendix A.2, supplement C05). A reduced word has no consecutive inverse pair, so $\Pi \not\equiv 0 \pmod 5$. If $h^{\Theta}(2) = I$, then $\Pi = 5^{n_-} I$, $5^{2n_-} = 5^{|h|}$, $|h| = 2n_- \geq 1$, so $n_- \geq 1$ and $\Pi \equiv 0$, a contradiction. Finally, for a real quaternion $q \neq 1$, $\det(I - q) = \operatorname{Norm}(1-q) > 0$. Two-term elements are $\alpha g(1 + \alpha^{-1}\beta g^{-1} h)$ with $\det \Theta(g) \neq 0$. □

Evidence & scope

- **Program:** residues mod 5 (36 products) and 3008 random reduced words, with a control (supplement C05). `scripts/auditoria_S11/s11_a1_cuaterniones.py`

Figure 3



The quaternion curve Θ. (a) For reduced words h of length at most 7 in x, y, z and their inverses, the angle on S^3 between $h^\Theta(2)/|h^\Theta(2)|$ and 1 (all words of length ≤ 5; for lengths 6 and 7, the 200 words of smallest angle and a random sample of 3800), with the minimum for each length taken over all 117 186 words; no word gives $h^\Theta(2) = I$ and no product Π vanishes modulo 5, as Lemma 4.5 proves for every length. (b) The corank at Θ over $\mathbb{C}(t)$ of the target $\hat{F}_d$ is $2d$ (Lemma 6.2(b)), while every $Q \in \Sigma^{(d-1)}$ has corank at most $2(d-1)$ (Lemma 6.2(a)). Two sample denominators are shown: the lift of $\hat{F}_{d-1}$ (with $\hat{F}_1 := p_1$), which attains the bound, and the border of $\hat{F}_{d-1} + e_{11}(1-xy)^{-1}$. Live version, the words appearing on S^3 length by length, in the blueprint.

For Θ, Lemma 4.5 plays the role of Lemma 4.3: there is no exceptional set, and no parameter to choose after Q.

Section 6 uses Θ through Theorem 3.4 with μ the corank at Θ (Definition 3.3, Example (b), with $K = \mathbb{C}(t)$).

5 The binomial hierarchy (Theorem B(i))

Question. Does the number of inverse binomials allowed in the denominators give a strict hierarchy inside E_2? Yes, at every step. The arc of §4.1 does the counting: at level $k+1$ a quadratic target has four Smith parts, every allowed denominator at most two.

Theorem 5.1. p. 17

> *Let $s = u^{-2} + v^{-2} + w^{-2}$. Then $s^{-1} \in \mathcal{U}_2$ and $s^{-1} \notin \mathcal{U}_1 = E_1[V_1^{-1}]$, where $V_1 = A + \sum A\ell^{-1}A$.*

Lean · ours, in part **show**

the Woodbury reduction of §5.1; added to the Lean repository after publication (21-oct-2026): s^{-1} as minus the border corner of the inverse of the arrow matrix [[diag(u², v², w²), 1], [1, 0]]

- `ThreeAxes.InverseIdentities.woodbury_corner`
- `ThreeAxes.ArrowInstances.s_inv_eq_neg_corner`
- `ThreeAxes.ArrowInstances.inv_eq_neg_corner`

✅ Checked when this page was built: ArrowInstances.lean, InverseIdentities.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — woodbury_corner, s_inv_eq_neg_corner, inv_eq_neg_corner: propext, Classical.choice, Quot.sound.

Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean ArrowInstances.lean`

`InverseIdentities.lean` (SHA-256 of the compiled file: `3e2df75becd9213c…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 3 results, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** part: the Woodbury reduction of §5.1; added to the Lean repository after publication (21-oct-2026): s^{-1} as minus the border corner of the inverse of the arrow matrix [[diag(u², v², w²), 1], [1, 0]] · **Cite:** Theorem 5.1, p. 17, doi:10.5281/zenodo.23481332

↳ Used in: Theorem B (p. 5), Theorem 5.2 (p. 26), Proposition 5.8 (p. 34), Theorem 7.1 (p. 41), Theorem 7.2 (p. 41), Corollary 7.10 (p. 48), 7.12 (p. 51), 8.1 (p. 52), 8.2 (p. 53)

Proof

$s \in \mathcal{F}_2$ because $u^{-2} = u^{-1} \cdot 1 \cdot u^{-1}$, so $s^{-1} \in \mathcal{U}_2$. Suppose $s^{-1} \in \mathcal{U}_1$. By §5.1 and Lemma 3.5, H_0^{-1} has entries in U_{Σ_1}. We check the hypothesis of Theorem 3.4 with $G = H = H_0$ and, for each Q, $\mu_Q = \nu_{k+1}$ along γ_k (Example (a)), with k chosen after Q (the case $G = H$, $\mu = \nu_j$ of Theorem 3.4).

- *H_0 is a full atom over P.* It is $H_{3,2}$, an atom of $\mathbb{C}\langle u, v, w\rangle = P$ by Lemma 5.7 below (the case $n = 3$, $p = 2$).
- *Target.* By Lemma 4.2(2) with $r = 1$, $\nu_{k+1}(H_0^{\gamma_k}) = 4$ for every k.
- *Denominators.* For $Q = Q_s$ or x_j, $\nu_{k+1} \le 2$. For $Q = \tilde{B}_f$, $0 \neq f \in V_1 = \mathcal{F}_1$, choose $k > e(f)$; Lemma 4.4 with $q = 1$ gives $\nu_{k+1}(\tilde{B}_f^{\gamma_k}) \le 2$.

Since $4 > 2$, Theorem 3.4 gives $H_0^{-1} \notin M_2(U_{\Sigma_1})$, a contradiction. □

Figure 4 shows this comparison on computed examples, together with the corresponding comparisons in the proofs of Theorems 5.2, 7.2 and 7.3.

Evidence & scope

- **Program:** the Woodbury reductions of s^-1 and g_r^-1 (§5.1), with a control (supplement C06). `scripts/auditoria_S6/s6_woodbury.py`

Theorem 5.2. p. 17

For every $r \ge 1$, $\mathcal{U}_{r-1} \subsetneq \mathcal{U}_r$. For $r \ge 3$ an explicit witness is φ_r^{-1}, where $\varphi_r \in \mathcal{F}_r$ is the sum of five products of r inverses of u, v, w given in (5.1).

↳ Used in: Theorem B (p. 5), 8.2 (p. 53)

Proof

For $r = 1$: the element $S_3 = u^{-1} + v^{-1} + w^{-1} \in V_1 = \mathcal{F}_1$ has $S_3^{-1} \notin E_1 = \mathcal{U}_0$. This is Theorem 3.4 with μ the corank at χ (Example (b), a homomorphism $P \to M_1(\mathbb{C})$; the case $G = H$, $\mu =$ corank of Theorem 3.4), with target $H_{3,1} = \operatorname{diag}(v, w) + u\mathbf{1}\mathbf{1}^{\mathsf{T}}$ (an atom, Lemma 5.7(1) with $n = 3$, $p = 1$): at χ, $H_{3,1}$ vanishes and has corank 2, while every $Q \in \Sigma_0$ has corank ≤ 1 there (the pivot T of Q_s evaluates at χ to an invertible triangular matrix, since $x_j \mapsto -1$, so Q_s^χ is equivalent to $I \oplus s^\chi$; and $x_j^\chi = -1$). For $r = 2$: Theorem 5.1, since $s \in \mathcal{F}_2$. (The case $r = 1$ also follows from [HS15, Thm 4.6] by two pivots; see §8.8.)

For $r \ge 3$: $\varphi_r \in \mathcal{F}_r$, so $\varphi_r^{-1} \in \mathcal{U}_r$. Suppose $\varphi_r^{-1} \in \mathcal{U}_{r-1}$. By Lemma 5.5 and Lemma 3.5, Q_r^{-1} has entries in $U_{\Sigma_{r-1}}$. Apply Theorem 3.4 with $G = H = Q_r$ (a full atom, Lemma 5.7(2)) and $\mu_Q = \nu_j$ along γ_k, $j = (r-1)k + 1$: given $Q \in \Sigma_{r-1}$, choose $k > \max(e(Q), r)$. Then $\nu_j(Q_r^{\gamma_k}) \ge 3$ by Lemma 5.6, while $\nu_j(Q^{\gamma_k}) \le 2$ by Lemma 4.4 (with $q = r - 1$) or by the remark after it for $Q \in \Sigma_0$. Contradiction. □

Corollary 5.3. p. 17

> *For every $r \geq 1$, $g_r = u^{-2r} + v^{-2r} + w^{-2r}$ satisfies $g_r^{-1} \in \mathcal{U}_{2r} \setminus \mathcal{U}_r$.*

↳ Used in: 8.2 (p. 53)

Proof

> $g_r \in \mathcal{F}_{2r}$ (u^{-2r} is a product of $2r$ inverses of u), so $g_r^{-1} \in \mathcal{U}_{2r}$. Suppose $g_r^{-1} \in \mathcal{U}_r$. By §5.1 and Lemma 3.5, $H_{3,2r}^{-1}$ has entries in U_{Σ_r}. Apply Theorem 3.4 with $G = H = H_{3,2r}$ (an atom, Lemma 5.7(1)) and $\mu_Q = \nu_j$ along γ_k, $j = rk + 1$: for $Q \in \Sigma_r$ choose $k > e(Q)$. By Lemma 4.2(2) all four parts of $H_{3,2r}^{\gamma_k}$ equal $r(k+1) \geq rk + 1$, so $\nu_j = 4$, while $\nu_j(Q^{\gamma_k}) \leq 2$ by Lemma 4.4 with $q = r$, or by the remark after it for $Q \in \Sigma_0$. Contradiction. □

The inclusion $\mathcal{U}_r \subsetneq \mathcal{U}_{2r}$ follows from Theorem 5.2; the content of Corollary 5.3 is the witness, which is harmonic and needs no auxiliary design. Theorem 5.1 is the case $r = 2$ of Theorem 5.2; over a field of characteristic p its witness fails (Remark 8.1). All three proofs go through Theorem 3.4. The arc is the exact arc $U_i^2 = t^{k+1} I$ (Section 4.1): all squares u^2, v^2, w^2 evaluate to the same scalar $t^{k+1} I$, so the four parts of a quadratic target reach level $k + 1$, while the binomials of an allowed denominator stay below it and its border adds at most two.

5.1 Reduction

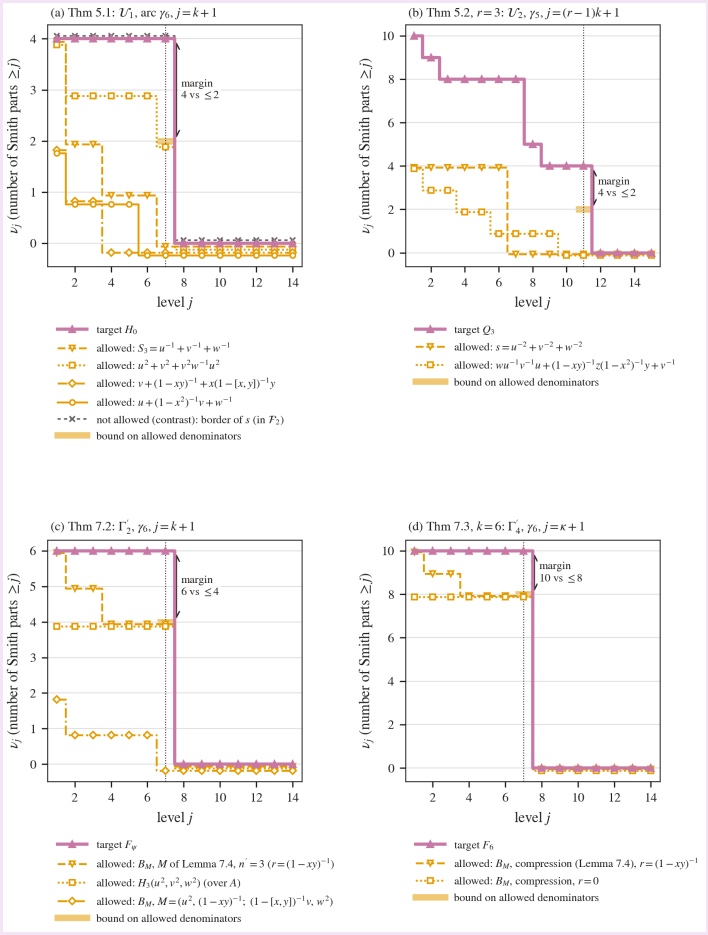
With $G = \operatorname{diag}(v^2, w^2)$ and $H_0 = G + u^2 \mathbf{1}\mathbf{1}^{\mathsf{T}}$, the Sherman–Morrison–Woodbury identity gives

$$H_0^{-1} = G^{-1} - G^{-1}\mathbf{1}\, s^{-1}\, \mathbf{1}^{\mathsf{T}} G^{-1}, \qquad s^{-1} = v^2 - v^2\, (H_0^{-1})_{11}\, v^2.$$

Hence, for any ring $\mathcal{U}$ with $E_1 \subseteq \mathcal{U} \subseteq D$: $s^{-1} \in \mathcal{U} \iff H_0^{-1} \in M_2(\mathcal{U})$. The same holds for g_r and $H_{3,2r}$, with u^{2r}, v^{2r}, w^{2r} in place of u^2, v^2, w^2, and for $S_3 = u^{-1} + v^{-1} + w^{-1}$ and $H_{3,1}$, with u, v, w.

5.2 Proof of Theorem 5.1

Figure 4



Smith profiles on the exact arc: ν_j is the number of Smith parts $\geq j$ (§3.2). In each panel the target has more parts at the marked level j than any allowed denominator can have (orange band: the bound of Lemma 4.4 in (a) and (b), of Lemma 7.7 in (c) and (d)). The orange curves are sample denominators, computed exactly over $\mathbb{F}_p[[t]]/t^N$ for two primes; they illustrate the bound and are not a maximum over Σ. Curves are offset vertically by at most 0.25 for legibility; all values are integers. (a) Theorem 5.1: target H_0 on γ_6; the grey curve, the border of $s \in \mathcal{F}_2$, is not an allowed denominator for $\mathcal{U}_1$. (b) Theorem 5.2 with $r=3$: target Q_3 on γ_5, level $(r-1)k+1=11$; here the target has 4 parts, Lemma 5.6 guarantees 3. (c) Theorem 7.2: target F_ψ on γ_6. (d) Theorem 7.3 with $k=6$ on γ_6: the curve is F_6, whose count at level $\kappa+1$ equals that of its atomic factor G_6 (Lemma 7.9(d)). Live version, each panel as a water level, in the blueprint.

The proof needs both halves of the comparison. If H is not an atom the count says nothing: $2u^{-2}+v^{-2}$ also gives 4 parts at level $k+1$, yet its inverse lies in E_1, because the matrix $\left(\begin{smallmatrix} 2u^2 & u^2 \\ u^2 & u^2+v^2 \end{smallmatrix}\right)$ of §5.1 for the terms u^{-2},u^{-2},v^{-2} is equivalent over P to $\mathrm{diag}(2u^2, \frac{1}{2}u^2+v^2)$. And with two axes the target is 1×1: u^2+v^2 has 2 parts at level $k+1$, against the bound 2, so nothing is excluded, as it must be, since $(u^{-2}+v^{-2})^{-1}=v^2(u^2+v^2)^{-1}u^2 \in E_1$.

Remark 5.4. (computations) p. 18

Computed exactly (rational arithmetic, truncated series) on the parabolic arc with a generic perturbation t^kE; we say a border is *excluded* when its count at level $k+1$ is at most 2. The border of s is never excluded and its profile contains that of H_0; the control $(u^{-2}+v^{-2}+w^{-1})^{-1}=u^2\Upsilon^{-1}v^2$ with $\Upsilon=u^2+v^2+v^2w^{-1}u^2 \in V_1$ has the target $H_c=\begin{pmatrix} u^2+v^2 & u^2 \\ u^2 & u^2+w \end{pmatrix}$, and the evaluated matrix $H_c^{\gamma_k}$ has the four parts $(1,k,k+1,k+1)$, only two at level $k+1$ (on γ_k this follows from $H_c^{\gamma_k} \sim \mathrm{diag}(2cI_2, W+\frac{c}{2}I_2)$ with $c=t^{k+1}$, $W=w^{\gamma_k}$ and $\det(W+\frac{c}{2}I_2)=-c+\frac{c^2}{4}$): as a target it would not exceed the bound on the allowed denominators, consistently with its inverse lying in $\mathcal{U}_1$; a border with twelve exceptional binomials, commutators of weight

2–4 that are trivial at χ, and Engel words is not excluded for $k = 10$ and is excluded for $k = 11, 12, 14$; here $e(f) = 11$ (the binomials $1 - [x, [x, y]]$ and $1 - [[x, y], [x, z]]$ have parts $(5, 11)$ and $(7, 11)$ under ρ_t), so the bound of Lemma 4.4 (proved on γ_k) predicts exclusion for $k \geq 12$, and at $k = 11$ the exclusion is a fact of the computation (supplement C07).

↳ Not used by another result of the paper.

Evidence & scope

- **Program:** profiles on the parabolic arc; the hard border (supplement C07). `scripts/smith_V1/parabolico_duro.py`

5.3 Consecutive strictness

Index the letters cyclically, $u \equiv 0$, $v \equiv 1$, $w \equiv 2 \pmod 3$. For $r \geq 3$ take five chains $\tau = 0, \ldots, 4$ with letter sequences $(\ell_{\tau,1}, \ldots, \ell_{\tau,r})$, $\ell_{\tau,i} \in \{u, v, w\}$ (each letter is itself a binomial, $u = 1 - (-1)x$):

- for $r = 3$: (u, u, u), (u, u, u), (v, v, v), (v, v, v), (w, w, w);
- for $r \geq 4$: $(b_\tau, b_\tau + 1, \ldots, b_\tau + r - 1)$ with $(b_0, \ldots, b_4) = (u, u, v, v, w)$ (consecutive letters distinct).

Let $(\alpha_0, \ldots, \alpha_4) = (w, v, w, u, v)$ and $(\omega_0, \ldots, \omega_4) = (\ell_{0,r} + 1,\ \ell_{0,r} + 2,\ 1,\ \ell_{2,r} + 1,\ \ell_{4,r} + 1)$, where the third entry is the scalar 1 and the others are letters mod 3. Define

$$\varphi_r = \sum_{\tau=0}^{4} \alpha_\tau \, \ell_{\tau,1}^{-1} \ell_{\tau,2}^{-1} \cdots \ell_{\tau,r}^{-1} \, \omega_\tau \ \in \mathcal{F}_r. \tag{5.1}$$

For example $\varphi_3 = wu^{-3}v + vu^{-3}w + wv^{-3} + uv^{-3}w + vw^{-3}u$ and $\varphi_4 = w\,u^{-1}v^{-1}w^{-1}u^{-1}\,v + v\,u^{-1}v^{-1}w^{-1}u^{-1}\,w + w\,v^{-1}w^{-1}u^{-1}v^{-1} + u\,v^{-1}w^{-1}u^{-1}v^{-1}\,w + v\,w^{-1}u^{-1}v^{-1}w^{-1}\,u$. By design α_τ differs from the first letter of chain τ and ω_τ from its last letter, and chains with the same letters have different α_τ and different ω_τ.

Let Q_r be the $(5r + 1) \times (5r + 1)$ matrix $\begin{pmatrix} L & \boldsymbol{\omega} \\ \boldsymbol{\alpha} & 0 \end{pmatrix}$ over P, with L the direct sum of the five chain blocks (letters $\ell_{\tau,i}$ on the diagonal, -1 above it), the row $\boldsymbol{\alpha}$ carrying α_τ at the first position of chain τ and the column $\boldsymbol{\omega}$ carrying ω_τ at its last position. (This is the border B_{φ_r} of §3.4, with interior coefficients 1.)

Lemma 5.5. p. 19

> *The Schur complement of L in Q_r is $-\varphi_r$. Over $\Lambda = \mathbb{C}[F(u, v, w)]$, Q_r is associated with $I \oplus (-\varphi_r)$. For every ring $\mathcal{U} \supseteq E_1$: $\varphi_r^{-1} \in \mathcal{U} \iff Q_r^{-1} \in M(\mathcal{U})$.*

↳ Used in: Theorem 5.2 (p. 26)

Proof

> L is triangular with letters (units of Λ) on the diagonal, and the inverse of the block of chain τ has $(1, r)$ entry $\ell_{\tau,1}^{-1} \cdots \ell_{\tau,r}^{-1}$; block elimination gives the first two claims. The block inverse of Q_r uses only L^{-1} (entries are products of inverse letters, in E_1), φ_r^{-1} and the entries of $\boldsymbol{\alpha}, \boldsymbol{\omega}$, which lie in A. □

Evidence & scope

- **Program:** phi_3, phi_4 generated from the design equal the printed ones; Schur(Q_r) = -phi_r (supplement C08). `scripts/auditoria_S12/s12_fr_simbolico.py`

Lemma 5.6. (target profile) p. 20

Let $r \geq 3$ and $k \geq r+1$, $\gamma = \gamma_k$. Each chain block of L^γ has exactly one part $\geq (r-1)k+1$. Hence $\nu_{(r-1)k+1}(L^\gamma) = 5$ and $\nu_{(r-1)k+1}(Q_r^\gamma) \geq 3$.

↳ Used in: Theorem 5.2 (p. 26), 8.1 (p. 52), 8.2 (p. 53)

Proof

A chain block is equivalent to $I \oplus \ell^\gamma_{\tau,r} \cdots \ell^\gamma_{\tau,1}$ (the -1's are unit pivots). For $r = 3$ the product is $U^3 = t^{k+1}U$, with parts $(k+2, 2k+1)$. For $r \geq 4$ write $U_i = t(\mathbf{a}_i\mathbf{b}_i - t^{k-1}E_{21})$ with $\mathbf{a}_i = (1, it)^\mathsf{T}$, $\mathbf{b}_i = (it, -1)$, so $\mathbf{b}_a\mathbf{a}_b = (a-b)t$. If the letters of the product have indices $i_1, \ldots, i_r$ with $i_j \neq i_{j+1}$, its leading term is $t^r \cdot t^{r-1} \prod_j (i_j - i_{j+1})\, \mathbf{a}_{i_1}\mathbf{b}_{i_r}$, whose $(1,2)$ entry has valuation exactly $2r-1$; corrections have valuation $\geq r+k-1 \geq 2r$. So the first invariant factor is t^{2r-1} and, as $\det U_i = -t^{k+1}$, the second is $t^{r(k+1)-(2r-1)}$. Now $rk - r + 1 \geq (r-1)k+1 \iff k \geq r$ and $2r-1 < (r-1)k+1$; for $r = 3$, $2k+1 \geq 2k+1 > k+2$. Five chains give $\nu(L^\gamma) = 5$, and Lemma 3.2(3) with $m = 2$ gives ≥ 3. □

Lemma 5.7. (atoms) p. 20

1. *For $n \geq 2$ and $p \geq 1$, $H_{n,p}$ is an atom of $\mathbb{C}\langle u_1, \ldots, u_n\rangle$.*
2. *For $r \geq 3$, Q_r is an atom of $P = \mathbb{C}\langle u, v, w\rangle$.*

Lean · ours, in part **show**

the eight conditions of (2) for Q_3 and Q_4

- `ThreeAxes.ChainPencil.R3.eightConditions`
- `ThreeAxes.ChainPencil.R4.eightConditions`

✅ Checked when this page was built: ChainPencilConditions.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — eightConditions, eightConditions: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean ChainPencilConditions.lean` (SHA-256 of the compiled file: `e266df18e48e9700…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 2 results, standard axioms only

Proof: in this note · **Covers:** not formalized · **Cite:** Theorem 5.2, p. 17, doi:10.5281/zenodo.23481332

↳ Used in: Theorem 5.2 (p. 26), Corollary 5.3 (p. 27), Proposition 5.8 (p. 34), 7.11 (p. 50), 8.1 (p. 52), 8.2 (p. 53)

Proof

Proof of (1). Let $R = [\mathbf{1} \mid I_{n-1}] \in M_{(n-1)\times n}(\mathbb{C})$, with columns $r_1 = \mathbf{1}$ and $r_i = e_{i-1}$ for $i \geq 2$, and let $\mathsf{U} = \operatorname{diag}(u_1, \ldots, u_n)$. Then $H_{n,p} = R\,\mathsf{U}^p R^\mathsf{T}$. The matrix $H_{n,p}$ is full, since $H_{n,p}(1, \ldots, 1) = I + \mathbf{1}\mathbf{1}^\mathsf{T}$ has determinant n. It is not invertible, since it vanishes at 0 (§2.2(a)). Put $N = np - 1$.

The pencil. For $p = 1$ let $\mathbf{L} = H_{n,1} = R\,\mathsf{U}R^\mathsf{T}$. For $p \geq 2$ take the vector $v = (v_0, v_1, \ldots, v_{p-1})$ with $v_0 \in \mathbb{C}^{n-1}$ and $v_k \in \mathbb{C}^n$, and let $\mathbf{L}$ be the $N \times N$ block matrix with

$$\mathbf{L}_{01} = R\,\mathsf{U}, \quad \mathbf{L}_{kk} = I_n \ (k \geq 1), \quad \mathbf{L}_{k,k+1} = -\mathsf{U} \ (1 \leq k \leq p-2), \quad \mathbf{L}_{p-1,0} = -\mathsf{U}R^\mathsf{T}$$

and zero blocks elsewhere. The block T on the indices $1, \ldots, p-1$ is $I - \mathsf{N}$, where N is block strictly upper triangular with U on the block superdiagonal. So

$T^{-1} = \sum_{m<p-1} \mathsf{N}^m$ is polynomial, and $(T^{-1})_{1,p-1} = \mathsf{U}^{p-2}$. The Schur complement of T is $-R\,\mathsf{U}\,\mathsf{U}^{p-2}(-\mathsf{U}R^{\mathsf{T}}) = H_{n,p}$. By §2.2(b),(c) it suffices to prove that $\mathbf{L}$ is an atom.

Translation. Write $\mathbf{L} = \mathbf{L}_0 + \sum_i u_i \mathbf{L}_i$. Substitute $u_i = 1 + z_i$, an affine automorphism (§2.2(d)). The new pencil has constant term $K = \mathbf{L}(1,\dots,1)$ and the same coefficients $\mathbf{L}_i$. Its determinant is $\det K = \det T(1) \cdot \det(RR^{\mathsf{T}}) = n \neq 0$.

Rank-p factorization of the coefficients. For $c = (c_1,\dots,c_p) \in \mathbb{C}^p$ let $\iota_i c$ be the vector with block 0 equal to $r_i c_1$, block k equal to $-e_i c_{k+1}$ ($1 \le k \le p-1$). Let

$$\varkappa_i v = (e_i^{\mathsf{T}} v_1, \dots, e_i^{\mathsf{T}} v_{p-1},\, r_i^{\mathsf{T}} v_0) \in \mathbb{C}^p.$$

Then $\mathbf{L}_i = \iota_i \varkappa_i$. The map ι_i is injective, because its p components land in different blocks and $r_i, e_i \neq 0$. The map $\varkappa_i$ is onto, for the same reason. For $p = 1$, $\iota_i = r_i$ and $\varkappa_i = r_i^{\mathsf{T}}$.

Monic. We have
$\sum_i \operatorname{im} \mathbf{L}_i = \sum_i \operatorname{im} \iota_i = \big(\sum_i \operatorname{span}(r_i)\big) \oplus \bigoplus_{k\ge1} \big(\sum_i \operatorname{span}(e_i)\big) = \mathbb{C}^N$, because R has rank $n-1$. Next, $\bigcap_i \ker \mathbf{L}_i = \bigcap_i \ker \varkappa_i$: this forces $v_k = 0$ for $k \ge 1$, and $R^{\mathsf{T}} v_0 = 0$, so $v_0 = 0$. Hence the row and the column of coefficients have rank N.

The transfer matrices. Let $\vartheta_i = K^{-1}\iota_i$, injective. To compute $v = \vartheta_i c$, solve $Kv = \iota_i c$:

- blocks $1 \le k \le p-2$ give $v_{k+1} = v_k + e_i c_{k+1}$;
- block $p-1$ gives $v_{p-1} = R^{\mathsf{T}} v_0 - e_i c_p$;
- block 0 gives $R v_1 = r_i c_1$.

Hence $RR^{\mathsf{T}} v_0 = r_i \sigma$ with $\sigma = \mathbf{1}^{\mathsf{T}} c$, and

$$v_k = G e_i\, \sigma - e_i \sum_{m>k} c_m \quad (1 \le k \le p-1), \qquad G = R^{\mathsf{T}}(RR^{\mathsf{T}})^{-1} R.$$

The matrix G is the orthogonal projector onto $\operatorname{im} R^{\mathsf{T}} = (\ker R)^{\perp}$. Since $\ker R = \langle \mathbf{n} \rangle$ with $\mathbf{n} = (1,-1,\dots,-1)$, we have $G = I - \mathbf{n}\mathbf{n}^{\mathsf{T}}/n$. Also $r_j^{\mathsf{T}} v_0 = e_j^{\mathsf{T}} G e_i\, \sigma$. Therefore

$$\varkappa_j \vartheta_i = g_{ji}\, \mathbf{1}\mathbf{1}^{\mathsf{T}} - \delta_{ij} \mathsf{N}_p, \qquad g_{ji} = \delta_{ij} - \frac{n_j n_i}{n}, \tag{5.2}$$

where $(\mathsf{N}_p)_{km} = 1$ for $m > k$ and 0 otherwise. In particular $g_{ji} = \pm 1/n \neq 0$ for $i \neq j$. (For $p = 1$, $\mathsf{N}_1 = 0$ and (5.2) reads $r_j^{\mathsf{T}}(RR^{\mathsf{T}})^{-1} r_i = g_{ji}$.)

Cyclicity. Let $M = a\mathbf{1}\mathbf{1}^{\mathsf{T}} - \mathsf{N}_p$ with $a \in \mathbb{C}$. Then $\mathbf{1}$ is a cyclic vector for M, and $\mathbf{1}^{\mathsf{T}}$ is cyclic for M acting on row vectors from the right.

- Since $Mx = -\mathsf{N}_p x + a(\mathbf{1}^{\mathsf{T}} x)\mathbf{1}$, induction gives $\operatorname{span}\{M^m \mathbf{1}\}_{m<q} = \operatorname{span}\{\mathsf{N}_p^m \mathbf{1}\}_{m<q}$ for every q.
- The vector $\mathsf{N}_p^m \mathbf{1}$ vanishes below position $p-m$ and has entry 1 there. So these vectors form a triangular basis.
- The row case is the same: $\mathbf{1}^{\mathsf{T}} \mathsf{N}_p^m$ vanishes at positions $\le m$ and equals 1 at position $m+1$.

No invariant subspace. Let $A_i = K^{-1}\mathbf{L}_i = \vartheta_i \varkappa_i$, and let $W \neq 0$ be invariant under all A_i. Put $M_i = \varkappa_i \vartheta_i = g_{ii}\mathbf{1}\mathbf{1}^{\mathsf{T}} - \mathsf{N}_p$. Then $A_i^m \vartheta_i = \vartheta_i M_i^m$, and $A_j \vartheta_i = g_{ji} \vartheta_j \mathbf{1}\mathbf{1}^{\mathsf{T}}$ for $j \neq i$.

1. Pick $w \in W \setminus 0$. By the monic condition some $c = \varkappa_i w \neq 0$, and $A_i w = \vartheta_i c \in W \setminus 0$.
2. Since $c \neq 0$ and $\mathbf{1}^{\mathsf{T}}$ is cyclic for M_i, there is m with $\beta = \mathbf{1}^{\mathsf{T}} M_i^m c \neq 0$. Fix $j \neq i$ (possible since $n \ge 2$). Then $W \ni A_j A_i^m \vartheta_i c = A_j \vartheta_i M_i^m c = g_{ji}\beta\, \vartheta_j \mathbf{1}$, so $\vartheta_j \mathbf{1} \in W$.
3. Then $A_j^m \vartheta_j \mathbf{1} = \vartheta_j M_j^m \mathbf{1} \in W$ for all m. Since $\mathbf{1}$ is cyclic for M_j, $\operatorname{im} \vartheta_j \subseteq W$.

4. For every $l \neq j$, $\vartheta_j e_1 \in W$ by step 3, and $A_l \vartheta_j e_1 = g_{lj} \vartheta_l \mathbf{1}\mathbf{1}^{\mathsf{T}} e_1 = g_{lj} \vartheta_l \mathbf{1} \in W$. So $\vartheta_l \mathbf{1} \in W$, and by step 3 $\operatorname{im} \vartheta_l \subseteq W$.

Hence $W \supseteq \sum_l \operatorname{im} \vartheta_l = K^{-1} \sum_l \operatorname{im} \iota_l = \mathbb{C}^N$. By Lemma 2.1 the translated pencil is an atom; so are $\mathbf{L}$ (§2.2(d)) and $H_{n,p}$ (§2.2(c)). □

Evidence & scope

- **Program:** the eight rank conditions of the design, r = 3..40 (supplement C09). `scripts/smith_V1/verifica_T160.py`
- **Program:** H_n,p: epic and full coefficient algebra modulo two primes (supplement C10). `scripts/smith_V1/atomo_epic.py`

The proof of (1) uses characteristic 0 only through $g_{ji} = \pm 1/n \neq 0$ and $\det K = n \neq 0$, which need $\operatorname{char} \nmid n$. This is consistent with Remark 8.1.

Proof of (2). Notation of §5.3: $P = \mathbb{C}\langle u, v, w\rangle$, $\Lambda = \mathbb{C}[F(u,v,w)] \supset P$, $n = 5r + 1$. Index the coordinates by (τ, i) ($\tau = 0, \dots, 4$, $i = 1, \dots, r$) and $*$. The nonzero entries of Q_r are:

- $Q_{(\tau,i),(\tau,i)} = \ell_{\tau,i}$;
- $Q_{(\tau,i),(\tau,i+1)} = -1$ for $i<r$;
- $Q_{(\tau,r),*} = \omega_\tau$, with $\omega_2 = 1$;
- $Q_{*,(\tau,1)} = \alpha_\tau$.

Write $Q_r = C_0 + uC_u + vC_v + wC_w$ with $C_a \in M_n(\mathbb{C})$. Then C_0 carries the -1's and the entry 1 at $((2,r),*)$, and C_j carries the positions occupied by the letter j.

Properties of the design (5.1), independent of $r \geq 3$. Let $b_\tau = \ell_{\tau,1}$ and $L_\tau = \ell_{\tau,r}$. Then $(b_\tau) = (u,u,v,v,w)$ and $(\alpha_\tau) = (w,v,w,u,v)$. Also $L_0 = L_1$, $L_2 = L_3 = L_0 + 1$ and $L_4 = L_0 + 2 \pmod 3$, for $r = 3$ and for $r \geq 4$ alike, since $L_\tau = b_\tau + r - 1$ for $r \geq 4$. Finally $(\omega_0, \omega_1, \omega_3, \omega_4) = (L_0+1, L_0+2, L_0+2, L_0)$. Hence:

- (P1) $\alpha_\tau \neq b_\tau$ for every τ;
- (P2) $\omega_\tau \neq L_\tau$ for every $\tau \neq 2$;
- (P3) for each letter i, α is injective on $\{\tau : b_\tau = i\}$ (these sets are $\{0,1\}, \{2,3\}, \{4\}$, with α-values $\{w,v\}, \{w,u\}, \{v\}$);
- (P4) for each letter i, ω is injective on $\{\tau \neq 2 : L_\tau = i\}$ (these sets are $\{0,1\}, \{3\}, \{4\}$);
- (P5) α takes at least two values;
- (P6) exactly one ω_τ is a scalar, namely $\omega_2 = 1$.

Claim 1. *φ_r is an atom of Λ, and Q_r is a full atom over Λ.*

Proof. The exponent sum $F(u,v,w) \to \mathbb{Z}$ grades $\Lambda = \bigoplus_m \Lambda_m$. Λ is a fir [Coh06, Cor. 7.11.8, p. 507], hence a domain. For $0 \neq g \in \Lambda$ let g_{top} and g_{bot} be its nonzero homogeneous components of largest and smallest degree, and $\operatorname{wd}(g)$ the difference of the degrees. In a graded domain $(gh)_{\mathrm{top}} = g_{\mathrm{top}} h_{\mathrm{top}}$ and $(gh)_{\mathrm{bot}} = g_{\mathrm{bot}} h_{\mathrm{bot}}$, so wd is additive. Units therefore have width 0.

The five terms of φ_r are group elements:

- $\alpha_2 \ell_{2,1}^{-1} \cdots \ell_{2,r}^{-1}$ of degree $1 - r$, alone in that degree by (P6);
- four of degree $2 - r$, each with coefficient $+1$; their sum is nonzero in the group algebra.

So $\operatorname{wd}(\varphi_r) = 1$, $\varphi_r \neq 0$, and φ_r is not a unit. Suppose $\varphi_r = gh$.

- If $\operatorname{wd}(g) = 0$, then $g\, h_{\mathrm{bot}} = (\varphi_r)_{\mathrm{bot}} =: m$, a group element. So $g\,(h_{\mathrm{bot}} m^{-1}) = 1$, and in a domain a one-sided inverse is two-sided: g is a unit.
- If $\operatorname{wd}(h) = 0$, symmetrically h is a unit.

So φ_r is an atom.

Over Λ the block L is triangular with units $\ell_{\tau,i}$ on the diagonal, hence invertible. Lemma 5.5 then gives $Q_r = E\,(L \oplus (-\varphi_r))\,F$ with E, F invertible, so Q_r is associated with $I_{5r} \oplus (-\varphi_r)$. Thus

$M(Q_r) \cong \Lambda/\Lambda\varphi_r$ is Tor-simple, and Q_r is an atom over Λ (§2.2(c)). Q_r is full because $Q_r(1,1,1)$ is invertible: $L(1)$ is unipotent and the Schur complement is $-\varphi_r(1) = -5$. $\square$

Claim 2 (the eight conditions). *For the design (5.1) and every $r \geq 3$:*

- (E^R) $\bigcap_{j\in\{u,v,w\}} \ker C_j = 0$, and ($\mathrm{E}^L$) $\bigcap_j \ker C_j^{\mathsf{T}} = 0$;
- (K_i^R) $\mathcal{K}_i^R := \ker C_0 \cap \bigcap_{j\neq i} \ker C_j = 0$, for $i = u, v, w$;
- (K_i^L) $\mathcal{K}_i^L := \ker C_0^{\mathsf{T}} \cap \bigcap_{j\neq i} \ker C_j^{\mathsf{T}} = 0$, for $i = u, v, w$.

Proof. Columns $e \in \mathbb{C}^n$ and rows f have coordinates $e_{\tau,i}$, e_* and $f_{\tau,i}$, f_*. We only use $r \geq 2$, so that $(\tau,1) \neq (\tau,r)$; the lemma assumes $r \geq 3$.

(E^R) Let $C_j e = 0$ for all j.

- For $i<r$, row (τ,i) of $C_{\ell_{\tau,i}} e$ is $e_{\tau,i}$.
- For $\tau \neq 2$, row (τ,r) of $C_{\omega_\tau} e$ is e_*, by (P2). Applied to $\tau = 0$ this gives $e_* = 0$.
- Row (τ,r) of $C_{L_\tau} e$ is $e_{\tau,r}$, using (P2) for $\tau \neq 2$ and $e_* = 0$.

So $e = 0$.

(E^L) Let $f C_j = 0$ for all j.

- For $i \geq 2$, column (τ,i) of $f C_{\ell_{\tau,i}}$ is $f_{\tau,i}$.
- Column $(\tau,1)$ of $f C_j$ is $[b_\tau = j] f_{\tau,1} + [\alpha_\tau = j] f_*$. Take $j = \alpha_0 \neq b_0$ (P1) to get $f_* = 0$. Then take $j = b_\tau$ to get $f_{\tau,1} = 0$.

(K_i^R) Let $e \in \mathcal{K}_i^R$. In $C_0 e$:

- row (τ,i'), $i'<r$, is $-e_{\tau,i'+1}$;
- row $(2,r)$ is e_*.

So only the coordinates $e_{\tau,1}$ survive. For $j \neq i$:

- row $(\tau,1)$ of $C_j e$ is $[b_\tau = j] e_{\tau,1}$, so $e_{\tau,1} = 0$ unless $b_\tau = i$;
- row $*$ of $C_j e$ is $\sum_{\tau:\,\alpha_\tau = j} e_{\tau,1}$.

On $\{\tau : b_\tau = i\}$ the values α_τ are $\neq i$ (P1) and pairwise distinct (P3). So each surviving $e_{\tau,1}$ is alone in the row-$*$ equation for $j = \alpha_\tau$, and $e = 0$.

(K_i^L) Let $f \in \mathcal{K}_i^L$. In $f C_0$:

- column $(\tau,i'+1)$, $i'<r$, is $-f_{\tau,i'}$;
- column $*$ is $f_{2,r}$.

So only $f_{\tau,r}$ ($\tau \neq 2$) and f_* survive. By (P5) there is τ with $j := \alpha_\tau \neq i$, and column $(\tau,1)$ of $f C_j$ is f_*; so $f_* = 0$. For $j \neq i$:

- column (τ,r) of $f C_j$ is $[L_\tau = j] f_{\tau,r}$, so $f_{\tau,r} = 0$ unless $L_\tau = i$;
- column $*$ is $\sum_{\tau\neq 2:\,\omega_\tau = j} f_{\tau,r}$.

On $\{\tau \neq 2 : L_\tau = i\}$ the values ω_τ are $\neq i$ (P2) and distinct (P4). So $f = 0$. $\square$

Claim 3 (letter factors). *Let $A = C_0 + \sum_{j\in\{u,v,w\}} C_j\, j$ be an $n \times n$ pencil over P with $\bigcap_j \ker C_j = 0$, and let i be a letter. If there is a nonzero homomorphism $M(A) \to P/Pi$, then $\mathcal{K}_i^R(A) \neq 0$.*

Proof.

1. *Setting.* A homomorphism $M(A) \to N$ is given by $y \in N^{n\times 1}$ with $Ay = 0$, and it is nonzero iff $y \neq 0$. The left ideal Pi is spanned by the words ending in i, so P/Pi has as basis the words not ending in i, graded by length.
2. *Left action.* Let a letter j act on such a word w. If $\mathsf{w} \neq 1$, then $j\mathsf{w}$ is again a basis word, and distinct pairs (j,w) give distinct words. If $\mathsf{w} = 1$, then $j \cdot 1$ is the basis word j for $j \neq i$, and 0 for $j = i$.
3. *Positive degree is impossible.* Let $0 \neq y = \sum_{\mathsf{w}} y_{\mathsf{w}} \mathsf{w}$ with $y_{\mathsf{w}} \in \mathbb{C}^n$, and let D be the largest length occurring. If $D \geq 1$, the component of Ay of length $D+1$ is $\sum_j \sum_{|\mathsf{w}|=D} (C_j y_{\mathsf{w}})\, j\mathsf{w}$. By

step 2 it vanishes only if $C_j y_{\mathsf{w}} = 0$ for all j, that is $y_{\mathsf{w}} \in \bigcap_j \ker C_j = 0$, a contradiction.

4. *Degree* 0. So $D = 0$ and $y = e \cdot \bar{1}$ with $0 \neq e \in \mathbb{C}^n$. Then $Ay = C_0 e \cdot \bar{1} + \sum_{j \neq i} C_j e \cdot \bar{j} = 0$. Since $\bar{1}$ and the $\bar{j}$ $(j \neq i)$ are linearly independent, $e \in \mathcal{K}_i^R$. □

The converse also holds: if $0 \neq e \in \mathcal{K}_i^R$, then $Ae = (C_i e)\, i$. Completing e to a basis splits off the right factor $\mathrm{diag}(i, 1, \ldots, 1)$. We do not need it.

Proof of (2), conclusion. Q_r is full, and it is not invertible over P because it is not invertible over Λ (Claim 1). Suppose $Q_r = FG$ with F, G full non-invertible $n \times n$ matrices over P.

- *One factor becomes invertible over* Λ. F and G stay full over Λ: if $F = AB$ through a smaller size over Λ, then $Q_r = A(BG)$ would not be full over Λ. As Q_r is an atom over Λ (Claim 1), F or G is invertible over Λ.

Case 1: G invertible over Λ.

1. P is a fir [Coh06, Cor. 2.5.2, p. 135], hence fully atomic [Coh06, Prop. 3.2.9, p. 197]. So $G = G' G_1$ with G_1 an atom over P.
2. Over Λ, $(G^{-1}G')G_1 = I$. Λ embeds in a field, so it is weakly finite, and G_1 is invertible over Λ. Thus G_1^{-1} has entries in the subring of D generated by P and u^{-1}, v^{-1}, w^{-1}, which is $U_{\{u,v,w\}}$.
3. By Lemma 3.1 (with $\Sigma = \{u, v, w\}$), G_1 is stably associated with u, v or w, say with i. Over $\mathbb{C}$ one can use instead [HKV22, Thm 2.12, p. 10]: $G_1(X)$ is invertible whenever X_u, X_v, X_w are invertible, so the free locus of G_1 lies in that of uvw, and the proof of [HKV22, Thm 2.12] (p. 11) matches G_1 with one atom of $u \cdot v \cdot w$.
4. Hence $M(G_1) \cong P/Pi$ (§2.2(c)). Since $P^{1\times n} Q_r \subseteq P^{1\times n} G_1$, there is a surjection $M(Q_r) \to M(G_1) \neq 0$.
5. By Claim 3, which applies by (E^R), $\mathcal{K}_i^R \neq 0$. This contradicts Claim 2.

Case 2: F invertible over Λ.

1. As in Case 1, $F = F_1 F'$ with F_1 an atom, invertible over Λ and stably associated with a letter i.
2. Let * be the anti-automorphism of P that reverses words. It fixes u, v, w and scalars, and extends to matrices by $A^* = (a_{ji}^*)$, so that $(AB)^* = B^* A^*$.
3. Then $Q_r^* = Q_r^{\mathsf{T}} = (F'G)^* F_1^*$, and F_1^* is stably associated with $i^* = i$.
4. Claim 3 applies to $Q_r^{\mathsf{T}} = C_0^{\mathsf{T}} + \sum_j C_j^{\mathsf{T}} j$ by (E^L), and gives $\mathcal{K}_i^L \neq 0$. This contradicts Claim 2.

Hence Q_r is an atom of P. □

Remarks on (2).

1. The atom argument uses only the first and last letters of the chains, α, ω and (P6). The condition "consecutive letters distinct" is used only in Lemma 5.6.
2. The six conditions (K_i^R), (K_i^L) are necessary as well: if one fails, the converse after Claim 3 splits off a letter (the cofactor is not invertible, since Q_r is not invertible over Λ). Conditions (E^R), (E^L) are not necessary: $\left(\begin{smallmatrix} x & 1 \\ y & 0 \end{smallmatrix}\right)$ is associated with $\mathrm{diag}(1, y)$, an atom, and is not monic.
3. Lemma 2.1 and [Coh06, Thm 5.8.8] are not needed for Q_r.

5.4 Proof of Theorem 5.2

5.5 Proof of Corollary 5.3

5.6 More axes

Proposition 5.8. p. 24

> *Let $N \geq n \geq 3$, $A = \mathbb{C}[F_N]$, $u_i = 1 + x_i$, and form E_1, V_1 over A. For $s_n = \sum_{i \leq n} u_i^{-2}$ we have $s_n^{-1} \in \mathcal{U}_2$ and $s_n^{-1} \notin E_1[V_1^{-1}]$.*

↳ Used in: 5.9 (p. 35)

Proof

The proof of Theorem 5.1 applies with three changes.

- The arc sends every letter to the exact arc, $u_i \mapsto U_{i-1}(t, t^k)$ for $i = 1, \dots, N$; Lemma 4.1, the trace law and Lemmas 4.3–4.4 hold verbatim for distinct indices. (Sending the extra letters to $-I$ does not work: $1 - [x_N, x_1]$ then evaluates to 0.)
- The target is $H_{n,2}$, with §5.1 read with u_2 in place of v. Its $2(n-1)$ parts all equal $k+1$, since $\det(\mathbf{1}\mathbf{1}^{\mathsf{T}} + I_{n-1}) = n$, against the bound 2 of Lemma 4.4.
- $H_{n,2}$ is an atom of $P_N = \mathbb{C}\langle x_1, \dots, x_N\rangle$. The translated pencil of the proof of Lemma 5.7(1) is a pencil over P_N whose coefficients at $x_{n+1}, \dots, x_N$ are zero. Zero coefficients change neither the ranks of the row and the column of coefficients nor the common invariant subspaces of the M_i. So Lemma 2.1 applies over P_N, and §2.2(b)–(d) give the claim as in Lemma 5.7(1). □

Evidence & scope

- **Program:** N-axis arc, 80 random borders, margin 2(n-2), with controls (supplement C11). `scripts/auditoria_S27/s27_n_ejes.py`

Remark 5.9. (beyond $\mathcal{U}_1$) p. 24

In the setting of Proposition 5.8, with Γ'_a and D' formed over $A = \mathbb{C}[F_N]$ (§2.4, §7.2), $s_n^{-1} \notin \Gamma'_{n-2}$ for $3 \le n \le N$; in particular $s_4^{-1} \notin \Gamma'_2 \supseteq D'$ for $N \ge 4$. The proof is that of Theorem 7.2 (§7.5) with $H_{n,2}$ in place of F_ψ. If $s_n^{-1} \in \Gamma'_{n-2}$, Woodbury with $\Delta_n = \operatorname{diag}(u_2^2, \dots, u_n^2)$ gives $H_{n,2}^{-1} = \Delta_n^{-1} - \Delta_n^{-1}\mathbf{1}s_n^{-1}\mathbf{1}^{\mathsf{T}}\Delta_n^{-1} \in M_{n-1}(U_{\Sigma'_{n-2}})$ (§5.1, and Lemma 3.6 with $a = n-2$, §7.2). $H_{n,2}$ is a full atom of P_N and its $2(n-1)$ parts at γ_k all equal $k+1$ (proof of Proposition 5.8). For $Q \in \Sigma'_{n-2}$ and $k > e(Q)$, $\nu_{k+1}(Q^{\gamma_k}) \le 2(n-2)$ (≤ 2 for Q_s, x_j and the borders of $\mathcal{U}_1$ by Lemma 4.4 and the remark after it, $\le 2(n-2)$ for $\tilde{B}_M$ by Lemma 7.7; their proofs use only Lemma 4.3, which holds on the arc with N axes), and Theorem 3.4 gives a contradiction. For $n = 3$ this is Proposition 5.8, since $\Gamma'_1 = \mathcal{U}_1$. Finally $D' \subseteq \Gamma'_2$ by Lemma 7.6, whose proof uses only that each element of $D_\pi = V_1 \cup D_{\omega+1}D_{\omega+1}$ (read as a set, as after Lemma 7.4) lies in V_1 or is pure. We do not claim here that $n[\pi]$ is a stopping step of the series over $\mathbb{C}[F_N]$ for $n \ge 4$; that would need Lemmas 2.2, 7.4 and 7.5 over F_N.

↳ Not used by another result of the paper.

6 The size chain (Theorem B(ii))

Question. Is the binomial hierarchy all of E_2, and does the size of the inverted matrices give a second hierarchy? No, and yes. The arc of §4.1 compares Smith parts at level $qk+1$, which grows with the number q of inverse binomials, so it cannot bound all of R_{bin} at once; the corank at the quaternion curve of §4.2 is bounded by $2a$ for every $a \times a$ border, whatever its depth.

With ξ, η, ζ, c as in §4.2, for $d \ge 2$ let

$$\hat{F}_d = (\xi^2 - \eta^2)I_d + (\eta^2 - \zeta^2)\mathsf{D}_d + c\,\mathsf{C}_d, \qquad \rho_d = (\hat{F}_d^{-1})_{11},$$

with $\mathsf{D}_d = \operatorname{diag}(3, 4, \dots, d+2)$ and C_d the cyclic permutation matrix with $(\mathsf{C}_d)_{i,i+1} = (\mathsf{C}_d)_{d,1} = 1$.

Theorem 6.1. p. 25

For every $d \ge 2$, $\rho_d \in A_d \cap E_2$ and $\rho_d \notin W_{d-1}$. Consequently

1. $\mathcal{U}_\infty = W_1 \cap E_2 \subsetneq E_2$; *a witness is* $\rho_2 = \left(p_1 - c\,p_2^{-1}c\right)^{-1}$ *with* $p_j = \xi^2 + (j+1)\eta^2 - (j+2)\zeta^2$, *the diagonal entries of* $\hat{F}_d$;
2. $A_{d-1} \cap E_2 \subsetneq A_d \cap E_2$ *and* $B_{d-1} \subsetneq B_d$ *for every* $d \geq 2$, *and* $\bigcup_b B_b = E_2$.

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the cycle inverse (6.1)–(6.2)

- `ThreeAxes.InverseIdentities.cycle_inv`
- `ThreeAxes.InverseIdentities.cycleMatrix_inv`

✅ Checked when this page was built: InverseIdentities.lean compiles with Lean 4.35.0, no errors, no sorry.
Axioms — cycle_inv, cycleMatrix_inv: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean InverseIdentities.lean` (SHA-256 of the compiled file: `3e2df75becd9213c…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 2 results, standard axioms only

Proof: in this note · **Covers:** not formalized · **Cite:** Proposition 5.8, p. 24, doi:10.5281/zenodo.23481332

↳ Used in: Theorem B (p. 5), Theorem 7.2 (p. 41), 8.2 (p. 53), 8.4 (p. 56)

Proof

Fix $d \geq 2$. $\hat{F}_d$ is a full $d \times d$ matrix over A (proof of (6.1)), so $\rho_d \in A_d$. By (6.2), $\sigma \in E_1$ and $\sigma \neq 0$, so $\rho_d = \sigma^{-1} \in E_2$.

Suppose $\rho_d \in W_{d-1}$. Since $E_1 \subseteq W_1 \subseteq W_{d-1}$ and T_d^{-1} has entries in E_1, (6.2) gives $\hat{F}_d^{-1} \in M_d(W_{d-1}) = M_d(U_{\Sigma^{(d-1)}})$. Apply Theorem 3.4 with $G = H = \hat{F}_d$ (a full atom, Lemma 6.2(c)), $\Sigma = \Sigma^{(d-1)}$ and, for every Q, μ_Q the corank at Θ (Example (b) with $K = \mathbb{C}(t)$; the case $G = H$, $\mu =$ corank of Theorem 3.4):
$\operatorname{corank} \hat{F}_d^{\Theta} = 2d > 2(d-1) \geq \operatorname{corank} Q^{\Theta}$ for every $Q \in \Sigma^{(d-1)}$ (Lemma 6.2(a),(b)). This is a contradiction, so $\rho_d \notin W_{d-1}$. (Figure 3(b) shows both sides of this comparison for $d \leq 6$.)

The consequences: $A_{d-1} \subseteq W_{d-1}$, so $\rho_d \in (A_d \cap E_2) \setminus A_{d-1}$ and $\rho_d \in B_d \setminus B_{d-1}$. For $d = 2$, $W_1 \cap E_2 = \mathcal{U}_\infty$ (§2.1), so $\rho_2 \in E_2 \setminus \mathcal{U}_\infty$. Finally $\bigcup_b B_b = E_2$: a nonzero $e \in E_1$ has a representation $e = \gamma - \mathbf{u}M^{-1}\mathbf{v}$ with M full over A [Coh06, Thm 7.1.2(d)], with Σ the full matrices over A, since the Σ-rational closure of A is a subfield of D containing A, hence all of D; then $N = \begin{pmatrix} M & \mathbf{v} \\ \mathbf{u} & \gamma \end{pmatrix}$ is full over A of some order n and $e^{-1} = (N^{-1})_{nn} \in A_n \subseteq W_n$; E_2 is generated by E_1 and such inverses, and the W_b increase. □

The strictness of the chain A_b alone does not need Lemma 4.5: with $\Sigma_A^{(b)} = \Sigma_0 \cup \{\text{lifts of full } a \times a \text{ matrices over } A,\ a \leq b\}$ one has $A_b = U_{\Sigma_A^{(b)}}$ (as in Lemma 3.5), and the lifts carry only the pivots x_j, which are invertible at Θ, so their corank is at most $2a$ directly. Lemma 4.5, whose proof amounts to the freeness of the group generated by $1 + 2\mathbf{i}, 1 + 2\mathbf{j}, 1 + 2\mathbf{k}$ modulo scalars, is needed only for W_b, that is, for $\mathcal{U}_\infty \subsetneq E_2$ and for $B_{d-1} \subsetneq B_d$. For the witnesses f_k the character χ already suffices (Remark 7.11); it does not detect ρ_d, since $\hat{F}_d^{\chi} = 8\mathsf{C}_d$ is invertible.

Evidence & scope

- **Program:** the formula for rho_d, the orientation of the cycle, F_d vanishes on Theta (supplement C12). `scripts/smith_V1/verifica_T162.py`

Theorem 6.1 is proved by a corank comparison at the quaternion curve Θ of §4.2: every binomial is invertible at Θ (Lemma 4.5), so every border of a full $a \times a$ matrix over R_{bin} has corank at most $2a$, while $\hat{F}_d^{\Theta} = 0$ has corank $2d$.

6.1 The witnesses

For $\lambda \in \mathbb{C}$ put $q_\lambda = \xi^2 + (\lambda-1)\eta^2 - \lambda\zeta^2 \in A$, and $p_j = q_{j+2} = \xi^2 + (j+1)\eta^2 - (j+2)\zeta^2$, the j-th diagonal entry $(\xi^2-\eta^2) + (j+2)(\eta^2-\zeta^2)$ of $\hat{F}_d$. Then

$$\hat{F}_d = \begin{pmatrix} p_1 & c & & \\ & p_2 & \ddots & \\ & & \ddots & c \\ c & & & p_d \end{pmatrix}, \qquad F^{(\lambda)} = \begin{pmatrix} q_\lambda & c \\ c & q_{\lambda+1} \end{pmatrix}.$$

Thus $\hat{F}_2 = F^{(\lambda)}$ with $\lambda = 3$. **The orientation of the cycle matters:** with c on the superdiagonal and in position $(d,1)$,

$$\rho_d = (\hat{F}_d^{-1})_{11} = \left[p_1 + (-1)^{d-1}\, c\, p_2^{-1} c\, p_3^{-1} c \cdots p_d^{-1} c\right]^{-1}. \tag{6.1}$$

With the transposed orientation (c on the subdiagonal and in $(1,d)$) formula (6.1) is false for $d \geq 3$.

Proof of (6.1). Write $\hat{F}_d = \begin{pmatrix} p_1 & \mathbf{b} \\ \mathbf{a} & T_d \end{pmatrix}$ with $\mathbf{b} = c\,e_1^{\mathsf{T}}$, $\mathbf{a} = c\,e_{d-1}$ and T_d upper bidiagonal with $p_2, \dots, p_d$ on the diagonal and c above it. Then T_d^{-1} has entries in E_1,

$$(T_d^{-1})_{1,d-1} = (-1)^{d-2} p_2^{-1} c\, p_3^{-1} \cdots c\, p_d^{-1}, \qquad \sigma = p_1 - \mathbf{b} T_d^{-1} \mathbf{a} = p_1 + (-1)^{d-1} c\, p_2^{-1} c \cdots p_d^{-1} c,$$

where σ is the Schur complement. The block inverse

$$\hat{F}_d^{-1} = \begin{pmatrix} \sigma^{-1} & -\sigma^{-1}\mathbf{b}T_d^{-1} \\ -T_d^{-1}\mathbf{a}\sigma^{-1} & T_d^{-1} + T_d^{-1}\mathbf{a}\sigma^{-1}\mathbf{b}T_d^{-1} \end{pmatrix} \tag{6.2}$$

gives $\rho_d = \sigma^{-1}$, once $\sigma \neq 0$. This holds because $\hat{F}_d(1,0,0) = I$ (at $(\xi,\eta,\zeta) = (1,0,0)$, $c = 0$), so $\hat{F}_d$ is full (a non-full matrix over P factors through a smaller size and is singular at every point); since T_d is invertible, the block LDU factorization gives $\sigma \neq 0$. □

Lemma 6.2. p. 25

> *(a) For $Q \in \Sigma^{(b)}$, $\operatorname{corank} Q^\Theta \leq 2b$. (b) $\hat{F}_d^\Theta = 0$, so $\operatorname{corank} \hat{F}_d^\Theta = 2d$; likewise $(F^{(\lambda)})^\Theta = 0$ for every λ. (c) For $d \geq 2$, $\hat{F}_d$ is a full atom over P; for $\lambda \in \mathcal{T} := \mathbb{C} \setminus \{-1,0,1,2\}$, $F^{(\lambda)}$ is a full atom over P.*

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the linear algebra of (c)

- `ThreeAxes.SizeChain.adjoin_cyc_diagD_eq_top`
- `ThreeAxes.SizeChain.det_matE_ne_zero`

✅ Checked when this page was built: SizeChainPencil.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — adjoin_cyc_diagD_eq_top, det_matE_ne_zero: propext, Classical.choice, Quot.sound. Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean SizeChainPencil.lean` (SHA-256 of the compiled file: `03090e8da8ae426b`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 2 results, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** part: the linear algebra of (c) · **Cite:** Lemma 6.2, p. 25, doi:10.5281/zenodo.23481332

↳ Used in: Theorem 6.1 (p. 35), 8.1 (p. 52), 8.4 (p. 56), 8.5 (p. 56)

Proof

(a) For Q_s ($s \in A$) the pivot block is triangular with $\Theta(x_j)$ on the diagonal, invertible, so corank Q_s^Θ = corank $s^\Theta \le 2$, also when $s^\Theta = 0$ (e.g. $s = c$). $\Theta(x_j)$ has corank 0. In $\tilde{B}_G^\Theta$ the triangular pivot evaluates to an invertible matrix ($\Theta(x_j)$ on its diagonal), and its Schur complement is B_G^Θ, since Θ is a ring homomorphism on A. In B_G^Θ the block L^Θ is block upper triangular with the 2×2 blocks ℓ^Θ on the diagonal, invertible by Lemma 4.5. Taking Schur complements does not change the corank, so corank $\tilde{B}_G^\Theta$ is the corank of a $2a \times 2a$ matrix over $\mathbb{C}(t)$, at most $2a \le 2b$. (No element of R_{bin} is evaluated at Θ; only the polynomial matrix $\tilde{B}_G$ is.)

1. $q_\lambda^\Theta = -t^2(1 + (\lambda - 1) - \lambda)I = 0$ and $c^\Theta = 0$.
2. We prove a slightly more general statement.

Claim. *Let $C, D \in M_d(\mathbb{C})$ satisfy:*

- *(H1) C and D have no common invariant subspace other than 0 and $\mathbb{C}^d$;*
- *(H2) D is invertible;*
- *(H3) $E := D - I - C^2$ is invertible.*

Then $F = (\xi^2 - \eta^2)I + (\eta^2 - \zeta^2)D + cC$, with $c = \xi\eta + \eta\xi$, is a full atom of $P = \mathbb{C}\langle \xi, \eta, \zeta \rangle$.

Proof. Let $\mathbf{L} = A_0 + \xi A_\xi + \eta A_\eta + \zeta A_\zeta$ be the $4d \times 4d$ pencil:

$$\mathbf{L} = \begin{pmatrix} 0 & \xi I + \eta C & \xi C + \eta(D - I) & -\zeta D \\ -\xi I & I & 0 & 0 \\ -\eta I & 0 & I & 0 \\ -\zeta I & 0 & 0 & I \end{pmatrix}.$$

Its lower right block is I_{3d}, and the Schur complement is $(\xi I + \eta C)\xi + (\xi C + \eta(D - I))\eta - \zeta D\zeta = \xi^2 I + cC + \eta^2(D - I) - \zeta^2 D = F$. So $\mathbf{L}$ is stably associated with F (§2.2(b)). F is full, because $F(1, 0, 0) = I$, and not invertible, because $F(0) = 0$.

Monic. Write vectors as (v_0, v_1, v_2, v_3) in blocks of size d. For a row f:

$$fA_\xi = (-f_1, f_0, f_0C, 0), \quad fA_\eta = (-f_2, f_0C, f_0(D - I), 0), \quad fA_\zeta = (-f_3, 0, 0, -f_0$$

If all three vanish, then $f = 0$. For a column v:

$$A_\xi v = (v_1 + Cv_2, -v_0, 0, 0), \quad A_\eta v = (Cv_1 + (D - I)v_2, 0, -v_0, 0), \quad A_\zeta v = (-Dv_1$$

If all three vanish, then $v_0 = 0$, $v_1 = -Cv_2$, $Ev_2 = 0$ and $Dv_3 = 0$, so $v = 0$ by (H2), (H3).

Normalization. Substitute $\xi = 1 + \xi'$ (§2.2(d)). The constant term becomes

$$K = \mathbf{L}(1, 0, 0) = \begin{pmatrix} 0 & I & C & 0 \\ -I & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \qquad K^{-1}(y_0, y_1, y_2, y_3) = (y_0 - y_1 - Cy_2,\ y_0$$

By Lemma 2.1 it suffices to show that $M_a = K^{-1}A_a$ ($a = \xi, \eta, \zeta$) have no common invariant subspace $W \neq 0$ other than $\mathbb{C}^{4d}$. Use the coordinates $z = (v_0,\ v_1 + Cv_2,\ v_2,\ v_3)$, an invertible change. A direct computation gives

$$M_\xi z = (z_0 + z_1,\ z_1,\ 0,\ 0), \qquad M_\zeta z = (-Dz_3,\ -Dz_3,\ 0,\ -z_0),$$
$$M_\eta z = (Cz_0 + Cz_1 + Ez_2,\ Cz_1 + Ez_2,\ -z_0,\ 0).$$

The polynomials $\mathsf{P} = 2M_\xi - M_\xi^2$ and $\mathsf{N} = M_\xi^2 - M_\xi$ in M_ξ act by $\mathsf{P}z = (z_0, z_1, 0, 0)$ and $\mathsf{N}z = (z_1, 0, 0, 0)$. Let B_k be the k-th block subspace.

1. *W meets B_0.* Let $0 \neq z \in W$.
   - If $z_1 \neq 0$, take $\mathsf{N}z$.
   - If $z_1 = 0 \neq z_0$, take $\mathsf{P}z$.
   - If $z_0 = z_1 = 0 \neq z_2$, take $\mathsf{N}M_\eta z = (Ez_2, 0, 0, 0)$, nonzero by (H3).
   - Otherwise take $\mathsf{N}M_\zeta z = (-Dz_3, 0, 0, 0)$, nonzero by (H2).
2. $B_0 \subseteq W$. Let $W_0 = \{a : (a, 0, 0, 0) \in W\} \neq 0$. On B_0 we have $\mathsf{P}M_\eta(a, 0, 0, 0) = (Ca, 0, 0, 0)$ and $\mathsf{N}M_\zeta^2(a, 0, 0, 0) = \mathsf{N}(Da, Da, 0, 0) = (Da, 0, 0, 0)$. So W_0 is invariant under C and D, and $W_0 = \mathbb{C}^d$ by (H1).
3. *The other blocks.* Let $a \in \mathbb{C}^d$.
   - $M_\zeta(a, 0, 0, 0) = (0, 0, 0, -a)$ gives $B_3 \subseteq W$.
   - $M_\eta(a, 0, 0, 0) = (Ca, 0, -a, 0)$, minus $(Ca, 0, 0, 0) \in B_0$, gives $B_2 \subseteq W$.
   - $M_\eta(0, 0, a, 0) = (Ea, Ea, 0, 0)$, minus $(Ea, 0, 0, 0)$, gives $(0, Ea, 0, 0) \in W$. By (H3), $B_1 \subseteq W$.

So $W = \mathbb{C}^{4d}$. By Lemma 2.1 and §2.2(c),(d), F is an atom. □

Evidence & scope

- **Program:** Schur complement of the 4d x 4d pencil and the criterion (supplement C13). `scripts/auditoria_S11/s11_a2_a4.py`
- **Program:** F_d atoms for d = 2, 3, 4: epic, algebra of dimension (4d)^2 (supplement C14). `scripts/auditoria_S16/s16_atomo_Fd.py`
- **Program:** coranks of borders at Theta: 0, 1, 2 (supplement C15). `scripts/auditoria_S10/s10_corango.py`

Now $P = \mathbb{C}\langle x, y, z\rangle = \mathbb{C}\langle \xi, \eta, \zeta\rangle$ via the affine automorphism $\xi = x - 1$, etc. (§2.2(d)). By the Claim it suffices to check (H1)–(H3).

$\hat{F}_d$: $D = \mathsf{D}_d = \mathrm{diag}(3, \ldots, d+2)$, $C = \mathsf{C}_d$.

- (H1) is classical. A subspace invariant under a diagonal matrix with distinct eigenvalues is spanned by the coordinate vectors e_i, $i \in I$: the projections onto the eigenlines are polynomials in D. Since $\mathsf{C}_d e_{i+1} = e_i$ and $\mathsf{C}_d e_1 = e_d$, the set I is closed under $i \mapsto i - 1 \pmod d$, so it is empty or everything.
- (H2) holds, since the diagonal entries are ≥ 3.
- (H3): $E = (\mathsf{D}_d - I)\big(I - (\mathsf{D}_d - I)^{-1}\mathsf{C}_d^2\big)$. Row i of $(\mathsf{D}_d - I)^{-1}\mathsf{C}_d^2$ has a single nonzero entry $1/(i+1)$, so the ∞-norm is $1/2<1$, and the Neumann series inverts the second factor.

$F^{(\lambda)}$: $d = 2$, $D = \mathrm{diag}(\lambda, \lambda + 1)$, $C = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The diagonal entries of F are $\xi^2 + (\lambda - 1)\eta^2 - \lambda\zeta^2 = q_\lambda$ and $q_{\lambda+1}$, and the off-diagonal ones are c, so $F = F^{(\lambda)}$.

- (H1): D has distinct eigenvalues, and neither coordinate line is invariant under C.
- (H2) $\iff$ $\lambda \notin \{0, -1\}$.
- (H3): $E = D - 2I = \mathrm{diag}(\lambda - 2, \lambda - 1)$, invertible iff $\lambda \notin \{1, 2\}$.

Note $\hat{F}_2 = F^{(3)}$. □

Remark. For $\lambda \in \{-1, 0, 1, 2\}$ the pencil is not monic (the column condition of the proof fails: $\ker D$ or $\ker E$ is nonzero). This says nothing about whether $F^{(\lambda)}$ is an atom: $\mathcal{T}$ is sufficient, not maximal.

6.2 Proof of Theorem 6.1

Remark 6.3. (a candidate second witness for part (1)) p. 27

An earlier route suggests a second witness. Let $\theta \in \mathbb{C}$ be transcendental, $\tau_i'^{\theta} = 1 + x_i + \theta x_{i+1}$ (indices mod 3) and $g_\theta = \sum_i (\tau_i'^{\theta})^{-2} \in E_1$. We expect $g_\theta^{-1} \in E_2 \setminus \mathcal{U}_\infty$ but do not prove it; it is not used. The argument would be the same corank comparison (4>2) at the point $X_i = \lambda_\theta I + T_i$, $\lambda_\theta = -1/(1+\theta)$, $T_i = (N_i - \theta N_{i+1} + \theta^2 N_{i+2})/(1+\theta^3)$, $N_i = \begin{pmatrix} it^2 & -t \\ i^2 t^3 & -it^2 \end{pmatrix}$, where $I + X_i + \theta X_{i+1} = N_i$ with $N_i^2 = 0$, so that the target $H_{3,2}(\tau_0', \tau_1', \tau_2')$ vanishes; the target is an atom because $(x_0, x_1, x_2) \mapsto (\tau_0'^{\theta}, \tau_1'^{\theta}, \tau_2'^{\theta})$ is affine with linear part of determinant $1 + \theta^3 \neq 0$. The missing step is that no allowed binomial is identically singular on the curve, $\det(I - \beta h(X(t))) \not\equiv 0$ for all $h \neq 1$ and $\beta \neq 0$; the route assumes θ transcendental, and for $\theta = 2, 1, \frac{1}{4}$ some binomials do become singular on the curve. We prefer the integer witness ρ_2.

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the identities I + X_i + θX_{i+1} = N_i, N_i² = 0

- `ThreeAxes.WeightedTrinomials.weighted_trinomial`
- `ThreeAxes.WeightedTrinomials.N_sq`

✅ Checked when this page was built: WeightedTrinomials.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — weighted_trinomial, N_sq: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean WeightedTrinomials.lean` (SHA-256 of the compiled file: `798b3d2a89739fff…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 2 results, standard axioms only

Cite: Remark 6.3, p. 27, doi:10.5281/zenodo.23481332

↳ Used in: 6.4 (p. 40)

Remark 6.4. (the symmetric curve fails) p. 28

With $\theta = 1$, i.e. $\tau_i' = 1 + x_i + x_{i+1}$, the analogous argument breaks: on the whole variety where $\tau_0', \tau_1', \tau_2'$ are nilpotent, the commutator $h = [xy, yx]$ (a word with exponent sum 0) is unipotent, so $1 - h$ degenerates. The reason is an identity in GL_2: if $\operatorname{tr} X = \operatorname{tr} Y = -1$ and $Z = I + X + Y$ satisfies $Z^2 = 0$, then $(XY)(YX)(XY)^{-1}(YX)^{-1} = I + (\det X^{-1} - \det Y^{-1})Z$. If $\tau_0', \tau_1', \tau_2'$ evaluate to nilpotent 2×2 matrices, then $\operatorname{tr}(I + X_i + X_{i+1}) = 0$ for $i = 0, 1, 2$, so every $\operatorname{tr} X_i = -1$, and the identity applies with $X = X_0$, $Y = X_1$ (supplement C26).

Lean · ours, in part **show**

the commutator identity

- `ThreeAxes.WeightedTrinomials.hidden_commutator`

✅ Checked when this page was built: WeightedTrinomials.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — hidden_commutator: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean WeightedTrinomials.lean` (SHA-256 of the compiled file: `798b3d2a89739fff…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 1 result, standard axioms only

Cite: Remark 6.4, p. 28, doi:10.5281/zenodo.23481332

↳ Not used by another result of the paper.

7 The Dubrovin–Gräter–Hanke series (Theorem C)

Question. Where do the two hierarchies meet the complexity series of §2.3, i.e. which strata are stopping steps? The steps b_{kXX}, for every $k \geq 3$. For this the binomial bound and the size bound must be combined (Lemma 7.7), and an atom is needed for every k (Lemma 7.9).

7.1 Results

The rings D', Γ_2, Γ_2' and Γ_a' are those of §2.4.

Theorem 7.1. p. 28

> *For every $k \geq 3$ the step $b_{kXX} = k[\pi]$ is a stopping step of the series of §2.3, for D and $\mathcal{M} = \mathbb{C}^\times F_3$ in Jakobs' formulation [Jak19, Def. 1.1.4]. When the inversion pointer reaches b_{kXX}, the ring built so far lies in $\Gamma_{k-2}' \cap E_2$; for $k = 3$ it is exactly $\mathcal{U}_1 = E_1[V_1^{-1}]$. A witness is $s = u^{-2} + v^{-2} + w^{-2}$ for $k = 3$, and $f_k = u^{-2} + w^{-2} + \sum_{i=1}^{k-2} y^i u^{-2} y^{-i}$ for $k \geq 4$ (ψ below is a second witness for $k = 4$). In particular infinitely many stopping steps occur while the ring built by the series is still inside E_2.*

↳ Used in: Theorem C (p. 5), 8.3 (p. 55)

Proof

> Let D_x be as in Lemma 7.5. For $k = 3$: $s \in D_\pi^{+3}$ and $s^{-1} \notin \mathcal{U}_1 = D_x$ (Theorem 5.1). For $k \geq 4$: $f_k \in D_\pi^{+k}$ and $f_k^{-1} \notin \Gamma_{k-2}' \supseteq D_x$ (Theorem 7.3); for $k = 4$ also $\psi \in D_\pi^{+4}$ and $\psi^{-1} \notin \Gamma_2'$ (Theorem 7.2). In each case the witness lies in no stratum born before b_{kXX}, since its inverse would then be in D_x; so its complexity is $\mathrm{cp} = b_{kXX}$, $D_{b_{kXX}}^{-1} \not\subseteq D_x$, and the Case 3 at which the pointer reaches b_{kXX} selects it. That Case 3 exists: otherwise every Case 3 would select an earlier stratum, and the union of the series, which is D [Jak19, Def. 1.1.4], would lie in Γ_{k-2}', against $f_k^{-1} \notin \Gamma_{k-2}'$ (or $s^{-1} \notin \mathcal{U}_1$). The ring reached after the step, $D_x[(D_\pi^{+k} \setminus 0)^{-1}]$, still lies in E_2, since $D_\pi^{+k} \subseteq E_1$. □

Evidence & scope

- **Program:** lexicographic order of N(O), 8 checks with a control (supplement C19). `scripts/auditoria_S15/orden_N.py`

That the series has infinitely many stopping steps in total already follows from [HS15, Thm 4.6]: Cases 1 and 2 of the series do not raise the inversion height and each Case 3 raises it by at most one, while the union of the series is D [Jak19, Def. 1.1.4] and the inversion height of A in D is infinite. What Theorem 7.1 adds is that infinitely many of them occur inside the second stage, and which strata they are.

Theorem 7.2. p. 28

> *Let $\psi = u^{-2} + v^{-2} + w^{-2} + y\,u^{-2}y^{-1}$. Then $\psi^{-1} \in \mathcal{U}_2$ and $\psi^{-1} \notin \Gamma_2'$. Since $D' \subseteq \Gamma_2 \subseteq \Gamma_2'$, this gives $\mathcal{U}_1 \subsetneq D' \subsetneq \mathcal{U}_2$. Moreover $\mathcal{U}_\infty$ and Γ_2 are incomparable.*

↳ Used in: Theorem 7.1 (p. 41), 7.12 (p. 51)

Proof

$u^{-2} = u^{-1} \cdot 1 \cdot u^{-1}$ and $yu^{-2}y^{-1}$ has outer coefficients $y, y^{-1} \in A$, so $\psi \in \mathcal{F}_2$. By Lemma 7.8, F_ψ is full, so $H_\psi = \Delta_\psi(I + \Delta_\psi^{-1}\mathbf{1}u^2\mathbf{1}^\mathsf{T})$ is full, and since $I + \mathbf{a}\mathbf{b}^\mathsf{T}$ is invertible iff $1 + \mathbf{b}^\mathsf{T}\mathbf{a}$ is, $u^2\psi \neq 0$; hence $\psi^{-1} \in \mathcal{U}_2$. Also $yu^{-2}y^{-1} = (yuy^{-1})^{-1}(yuy^{-1})^{-1}$ with $yuy^{-1} = 1 + yxy^{-1}$ a binomial, so $\psi \in D_\pi^{+4}$.

Suppose $\psi^{-1} \in \Gamma_2'$. Since $H_\psi = \Delta_\psi + \mathbf{1}u^2\mathbf{1}^\mathsf{T}$ and $\psi = u^{-2} + \mathbf{1}^\mathsf{T}\Delta_\psi^{-1}\mathbf{1}$, Woodbury gives $H_\psi^{-1} = \Delta_\psi^{-1} - \Delta_\psi^{-1}\mathbf{1}\psi^{-1}\mathbf{1}^\mathsf{T}\Delta_\psi^{-1}$ with $\Delta_\psi^{-1} \in M_3(E_1)$, and $F_\psi^{-1} = \mathrm{diag}(1, 1, y^{-1})H_\psi^{-1} \in M_3(\Gamma_2') = M_3(U_{\Sigma_2'})$ (Lemma 3.6). Apply Theorem 3.4 with $G = H = F_\psi$ and $\mu_Q = \nu_{k+1}$ along γ_k (the case $G = H$, $\mu = \nu_j$ of Theorem 3.4). Given $Q \in \Sigma_2'$, choose k larger than $e(Q)$: then $\nu_{k+1}(Q^{\gamma_k}) \leq 2$ for $Q = Q_s$ or x_j and for the borders of $\mathcal{U}_1$ (Lemma 4.4 with $q = 1$), and ≤ 4 for $Q = \tilde{B}_M$ (Lemma 7.7). Since $\nu_{k+1}(F_\psi^{\gamma_k}) = 6$ for every k (Lemma 7.8), this contradicts Theorem 3.4. Hence $\psi^{-1} \notin \Gamma_2'$.

$D' \subseteq \Gamma_2 \subseteq \Gamma_2'$ by Lemma 7.6, so $\psi^{-1} \in \mathcal{U}_2 \setminus D'$. Each element of D_π lies in $\mathcal{F}_2$, so $D' \subseteq \mathcal{U}_2$; and $s = u^{-2} + v^{-2} + w^{-2} \in D_\pi^{+3}$ gives $s^{-1} \in D' \setminus \mathcal{U}_1$ by Theorem 5.1. Finally $\psi^{-1} \in \mathcal{U}_2 \subseteq \mathcal{U}_\infty$ and $\psi^{-1} \notin \Gamma_2$, while $\rho_2 \in \Gamma_2$ ($\hat{F}_2$ is a full 2×2 matrix over A) and $\rho_2 \notin \mathcal{U}_\infty$ (Theorem 6.1). Both rings lie in E_2: $\mathcal{U}_\infty \subseteq E_2$ (§2.1), and for a full 2×2 matrix over A a pivot (after a permutation if needed) leaves a Schur complement in E_1, so the entries of its inverse lie in E_2. □

Theorem 7.3. p. 28

For every $k \geq 4$, $f_k = u^{-2} + w^{-2} + \sum_{i=1}^{k-2} y^i u^{-2} y^{-i}$ is a sum of k elements of D_π, and $f_k^{-1} \notin \Gamma_{k-2}'$.

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added to the Lean repository after publication (21-oct-2026): f_k⁻¹ as minus a corner of the inverse of an arrow matrix (the instance of Appendix A.1)

- `ThreeAxes.ArrowInstances.f_inv_eq_neg_corner`

✅ Checked when this page was built: ArrowInstances.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — f_inv_eq_neg_corner: propext, Classical.choice, Quot.sound.

Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean ArrowInstances.lean` (SHA-256 of the compiled file: `191ca5ad3d1c83de`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 1 result, standard axioms only

Proof: in this note · **Covers:** not formalized · **Cite:** Theorem 7.1, p. 28, doi:10.5281/zenodo.23481332

↳ Used in: Theorem C (p. 5), Theorem 7.1 (p. 41), Corollary 7.10 (p. 48), 7.12 (p. 51), 8.2 (p. 53)

Proof

Each summand of f_k is the inverse of the square of w or of a binomial $y^i u y^{-i} = 1 + y^i x y^{-i}$, so $f_k \in D_\pi^{+k} \cap \mathcal{F}_2$, and $f_k \neq 0$ (Lemma 7.9(a)). Suppose $f_k^{-1} \in \Gamma_{k-2}'$. Woodbury gives $F_k^{-1} = \mathrm{diag}(y^{-b})\left(\Delta_k^{-1} - \Delta_k^{-1}\mathbf{1}f_k^{-1}\mathbf{1}^\mathsf{T}\Delta_k^{-1}\right)$ with Δ_k^{-1} over E_1, so F_k^{-1} has entries in $\Gamma_{k-2}' = U_{\Sigma_{k-2}'}$ (Lemma 3.6 for $a = k - 2$). Given $Q \in \Sigma_{k-2}'$, choose $\kappa > e(Q)$; then $\nu_{\kappa+1}(Q^{\gamma_\kappa}) \leq 2(k-2)$ (Lemma 7.7, Lemma 4.4), while $\nu_{\kappa+1}(G_k^{\gamma_\kappa}) = 2(k-1)$ (Lemma 7.9(d)). Theorem 3.4, with $G = G_k$, $H = F_k$ and $\mu_Q = \nu_{\kappa+1}$ along γ_κ, gives a contradiction. □

Theorem 7.3 with $k = 4$ and Remark 8.2 give $f_4^{-1} \in \mathcal{U}_2 \setminus \Gamma_2'$, the separation of Theorem 7.2 with another witness; ψ has a nine-dimensional linear certificate of atomicity (Lemma 7.8), while f_k needs the grading argument of Lemma 7.9.

The proof of Theorem 7.3 needs an atom for every k. On P the harmonic matrices need not be atoms: they can split off factors stably associated with a letter (Remark 8.3). We avoid this in two ways: we choose targets in the group algebra of the free group on u, y, w, where a grading argument proves atomicity uniformly (Lemma 7.9), and we apply the obstruction to one atomic factor instead of to the whole matrix (Theorem 3.4).

7.2 The ring reached at $k[\pi]$

Lemma 3.6 holds with 2 replaced by a (its proof does not use $a \le 2$), with $\Sigma'_a = \Sigma_1 \cup \{\tilde{B}_M : M \text{ full over } V_1 \text{ of size} \le a\}$; over $\mathbb{C}[F_N]$ the same holds with Σ_0, Σ_1 formed with $x_1, \dots, x_N$.

Lemma 7.4. (compression at level k) p. 29

> *Let $k \ge 3$ and let $0 \ne f \in D_\pi^{+n} + V_1$ with $n \le k-1$. Then $f^{-1} \in \Gamma'_{k-2}$.*

Lean · ours, in part **show**

the inverse identities (fullness assumed)

- `ThreeAxes.InverseIdentities.compression_inv`
- `ThreeAxes.InverseIdentities.three_term_inv`
- `ThreeAxes.InverseIdentities.one_term_inv`

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Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 3 results, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** part: the inverse identities (fullness assumed) · **Cite:** Lemma 7.4, p. 29, doi:10.5281/zenodo.23481332

↳ Used in: Lemma 7.5 (p. 44), Lemma 7.6 (p. 44), Corollary 7.10 (p. 48), 7.11 (p. 50), 7.12 (p. 51), 8.3 (p. 55)

Proof

Write $f = d_1^{-1} + \cdots + d_{n'}^{-1} + r$ with $n' \le n$ pure summands and $r \in V_1$ (§2.3; V_1 is closed under sums). If $n' \le 2$, then $f = d_1^{-1} + d_2^{-1} + r$, $d_1^{-1} + r$ or r, and

$$f^{-1} = d_2(d_1 r d_2 + d_1 + d_2)^{-1} d_1, \qquad (1 + d_1 r)^{-1} d_1 \qquad \text{or} \qquad r^{-1},$$

with $d_1 r d_2 + d_1 + d_2,\ 1 + d_1 r \in V_1$, so $f^{-1} \in \mathcal{U}_1 \subseteq \Gamma'_{k-2}$. If $n' \ge 3$, then $k \ge 4$. Put $Y = d_1 r d_2 + d_1 + d_2 \in V_1$ and

$$M = \begin{pmatrix} Y & d_1 \mathbf{1}_{n'-2}^{\mathsf{T}} \\ -\mathbf{1}_{n'-2} d_2 & \operatorname{diag}(d_3, \dots, d_{n'}) \end{pmatrix} \in M_{n'-1}(V_1).$$

The Schur complement of the diagonal block is $Y + d_1\big(\sum_{i \ge 3} d_i^{-1}\big) d_2 = d_1 f d_2 \ne 0$, so M is full and $f^{-1} = d_2 (M^{-1})_{11} d_1$, with $n' - 1 \le k - 2$. □

Evidence & scope

- **Program:** f^-1 = d_2 (M^-1)_11 d_1 for m = 2..7, with controls (supplement C20). `scripts/auditoria_S17/s17_p2_compresion.py`

The hypotheses of Lemma 2.2(4),(5) hold. If $D_\pi^{+j} \subseteq D_\pi^{+(j-1)} + V_1$ for some $j \ge 2$, then by induction $D_\pi^{+n} \subseteq D_\pi^{+(j-1)} + V_1$ for all $n \ge j$. For $j \le 3$ this puts $s \in D_\pi^{+2} + V_1$, and then $s^{-1} \in \mathcal{U}_1$ by

Lemma 7.4, against Theorem 5.1. For $j \geq 4$ it puts $f_j \in D_\pi^{+(j-1)} + V_1$, and then $f_j^{-1} \in \Gamma'_{j-2}$ by Lemma 7.4, against Theorem 7.3. If $D_{\omega+1}D_{\omega+1} \subseteq V_1$, then V_1 would be a ring containing s, and $s^{-1} \in \mathcal{U}_1$. (Theorems 5.1, 7.3 and Lemma 7.4 do not use the series; in them D_π may be read as the set $V_1 \cup D_{\omega+1}D_{\omega+1}$, which is defined without the hypotheses of Lemma 2.2(4),(5).)

Lemma 7.5. (the ring reached at $k[\pi]$) p. 29

> *Let $k \geq 3$, and let D_x be the ring built by the series when the inversion pointer reaches b_{kXX}. Then D_x is generated by $\mathcal{U}_1$ and the inverses of the nonzero elements of $D_\pi^{+n} + V_1$, $n \leq k-1$; $\mathcal{U}_1 \subseteq D_x \subseteq \Gamma'_{k-2} \cap E_2$; and for $k = 3$, $D_x = \mathcal{U}_1$.*

↳ Used in: Theorem 7.1 (p. 41), Corollary 7.10 (p. 48), 7.12 (p. 51)

Proof

> By Lemma 2.2, every stratum born before $k[\pi]$ is contained in A, in V_1, or in $D_\pi^{+n} + V_1$ with $n \leq k-1$. By §2.3, at the Case 3 where the pointer reaches b_{kXX} the current ring contains D_γ^{-1} for every earlier stratum γ, whether γ was a stopping step (its inverses were adjoined there) or not (the pointer passed it at a Case 3 where its inverses already lay in the ring); and between two Case 3 steps Cases 1 and 2 only add sums and products. So D_x is generated by A and the inverses of the nonzero elements of all earlier strata. Conversely, for $n \leq k-1$ the elements of $D_\pi^{+n} + V_1$ are the sums indexed by the multiset $n[\pi] + [\sigma]{<}k[\pi]$; since Case 1 selects the least multiset with a new sum (Lemma 2.2(5)), they already lie in the set built before b_{kXX}, so the inverses of its nonzero elements lie in D_x. These inverses lie in E_1, in $\mathcal{U}_1$, or in Γ'_{k-2} by Lemma 7.4; and D_x contains A^{-1} and V_1^{-1}, since $D_\omega = A$ and $D_\sigma = V_1$ are earlier strata. This gives the first two claims, and the third because $\Gamma'_1 = \mathcal{U}_1$. Finally $\mathcal{U}_1 \subseteq E_2$ and $D_\pi^{+n} + V_1 \subseteq E_1$, so the inverses of its nonzero elements lie in E_2. □

Evidence & scope

- **Program:** order of the strata: 6480 multisets, k = 4..8, with a control (supplement C21). `scripts/auditoria_S17/s17_p4_orden.py`

Lemma 7.6. ($D' \subseteq \Gamma_2$) p. 29

> *Let $e \in D_\pi^{+3}$ be nonzero.*
>
> 1. *If $e = d_1^{-1} + d_2^{-1} + d_3^{-1}$ is a sum of three pure elements, then*
> $$H_3 = \begin{pmatrix} d_1 + d_2 & d_1 \\ d_1 & d_1 + d_3 \end{pmatrix} \in M_2(A) \text{ is full and } e^{-1} = d_1 - (d_1\ d_1)\, H_3^{-1} \binom{d_1}{d_1} \in \Gamma_2$$
> .
> 2. *Otherwise $e^{-1} \in \mathcal{U}_1$.*

Lean · ours, in part show

the inverse identities (fullness assumed)

- `ThreeAxes.InverseIdentities.woodbury_sum`
- `ThreeAxes.InverseIdentities.three_term_inv`

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Axioms — woodbury_sum, three_term_inv: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean InverseIdentities.lean` (SHA-256 of the compiled file: `3e2df75becd9213c`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 2 results, standard axioms only

Proof: cited, [}\pi {] (not ours) · **Covers:** not formalized · **Cite:** Lemma 7.5, p. 29, doi:10.5281/zenodo.23481332

↳ Used in: Theorem 7.2 (p. 41)

Proof

(1) $H_3 = \Delta + \mathbf{1} d_1 \mathbf{1}^{\mathsf{T}}$ with $\Delta = \operatorname{diag}(d_2, d_3)$, and $e = d_1^{-1} + \mathbf{1}^{\mathsf{T}} \Delta^{-1} \mathbf{1}$; Woodbury gives the formula, and H_3 is full iff $1 + d_1(d_2^{-1} + d_3^{-1}) = d_1 e \neq 0$. (2) With two pure summands and one in V_1, $e = p'^{-1} + q'^{-1} + r$ and $e^{-1} = q'(p'rq' + p' + q')^{-1}p'$ with $p'rq' + p' + q' \in V_1$ (Lemma 7.4); with one pure summand, $(d^{-1} + r)^{-1} = (1 + dr)^{-1} d$ with $1 + dr \in V_1$; with none, $e \in V_1$. □

Evidence & scope

- **Program:** three-term inverse, Woodbury, intermediate strata, with controls (supplement C16). `scripts/auditoria_S16/s16_prop51_lemas6.py`

7.3 The bound $2a$

We use the presentations B_M, $\tilde{B}_M$ of §3.4 and $e(M)$ of §4.1.

Lemma 7.7. (the bound $2a$) p. 29

For every $j \geq 1$, $\nu_j(B_M^{\gamma_k}) \leq 2a + \sum_s \nu_j(\ell_s^{\gamma_k})$. In particular $\nu_{k+1}(\tilde{B}_M^{\gamma_k}) \leq 2a$ for $k > e(M)$.

↳ Used in: Theorem 7.2 (p. 41), Theorem 7.3 (p. 42), 7.12 (p. 51)

Proof

Over $R_j = \mathbb{C}[[t]]/t^j$, a square matrix N of size n satisfies $\nu_j(N) = n - \mu(\operatorname{im} N)$, with μ the minimal number of generators. Let $n = 2a + 2m$ be the size of B_M^{γ} and pr the projection onto the last $2m$ coordinates. Then $\operatorname{pr}(\operatorname{im} B_M) = \operatorname{im}[-C\ \Delta] \supseteq \operatorname{im} \Delta$. An image of a module needs no more generators, and neither does a submodule, viewing R_j-modules as $\mathbb{C}[[t]]$-modules with the same μ ($R_j = \mathbb{C}[[t]]/t^j$ is not a principal ideal domain, but $\mathbb{C}[[t]]$ is), so $\mu(\operatorname{im} B_M) \geq \mu(\operatorname{im} \Delta)$ and $\nu_j(B_M) \leq 2a + 2m - \mu(\operatorname{im} \Delta) = 2a + \nu_j(\Delta) = 2a + \sum_s \nu_j(\ell_s)$. For $k > e(M)$, every $\ell_s^{\gamma_k}$ has largest part at most k (Lemma 4.3: units, exceptional binomials with parts $(1, k)$, the others with parts $< k$), so $\nu_{k+1}(\ell_s^{\gamma_k}) = 0$. The pivots of the lift are units on the arc. □

Evidence & scope

- **Program:** the bound 2a on 588 presentations, k = 4..10, with controls (supplement C17). `scripts/auditoria_S14/s14_falsa.py`

For $a = 1$ this is the bound 2 of Lemma 4.4 with $q = 1$. The bound 4 is attained: $\operatorname{diag}(u^2, v^2)$ and $H_3(u^2, v^2, w^2)$ have four parts equal to $k + 1$. A presentation as in Lemma 7.4 places d_1, d_2, d_3, whose evaluations may have parts $k + 1$, in the matrix block, where they are bounded by size; a presentation that put them in Δ would only give ≤ 8.

7.4 The target for ψ

Let $\Delta_\psi = \operatorname{diag}(v^2, w^2, yu^2y^{-1})$, $H_\psi = \Delta_\psi + u^2 J_3$ (J_3 the all-ones matrix) and

$$F_\psi = H_\psi \operatorname{diag}(1, 1, y) = \begin{pmatrix} u^2 + v^2 & u^2 & u^2 y \\ u^2 & u^2 + w^2 & u^2 y \\ u^2 & u^2 & yu^2 + u^2 y \end{pmatrix} \in M_3(P).$$

Lemma 7.8. p. 30

> *(a) F_ψ is a full atom over P. (b) For every $k \geq 1$, $F_\psi^{\gamma_k} = t^{k+1}\big((I_3 + J_3) \otimes I_2\big)\,\mathrm{diag}(I, I, Y)$ with $Y = y^{\gamma_k}$, $\det Y = 1 - t^{k+1}$; hence all six parts of $F_\psi^{\gamma_k}$ equal $k+1$ and $\nu_{k+1}(F_\psi^{\gamma_k}) = 6$.*

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the integer data of the certificate of (a)

- `ThreeAxes.AtomCertificate.Q2_mul_K`
- `ThreeAxes.AtomCertificate.det_Q2`
- `ThreeAxes.AtomCertificate.Kv_identity`
- `ThreeAxes.AtomCertificate.det_matR`
- `ThreeAxes.AtomCertificate.det_matO`

✅ Checked when this page was built: AtomCertificate.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — Q2_mul_K, det_Q2, Kv_identity, det_matR, det_matO: propext, Classical.choice, Quot.sound. Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean AtomCertificate.lean` (SHA-256 of the compiled file: `948997ae47b68a33`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 5 results, standard axioms only

Proof: in this note · **Covers:** not formalized · **Cite:** Lemma 7.7, p. 29, doi:10.5281/zenodo.23481332

↳ Used in: Theorem 7.2 (p. 41), Lemma 7.9 (p. 47), 8.1 (p. 52)

Proof

(b) $u^2, v^2, w^2 \mapsto t^{k+1}I$ (Lemma 4.1), central; $\det(I_3 + J_3) = 4$; and $\det(U - I) = 1 - \operatorname{tr} U + \det U = 1 - t^{k+1}$ for $U = v^{\gamma_k}$, a unit of $\mathbb{C}[[t]]$.

1. Let $Q = Q_0 + uQ_u + vQ_v + wQ_w$ be the 9×9 pencil with nonzero entries (row, column, coefficient)

- Q_0: $(i, i, 1)$ for $i = 4, \dots, 9$;
- Q_u: $(1,4,1)$, $(2,4,1)$, $(3,4,1)$, $(3,7,-1)$, $(4,1,-1)$, $(4,2,-1)$, $(4,3,1)$, $(4,9,1)$, $(7,3,-1)$, $(8,7,1)$;
- Q_v: $(1,5,1)$, $(3,8,-1)$, $(5,1,-1)$, $(9,3,1)$;
- Q_w: $(2,6,1)$, $(6,2,-1)$.

Its block Q_{22} (rows and columns 4–9) is $I + N$ with $N^2 = 0$, so $Q_{22}^{-1} = I - N$ is polynomial, the Schur complement of Q_{22} is exactly $+F_\psi$, and $Q \sim F_\psi \oplus I_6$ over P. The pencil is epic (the stacked coefficient matrices have rank 9 both ways). At $u = v = w = 2$, $\det Q(2,2,2) = 256$, and $K_i = 16\,Q(2,2,2)^{-1}Q_i$ ($i = u, v, w$) are integer matrices. With $\mathbf{r} = (0,0,-1,0,0,0,-2,4,1)^{\mathsf{T}}$ and $\boldsymbol{\ell} = (2,0,-2,0,2,0,0,3,0)$,

$$K_v^2\big(K_v^2 - 12K_v + 48I\big) = 2048\,\mathbf{r}\boldsymbol{\ell}, \tag{7.1}$$

and, writing $K_\omega = K_{\omega_1}K_{\omega_2}\cdots$ for a word ω (left to right),

$$\det\big[K_\omega \mathbf{r}\big]_{\omega \in \mathcal{R}} = 5 \cdot 2^{56}, \qquad \det\big[\boldsymbol{\ell} K_\omega\big]_{\omega \in \mathcal{O}} = 2^{49}, \tag{7.2}$$

with $\mathcal{R} = (\varnothing, u, uu, vu, wu, uuu, vuu, uvu, vvu)$ and $\mathcal{O} = (\varnothing, u, w, uu, uv, wu, ww, uuu, uuv)$. By (7.1) the rank-one matrix $\mathbf{r}\boldsymbol{\ell}$ is a polynomial without constant term in K_v, so it lies in the algebra generated by the K_i; by (7.2) the vectors $K_\omega \mathbf{r}$ and the covectors $\boldsymbol{\ell} K_{\omega'}$ form bases, so the products $K_\omega(\mathbf{r}\boldsymbol{\ell})K_{\omega'}$ span M_9. The translated pencil is monic with invertible constant term $Q(2,2,2)$, and its normalized coefficients generate M_9, so it is an atom by Lemma 2.1;

the shift is an automorphism (§2.2(d)) and $Q \sim F_\psi \oplus I_6$ (§2.2(b),(c)), so F_ψ is an atom. $Q(2,2,2)$ is invertible, so Q, and with it F_ψ, is full (§2.2(a),(b)). □

Evidence & scope

- **Program:** the 9 x 9 certificate: Schur = +F, det 256, (7.1), (7.2), in exact arithmetic (supplement C18). `scripts/auditoria_S16/s16_cert9.py`

7.5 Proof of Theorem 7.2

7.6 The targets for every k: proof of Theorem 7.3

For $m \geq 2$ let

$$f_k = u^{-2} + w^{-2} + \sum_{i=1}^{m} y^i u^{-2} y^{-i} \qquad (k = m+2),$$

$\Delta_k = \operatorname{diag}(w^2, yu^2y^{-1}, \dots, y^m u^2 y^{-m})$, $H_k = \Delta_k + u^2 J_{k-1}$ and $F_k = H_k \operatorname{diag}(1, y, \dots, y^m) \in M_{k-1}(P)$, whose entries are $u^2y^b + \delta\, y^b u^2$ ($y^0 = 1$ and w^2 in place of y^0u^2 for the first index). Let $\Lambda_y = P[u^{-1}, y^{-1}, w^{-1}] \subset D$. The automorphism of P fixing y with $x \mapsto 1+x$, $z \mapsto 1+z$ extends to D (by the universal property of the universal field of fractions) and maps A onto Λ_y, so $\Lambda_y = \mathbb{C}[F(u,y,w)]$ is the group algebra of the free group on u, y, w; it is a fir [Coh06, Cor. 7.11.8], hence a weakly finite domain, and its units are trivial [Jak19, Thm 1.3.10].

Lemma 7.9. p. 31

(a) f_k is an atom of Λ_y. (b) F_k is full, and an atom over Λ_y. (c) No atomic factor of F_k over P is stably associated with u or with w. (d) In a complete factorization of F_k over P exactly one atomic factor G_k is not stably associated with y, and $\nu_{\kappa+1}(G_k^{\gamma_\kappa}) = 2(k-1)$ for every $\kappa \geq 1$.

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the two evaluation points of (c) (u and u² evaluate alike there)

- `ThreeAxes.BinomialSizeFamily.det_scalar_point`
- `ThreeAxes.BinomialSizeFamily.idemBlock_isUnit`

✅ Checked when this page was built: BinomialSizeFamily.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — det_scalar_point, idemBlock_isUnit: propext, Classical.choice, Quot.sound. Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean BinomialSizeFamily.lean` (SHA-256 of the compiled file: `01db4f029c94cedf`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 2 results, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** part: the two evaluation points of (c) (u and u² evaluate alike there) · **Cite:** Lemma 7.9, p. 31, doi:10.5281/zenodo.23481332

↳ Used in: Theorem 7.3 (p. 42), Corollary 7.10 (p. 48), 7.11 (p. 50), 8.1 (p. 52)

Proof

(a) Grade Λ_y by the exponent of w and put $T = \sum_{i=0}^{m} y^i u^{-2} y^{-i}$, so that $f' = w f_k w = 1 + wTw$ has components in degrees 0 and 2 only. Let $f' = gh$. The extreme components multiply, and the lowest ones give $g_a h_c = 1$, so g_a is a unit, i.e. a scalar times a group element; replacing g, h by $gg_a^{-1}, g_a h$ we may assume $g_a = h_c = 1$ in degree 0. The widths of g and h add up to 2. If one of them is 0, that factor is 1. Otherwise $g = 1 + g_1$, $h = 1 + h_1$ with $g_1 + h_1 = 0$ and $g_1 h_1 = wTw$, so $-g_1^2 = wTw$. For a bi-invariant order of the free group the leading word of a product is

the product of the leading words, so the leading word of wTw, which is one of the distinct reduced words $W_i = wy^iu^{-2}y^{-i}w$, would be a square. But W_i is cyclically reduced, so a square root of it would be cyclically reduced and $W_i = hh$ as words; the first letter of W_i is w and the letter in position $i+3$ is u^{-1}. Hence f', which is not a unit, is an atom, and so is $f_k = w^{-1}f'w^{-1}$.

1. Over Λ_y, Δ_k is invertible and $H_k = \Delta_k(I + \mathbf{a}\mathbf{b}^{\mathsf{T}})$ with $\mathbf{a} = \Delta_k^{-1}\mathbf{1}$, $\mathbf{b}^{\mathsf{T}} = u^2\mathbf{1}^{\mathsf{T}}$; $I + \mathbf{a}\mathbf{b}^{\mathsf{T}}$ is stably associated with $1 + \mathbf{b}^{\mathsf{T}}\mathbf{a} = u^2 f_k$. Since $\operatorname{diag}(1, y, \dots, y^m)$ and u^2 are units of Λ_y, F_k is stably associated with $f_k \neq 0$, hence full, and stable association preserves atoms [Coh06, Cor. 0.5.5, Thm 3.2.3 and p. 195].
2. If an atomic factor were stably associated with u, then at any matrix point X where $u(X)$ is singular, that factor, and hence $F_k(X)$, would be singular (evaluate the stable association and the factorization). Take $n = m+1$, $u \mapsto U = \operatorname{diag}(0, 1, \dots, 1)$, $w \mapsto I$ and $y \mapsto Y$ the cyclic permutation matrix, so that the evaluations of u^2 and of $y^iu^2y^{-i}$ are $I - e_ie_i^{\mathsf{T}}$ $(i = 0, \dots, m)$. A vector $\mathbf{v}$ in the kernel of H_k gives, with $s = (I - e_0e_0^{\mathsf{T}})\sum \mathbf{v}$, $\mathbf{v}_w = -s$, $\mathbf{v}_i = -s + c_ie_i$ with $s \perp e_0, \dots, e_m$ and $(m+2)s = \sum c_ie_i$; since $e_0, \dots, e_m$ is a basis, $s = 0$, so $\sum c_ie_i = 0$, $c_i = 0$ and $\mathbf{v} = 0$ (no division is needed). So F_k is invertible at a point where u is singular. For w a scalar point suffices: at $u = y = 1$, $w = 0$, $F_k = H_k = \operatorname{diag}(0, 1, \dots, 1) + J_{k-1}$, and subtracting its first row from the others leaves the rows $e_1^{\mathsf{T}}, \dots, e_m^{\mathsf{T}}$, so $\det F_k = 1$.
3. Let $F_k = G_1 \cdots G_s$ be a complete factorization over P. Over a weakly finite ring a product of square matrices is invertible only if every factor is: if XY is invertible, X has the right inverse $Y(XY)^{-1}$, so X is invertible, and then so is Y. By (b), F_k is not invertible over Λ_y, so some G_{i_0} is not. If another G_{i_1} were not invertible over Λ_y, say $i_1 > i_0$, then $F_k = XY$ with $X = G_1 \cdots G_{i_0}$ and $Y = G_{i_0+1} \cdots G_s$ full and both non-invertible over Λ_y, against (b). So each G_i, $i \neq i_0$, is invertible over $\Lambda_y = U_{\{u,y,w\}}$. By Lemma 3.1 with $\Sigma = \{u, y, w\}$ (atoms of P, being of degree one), it is stably associated with u, y or w, and by (c) with y. On the other hand, G_{i_0} is not stably associated with y, since it would then be invertible over Λ_y (converse part of Lemma 3.1). On the arc, $\det y^{\gamma_\kappa} = 1 - t^{\kappa+1}$ (as in Lemma 7.8(b)), so y^{γ_κ}, and by stable association every $G_i^{\gamma_\kappa}$ with $i \neq i_0$, is invertible over $\mathbb{C}[[t]]$. Hence $\nu_{\kappa+1}(G_{i_0}^{\gamma_\kappa}) = \nu_{\kappa+1}(F_k^{\gamma_\kappa})$. Each $y^iu^2y^{-i}$ evaluates to $t^{\kappa+1}I$ (Lemma 4.1: $u^2 \mapsto t^{\kappa+1}I$ is central), so $F_k^{\gamma_\kappa} = t^{\kappa+1}\big((I_{k-1} + J_{k-1}) \otimes I_2\big)\operatorname{diag}(Y^b)$ with $\det(I + J) = k \neq 0$, and all $2(k-1)$ parts equal $\kappa + 1$. □

Evidence & scope

- **Program:** words W_i are not squares, root lemma, F_k stably associated with f_k (supplement C22). `scripts/auditoria_S18/s18_comprobaciones.py`
- **Program:** F_k invertible where u (or w) is singular, k = 3..10 (supplement C23). `scripts/kXX_todo_k/paso_C_puntos.py`

Lemma 7.9(d) is the input that Theorem 3.4 needs when the obstruction is applied to one atomic factor $G = G_k$ of $H = F_k$ (the case $\mu = \nu_j$ with a proper atomic factor of Theorem 3.4).

Corollary 7.10. p. 32

(a) For every $d \geq 2$ and $r \geq 1$, $A_{d-1} \cap \mathcal{U}_r \subsetneq A_d \cap \mathcal{U}_r$; a witness is $e_d = q_d^{-1}$ with $q_d = w^{-1} + \sum_{i=0}^{d-1} y^iu^{-1}y^{-i}$, and $e_d \in (A_d \cap \mathcal{U}_1) \setminus A_{d-1}$. The range $r \geq 1$ is optimal, since $A_d \cap \mathcal{U}_0 = E_1$ for every d. (b) The bound of Lemma 7.4 is optimal: for $n \geq 3$ there is a sum f of n pure elements with $f^{-1} \notin \Gamma'_{n-2}$. (c) Let $\mathcal{D}_k$ be the ring D_x of Lemma 7.5, reached when the inversion pointer reaches b_{kXX} (it does, §7.7). Then $\mathcal{U}_1 = \mathcal{D}_3 \subsetneq \mathcal{D}_4 \subsetneq \mathcal{D}_5 \subsetneq \cdots$ and $\bigcup_k \mathcal{D}_k = \mathcal{U}_2$.

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the identities and the two evaluation points of (a); the identity used in (c); added to the Lean repository after publication (21-oct-2026): e_d = q_d⁻¹ as minus a corner of the inverse of an arrow matrix

- `ThreeAxes.BinomialSizeFamily.rankOne_mul_inv`
- `ThreeAxes.BinomialSizeFamily.rankOneInv_mul`
- `ThreeAxes.BinomialSizeFamily.inv_q_eq`
- `ThreeAxes.BinomialSizeFamily.conj_add_mul_pow`
- `ThreeAxes.BinomialSizeFamily.det_scalar_point`
- `ThreeAxes.BinomialSizeFamily.idemBlock_isUnit`
- `ThreeAxes.PhiCompression.inv_mul_conj`
- `ThreeAxes.ArrowInstances.e_inv_eq_neg_corner`

✅ Checked when this page was built: ArrowInstances.lean, BinomialSizeFamily.lean, PhiCompression.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — rankOne_mul_inv, rankOneInv_mul, inv_q_eq, conj_add_mul_pow, det_scalar_point, idemBlock_isUnit, inv_mul_conj, e_inv_eq_neg_corner: propext, Classical.choice, Quot.sound.

Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean ArrowInstances.lean BinomialSizeFamily.lean PhiCompression.lean` (SHA-256 of the compiled file: `01db4f029c94cedf`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 8 results, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** part: the identities and the two evaluation points of (a); the identity used in (c); added to the Lean repository after publication (21-oct-2026): e_d = q_d⁻¹ as minus a corner of the inverse of an arrow matrix · **Cite:** Corollary 7.10, p. 32, doi:10.5281/zenodo.23481332

🆕 **Classical and new:** the refinement of the size chain inside $\mathcal{U}_1$ has not been studied, to the paper's knowledge (§8.8).

↳ Used in: 7.11 (p. 50), 8.1 (p. 52), 8.2 (p. 53)

Proof

(a) Let $d \geq 2$, $\Delta = \operatorname{diag}(w, yuy^{-1}, \dots, y^{d-1}uy^{-(d-1)})$, $H = \Delta + uJ_d$ and $L = H\operatorname{diag}(1, y, \dots, y^{d-1}) \in M_d(P)$, whose entries are $uy^b + \delta\, y^b u$ (w in place of y^0u for the first index). This is the construction of §7.6 with u, w in place of u^2, w^2, and Lemma 7.9(a)–(c) and the first part of (d) hold for q_d and L with the same proofs. In (a), $wq_d = 1 + wT$ with $T = \sum_{i=0}^{d-1} y^i u^{-1} y^{-i}$ has components in degrees 0 and 1 only, so one of two factors has width 0, hence is a unit, and the argument with squares is not needed. In (b), $1 + u\mathbf{1}^{\mathsf{T}}\Delta^{-1}\mathbf{1} = uq_d$. In (c), with $n = d$, the matrix U is idempotent, so u and u^2 have the same evaluation, and the two points give the same matrices as there. Each summand of q_d is an inverse binomial ($w^{-1} = (1+z)^{-1}$, $y^i u^{-1} y^{-i} = (1 + y^i x y^{-i})^{-1}$), so $q_d \in V_1$, $q_d \neq 0$ by (a), and $e_d \in \mathcal{U}_1 \subseteq \mathcal{U}_r$. Woodbury gives $e_d = u - u\mathbf{1}^{\mathsf{T}}H^{-1}\mathbf{1}u \in A_d$, since H is a full $d \times d$ matrix over A. If $e_d \in A_{d-1}$, then $H^{-1} = \Delta^{-1} - \Delta^{-1}\mathbf{1}e_d\mathbf{1}^{\mathsf{T}}\Delta^{-1}$, with Δ^{-1} over $E_1 = A_1$, and $L^{-1} = \operatorname{diag}(y^{-b})H^{-1}$ have entries in A_{d-1}. Every entry of L vanishes at χ, and the atomic factors of L other than the exceptional one, G, are stably associated with y, which is invertible at χ; so corank $G^{\chi} = d$, while the matrices of $\Sigma_A^{(d-1)}$ have corank at most $d-1$ at χ (Remark 7.11(i)). Theorem 3.4, with G, $H = L$ and μ the corank at χ, gives a contradiction. Finally $A_d \cap \mathcal{U}_0 = A_d \cap E_1 = E_1$, since $E_1 = A_1 \subseteq A_d$. (b) s for $n = 3$ (Theorem 5.1, $\Gamma_1' = \mathcal{U}_1$) and f_n for $n \geq 4$ (Theorem 7.3). (c) Expanding the coefficients into monomials, a term of an element of $\mathcal{F}_2$ with two inverse binomials is $m_0\ell_1^{-1}m_1\ell_2^{-1}m_2 = (\ell_1'^{-1}m_0m_1)(\ell_2^{-1}m_2)$ with $\ell_1' = m_0\ell_1m_0^{-1}$, a product of two elements of $D_{\omega+1}$ (Lemma 2.2(2)); terms with fewer inverses lie in $V_1 \subseteq D_\pi$. With $D_\pi \subseteq \mathcal{F}_2$ this gives $\mathcal{F}_2 = \bigcup_n D_\pi^{+n}$. By Lemma 7.5, $\mathcal{D}_k \subseteq \mathcal{D}_{k+1}$ and $\mathcal{D}_k \subseteq \mathcal{U}_2$, and $\bigcup_k \mathcal{D}_k$ contains E_1 and the inverse of every nonzero element of $\mathcal{F}_2$, so it is $\mathcal{U}_2$. Finally $s \in D_\pi^{+3}$ gives $s^{-1} \in \mathcal{D}_4 \setminus \mathcal{D}_3$ (Theorem 5.1), and for $k \geq 4$, $f_k \in D_\pi^{+k}$ gives $f_k^{-1} \in \mathcal{D}_{k+1} \setminus \mathcal{D}_k$, since $\mathcal{D}_k \subseteq \Gamma_{k-2}'$ (Lemma 7.5 and Theorem 7.3). □

Evidence & scope

- **Program:** Woodbury for e_d and H⁻¹, polynomial entries of L, det of L at the idempotent and scalar points (d = 2..10), L vanishes at χ, with controls. `scripts/V168/verifica_qd.py`
- **Program:** the two Woodbury identities in exact rational arithmetic, d = 2..6, with a control. `scripts/V168/verifica_qd_exacta.py`

Part (a) refines Theorem 6.1 in another direction: its witnesses lie in $\mathcal{U}_1$, while $\rho_d \notin \mathcal{U}_\infty$ (Theorem 6.1, since $\mathcal{U}_\infty = W_1 \subseteq W_{d-1}$). Inside $\mathcal{U}_2$ the witnesses s^{-1} ($d = 2$, by $s^{-1} = v^2 - v^2(H_0^{-1})_{11}v^2 \in A_2$ of §5.1) and f_{d+1}^{-1} ($d \geq 3$, by $f_{d+1}^{-1} = u^2 - u^2\mathbf{1}^\mathsf{T}H_{d+1}^{-1}\mathbf{1}u^2 \in A_d$) give more: $s^{-1} \notin \mathcal{U}_1$ and $f_{d+1}^{-1} \notin \Gamma'_{d-1}$ (Theorems 5.1, 7.3), which $e_d \in \mathcal{U}_1 \subseteq \Gamma'_1$ cannot give.

Remark 7.11. (matrix size and number of binomial inverses) p. 33

Let $\mathrm{sz}(e)$ be the least b with $e \in A_b$.

1. For the chain A_b alone the character χ suffices. By §6.2, $A_b = U_{\Sigma_A^{(b)}}$ with $\Sigma_A^{(b)} = \Sigma_0 \cup \{\tilde{G} : G \text{ full over } A \text{ of size } a \leq b\}$, $\tilde{G}$ the polynomial lift with pivots x_j, and corank $\tilde{G}^\chi = \mathrm{corank}\, G^\chi \leq a$ since $x_j^\chi = -1$; also corank $Q_s^\chi \leq 1$. Every entry of F_k vanishes at χ ($u^\chi = w^\chi = 0$), and the atomic factors of F_k other than G_k are stably associated with y, which is invertible at χ (Lemma 7.9(d)); so $\mathrm{corank}\, G_k^\chi = k - 1$. For $k \geq 4$, if $f_k^{-1} \in A_{k-2}$, then $F_k^{-1} \in M(A_{k-2})$ by Woodbury (as in §7.6, with Δ_k^{-1} over $E_1 = A_1$), and Theorem 3.4 with μ the corank at χ gives a contradiction. Likewise $H_0^\chi = 0$, with H_0 an atom (Lemma 5.7(1)), gives $s^{-1} \notin A_1$, and $H_{3,2r}^\chi = 0$, with $H_{3,2r}$ an atom, gives $g_r^{-1} \notin A_1$. So the exclusions $f_k^{-1} \notin A_{k-2}$, $s^{-1} \notin A_1$ and $g_r^{-1} \notin A_1$ do not need the arc either; the proof of Corollary 7.10(a) uses this argument.
2. Let $\mathrm{bin}(e)$ be the least r with $e \in \mathcal{U}_r$, for $e \in \mathcal{U}_\infty$ (and $\mathrm{bin}(e) = \infty$ otherwise). By (i), §5.1 and Woodbury: $\mathrm{sz}(s^{-1}) = \mathrm{sz}(g_r^{-1}) = 2$, $\mathrm{sz}(f_k^{-1}) = k - 1$ ($k \geq 4$), $\mathrm{sz}(e_d) = d$ with $\mathrm{bin}(e_d) = 1$ ($d \geq 2$, Corollary 7.10(a)), $\mathrm{sz}(\rho_d) = d$ ($d \geq 2$) and $\mathrm{sz}(\varphi_r^{-1}) = 5$ for every $r \geq 3$ (Remark 8.2). The two complexities are independent: $\varphi_r^{-1} \in \mathcal{U}_r \setminus \mathcal{U}_{r-1}$ has size 5 for every $r \geq 3$, while $e_d \in \mathcal{U}_1 \setminus \mathcal{U}_0$ has size d and $f_k^{-1} \in \mathcal{U}_2 \setminus \mathcal{U}_1$ has size $k - 1$: neither of sz, bin is bounded in terms of the other, and the size is unbounded already for bin $= 1$.
3. For $n \geq 2$ and $d_1, \ldots, d_n \in A \setminus 0$ with $f = \sum_i d_i^{-1} \neq 0$, $f^{-1} \in A_{n-1}$: the compression of Lemma 7.4 with $r = 0$ is a matrix over A. The size $n - 1$ is attained for every $n \geq 3$ already by a sum of n inverse binomials, q_{n-1} (Corollary 7.10(a)), and also by s for $n = 3$ and by f_n for $n \geq 4$, although f itself lies in $A_1 = E_1$. For $n = 2$, $f^{-1} = d_2(d_1 + d_2)^{-1}d_1$ is the parallel sum of Anderson and Duffin [AD69]; we did not find the compression for general n, or its optimality, stated. When the d_i are non-units, pairwise totally coprime and contained in a proper ideal $\mathfrak{c}$ with $A/\mathfrak{c}$ weakly finite, f has depth n [Coh06, Prop. 7.7.8] (without the ideal this fails: $(x+1)^{-1} - (x+2)^{-1} = ((x+1)(x+2))^{-1}$), a statement about f rather than about the size of the matrices needed for f^{-1}.

Lean · ours, in part **show**

the compression of (iii)

- `ThreeAxes.PhiCompression.harmonic_compression_inv`

Checked when this page was built: PhiCompression.lean compiles with Lean 4.35.0, no errors, no sorry.
Axioms — harmonic_compression_inv: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean PhiCompression.lean` (SHA-256 of the compiled file: `f5da776cad749a60…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 1 result, standard axioms only

Cite: Remark 7.11, p. 33, doi:10.5281/zenodo.23481332

🆕 **Classical and new:** for n = 2 the compression is the parallel sum of Anderson and Duffin [AD69]; the paper did not find the compression for general n, or its optimality, stated.

↳ Used in: 8.2 (p. 53)

7.7 Proof of Theorem 7.1

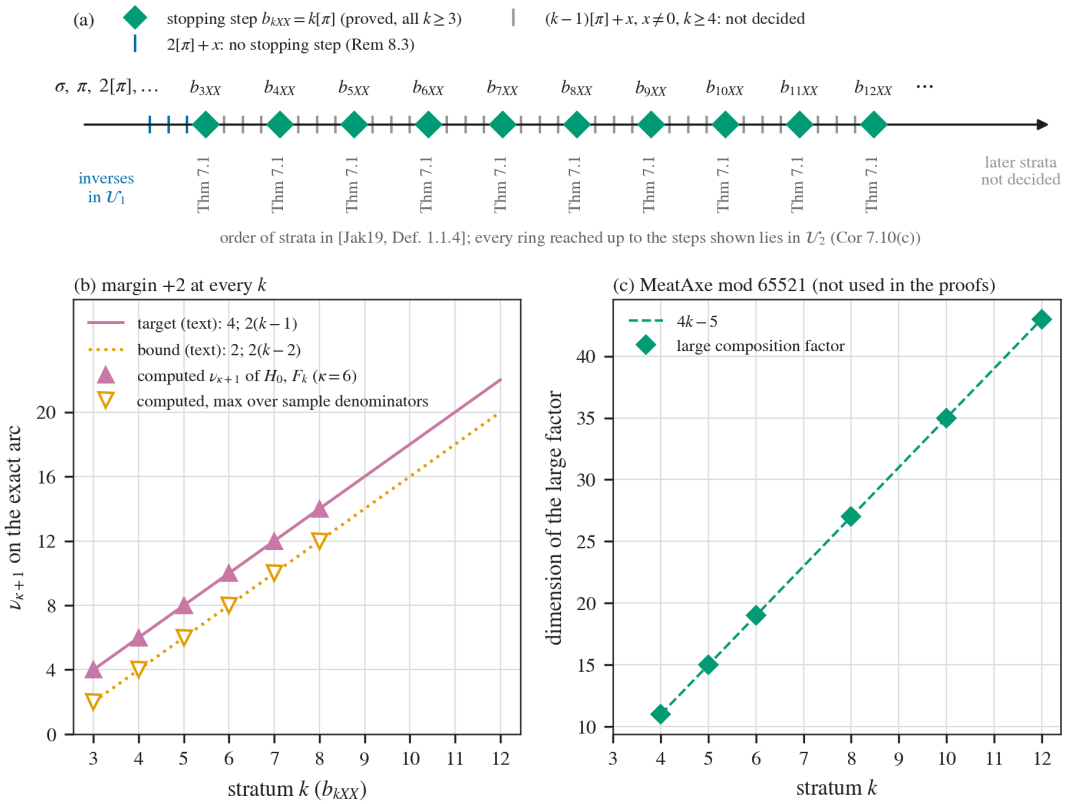
Remark 7.12. (dependencies of Theorem 7.1) 📕 p. 33

The theorem rests on: the reading of [Jak19, Defs. 1.1.4–1.1.5] in §2.3 and Lemma 2.2, including the order of the index monoid; Lemmas 7.4 and 7.5; Theorems 5.1 and 7.3 (Theorem 7.2 only for the second witness ψ), and through them the criterion (Theorem 3.4), the localization lemma (Lemma 3.1), the binomials on the arc (Lemma 4.3), the chain bound (Lemma 4.4), the bound $2a$ (Lemma 7.7) and the atom criteria (Lemma 2.1, Lemma 7.9). Like the series, it is relative to $\mathcal{M} = \mathbb{C}^{\times} F_3$ [Jak19, Rem. 1.1.6]. It does not decide whether the intermediate strata $(k-1)[\pi] + x$ with $x \neq 0$, $k \geq 4$, are stopping steps; for $k = 3$ they are not (Remark 8.3).

Figure 5 places these stopping steps in the order of the strata and shows the margin behind each of them.

↳ Not used by another result of the paper.

Figure 5



The Dubrovin–Gräter–Hanke series inside E_2. (a) Order of the strata [Jak19, Def. 1.1.4]: every $b_{kXX} = k[\pi]$, $k \geq 3$, is a stopping step (Theorem 7.1); the strata $2[\pi] + x$ are not stopping steps, and the strata $(k-1)[\pi] + x$ with $x \neq 0$, $k \geq 4$, are not decided (Remark 8.3). (b) The margin behind each step: the target (H_0 for $k = 3$; F_k for $k \geq 4$, with the count of its atomic factor G_k) has $2(k-1)$ Smith parts at level $\kappa + 1$ and every allowed denominator at most $2(k-2)$. Lines: values proved in the text; markers: exact computations with $\kappa = 6$, the lower marker being the maximum over sample denominators. (c) For comparison only: the size of the large composition factor of the linear pencil of F_k, computed with MeatAxe modulo 65521, is $4k-5$; this is not used in the proofs, and absolute irreducibility was confirmed only for $k = 6$ and 12.

8 Remarks and open problems

Question. What is left: are the two parameters, number of binomials and matrix size, independent; what happens in characteristic p; which later strata are stopping steps?

Remark 8.1. (characteristic p) p. 35

Over a field of characteristic p, $(1+x)^p = 1 + x^p$, so $u^{-2} = u^{p-2}(1+x^p)^{-1} \in V_1$, since $1 + x^p$ is a binomial. Hence $s \in V_1$ and $s^{-1} \in E_1[V_1^{-1}]$: the witness of Theorem 5.1 fails in characteristic p. Whether $\mathcal{U}_1 \subsetneq \mathcal{U}_2$ holds in characteristic p is not studied here. For this witness the first step that breaks is Lemma 4.3(2): the binomial $1 + x^p = (1+x)^p$ is exceptional but $1 + X^p$ has a multiple root, so its parts on the arc exceed k (indeed $U^p = t^{(k+1)(p-1)/2}U$ for odd p) and the chain bound of Lemma 4.4 is lost. Characteristic 0, or the field $\mathbb{C}$, is used at least at the following places:

- Lemma 4.2(2): $\det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3$ (the consequence $\nu = 4$ survives by Lemma 4.2(1));

- Lemma 4.3(2) (simple roots of $1-\beta X^n$) and Lemma 4.3(3) (the factors $\prod n_i$ and $\prod(j_{i+1}-j_i)$ of the trace law; in characteristic 2 the exact arc also identifies u and w);
- Lemma 5.6: the differences $i_j - i_{j+1} = \pm 1, \pm 2$;
- Lemma 5.7(1): $g_{ji} = \pm 1/n$ and $\det K = n$; Lemma 5.7(2): $\varphi_r(1) = 5$;
- Lemma 4.5: real quaternions at $t = 2$ and $\sqrt{-1}$;
- Lemma 6.2(c): the distinct eigenvalues $3, \dots, d+2$ and the Neumann series;
- Lemma 7.8: $\det Q(2,2,2) = 256$, the factor 16, (7.1), (7.2) and $\det(I_3 + J_3) = 4$;
- Lemma 7.9(d): $\det(I_{k-1} + J_{k-1}) = k$, for all parts to equal $\kappa + 1$ (if $p \mid k$, two parts are infinite and the count $\nu_{\kappa+1} = 2(k-1)$ survives, since ν_j counts infinite parts; what is lost in characteristic p is the bound on the allowed denominators, Lemmas 4.3–4.4).

By contrast, Lemma 3.1 holds over any field by its algebraic proof (§3.1), and the proofs of Lemma 7.9(a)–(c) and of the first part of (d) (exactly one atomic factor of F_k not stably associated with y) use no division by an integer and no property of $\mathbb{C}$, the free group algebra Λ_y having trivial units over any field. Computations modulo $p = 2, 3, 5, 7, 11$ for $k = 4, 5, 6$ agree; they are not used, and the tool that produced them miscounts the one-dimensional factors. The same holds for the proof of Corollary 7.10(a), whose other inputs are Theorem 3.4 and the presentation of A_b in §6.2; we have not restated §§2–3 over a general field.

↳ Not used by another result of the paper.

Remark 8.2. (two parameters) p. 35

The hierarchy $\mathcal{U}_r$ (number of inverse binomials) and the rings $\Gamma_b = \mathcal{U}_1[\text{entries of inverses of full } a \times a \text{ matrices over } A,\ a \le b]$ (matrix size) are not nested: $\psi^{-1} \in \mathcal{U}_2 \setminus \Gamma_2$ and $\rho_2 \in \Gamma_2 \setminus \mathcal{U}_\infty$, and more generally, for all $r, b \ge 2$, $\mathcal{U}_r$ is incomparable with Γ_b, Γ'_b and $A_b \cap E_2$ ($f_{b+2}^{-1} \in \mathcal{U}_2 \setminus \Gamma'_b$ by Theorem 7.3, and $\rho_2 \in (A_2 \cap E_2) \setminus \mathcal{U}_\infty$ by Theorem 6.1, with $A_b \subseteq \Gamma_b \subseteq \Gamma'_b$); while $s^{-1} \in (\mathcal{U}_2 \cap \Gamma_2) \setminus \mathcal{U}_1$ by the identity $s^{-1} = v^2 - v^2 (H_0^{-1})_{11} v^2$ of §5.1. In the plane of the rings $\mathcal{U}_r \cap \Gamma'_b$ (§2.4) the rows $r \ge 2$ are strict (with Γ_b as well, by the same witnesses). Note $\Gamma'_1 = \mathcal{U}_1$. For every $r \ge 2$, $s^{-1} \in (\mathcal{U}_r \cap \Gamma'_2) \setminus \Gamma'_1$ (Theorem 5.1). For $b \ge 2$ and $r \ge 2$, $f_{b+2}^{-1} \in (\mathcal{U}_r \cap \Gamma'_{b+1}) \setminus \Gamma'_b$ by Theorem 7.3, because $f_{b+2} \in \mathcal{F}_2$ and $f_{b+2}^{-1} = u^2 - u^2 \mathbf{1}^{\mathsf{T}} H_{b+2}^{-1} \mathbf{1}\, u^2$ with H_{b+2} a full $(b+1) \times (b+1)$ matrix over A. So $\mathcal{U}_r \cap \Gamma'_1 \subsetneq \mathcal{U}_r \cap \Gamma'_2 \subsetneq \cdots$, with union $\mathcal{U}_r$, since $\bigcup_b \Gamma'_b \supseteq \bigcup_b A_b = D$ [Mal84a, Thm 3]. The row $r = 1$ is constant, since $\Gamma'_1 = \mathcal{U}_1$; the chain $A_d \cap \mathcal{U}_1$, which is strict (Corollary 7.10(a)), is not visible in this plane, since every Γ'_b contains $\mathcal{U}_1$. In the columns (b fixed) the inclusions are as follows. Since $\mathcal{U}_1 \subseteq \Gamma'_b$, $\mathcal{U}_r \cap \Gamma'_1 = \mathcal{U}_1$ for every $r \ge 1$, and $\mathcal{U}_0 \cap \Gamma'_b = E_1 \subsetneq \mathcal{U}_1 = \mathcal{U}_1 \cap \Gamma'_b$ for every b (Theorem 5.2, $r = 1$). For $b \ge 2$, s^{-1} separates $r = 1$ from $r = 2$, and $g_r^{-1} \in (\mathcal{U}_{2r} \cap \Gamma_2) \setminus \mathcal{U}_r$ for every $r \ge 1$ by Corollary 5.3 and the identity of §5.1 for $H_{3,2r}$; so no column $b \ge 2$ is eventually constant. For $r \ge 3$ let $d_\tau = \ell_{\tau,r} \cdots \ell_{\tau,1} \in P$, so that $\varphi_r = \sum_\tau \alpha_\tau d_\tau^{-1} \omega_\tau$, and let $T = \{0, 1, 3, 4\}$, $D_T = \operatorname{diag}(d_\tau)_{\tau \in T}$, $\boldsymbol{\omega}_T$ the column of the ω_τ and $\boldsymbol{\alpha}_T$ the row of the α_τ, $\tau \in T$. Since $\omega_2 = 1$ and $\alpha_2 = w$ (§5.3), the matrix

$$M_r = \begin{pmatrix} D_T & -\boldsymbol{\omega}_T d_2 \\ \boldsymbol{\alpha}_T & w \end{pmatrix} \in M_5(P)$$

has Schur complement $w + \sum_{\tau \in T} \alpha_\tau d_\tau^{-1} \omega_\tau d_2 = \varphi_r d_2$ with respect to D_T, so it is full and $\varphi_r^{-1} = d_2 (M_r^{-1})_{55} \in A_5$. Pivoting on w instead leaves $S_r = D_T + \boldsymbol{\omega}_T d_2 w^{-1} \boldsymbol{\alpha}_T \in M_4(V_1)$, which is full, and $\varphi_r^{-1} = d_2 w^{-1} - d_2 w^{-1} \boldsymbol{\alpha}_T S_r^{-1} \boldsymbol{\omega}_T d_2 w^{-1} \in \Gamma'_4$. By Theorem 5.2, $\varphi_r^{-1} \in (\mathcal{U}_r \cap \Gamma'_4) \setminus \mathcal{U}_{r-1}$

, so every column $b \geq 4$ is strict at every step (and every column $b \geq 5$ with Γ_b in place of Γ'_b). Whether $\mathcal{U}_{r-1} \cap \Gamma'_b \subsetneq \mathcal{U}_r \cap \Gamma'_b$ for $r \geq 3$ and $b \in \{2, 3\}$ is open.

The same witnesses separate sizes inside E_2. Up to permutations of rows and columns, M_r is the Schur complement in Q_r of the square block on the rows and columns of the $5(r-1)$ entries -1 of the chains and of the entry $\omega_2 = 1$; that block is triangular with diagonal ± 1, hence invertible over P, so M_r is stably associated with Q_r and is an atom (Lemma 5.7(2)). Every entry of M_r is a product involving a letter, so $M_r^{\chi} = 0$ and corank $M_r^{\chi} = 5$. Now $\Gamma_4 = U_\Sigma$ with $\Sigma = \Sigma_1 \cup \Sigma_A^{(4)}$ (Lemma 3.5, §6.2), and if $\varphi_r^{-1} \in \Gamma_4$ then $M_r^{-1} \in M_5(\Gamma_4)$ (block inverse with respect to D_T, whose inverse is over E_1). Apply Theorem 3.4 with target M_r and μ_Q depending on Q: for Q_s, x_j and the lifts in $\Sigma_A^{(4)}$, the corank at χ, at most 4 (the pivots x_j evaluate to -1) against 5; for the borders of $\mathcal{U}_1$, $\nu_{(r-1)k+1}$ on γ_k with $k > \max(e(Q), r)$, at most 2 (Lemma 4.4) against at least 3 (Lemma 5.6; stable association preserves Smith profiles). Hence $\varphi_r^{-1} \in (A_5 \cap \Gamma'_4) \setminus \Gamma_4$ for $r \geq 3$, and in particular $\varphi_r^{-1} \notin A_4$. So $\Gamma_4 \subsetneq \Gamma'_4$ by elements of inversion height two. Figure 6 shows the plane for $r, b \leq 5$ and places the witnesses by exact matrix size and number of binomial inverses (Remark 7.11).

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the compressions of φ_r (identities); added to the Lean repository after publication (21-oct-2026): φ_r⁻¹ as minus a corner of the inverse of the arrow matrix with d_τ = ℓ_{τ,r} ⋯ ℓ_{τ,1}

- `ThreeAxes.PhiCompression.phi_inv_eq_corner`
- `ThreeAxes.PhiCompression.phi_inv_woodbury`
- `ThreeAxes.ArrowMatrix.arrowInv_corner`
- `ThreeAxes.ArrowInstances.phi_inv_eq_neg_corner`
- `ThreeAxes.ArrowInstances.chainDiag_inv`

✅ Checked when this page was built: ArrowInstances.lean, ArrowMatrix.lean, PhiCompression.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — phi_inv_eq_corner, phi_inv_woodbury, arrowInv_corner, phi_inv_eq_neg_corner, chainDiag_inv: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean ArrowInstances.lean ArrowMatrix.lean PhiCompression.lean` (SHA-256 of the compiled file: `f5da776cad749a60…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 5 results, standard axioms only

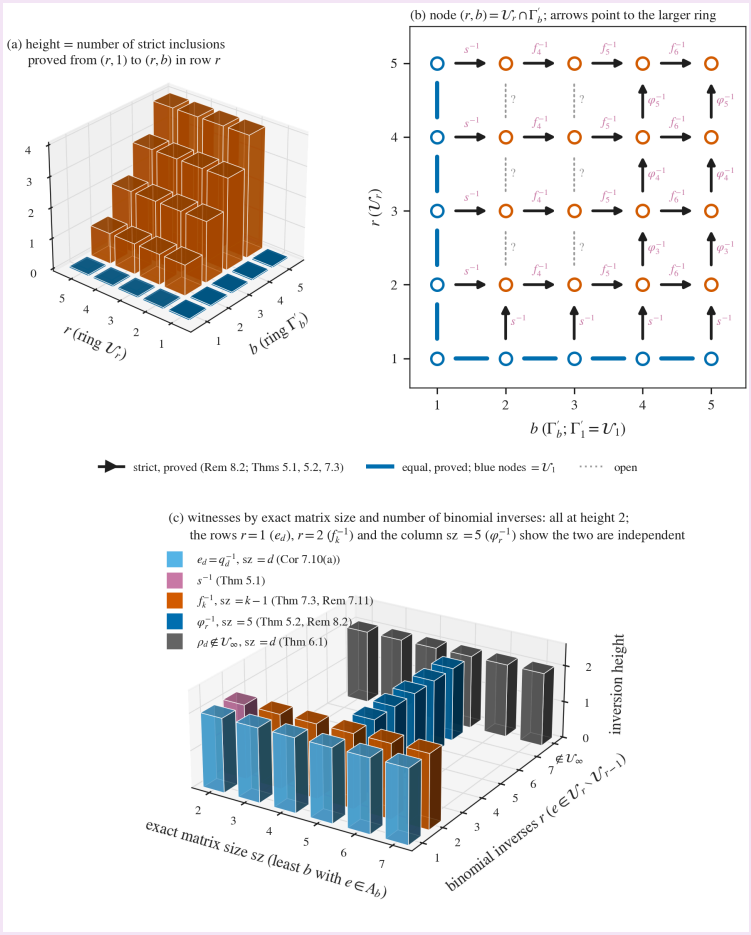
Cite: Remark 7.12, p. 33, doi:10.5281/zenodo.23481332

↳ Not used by another result of the paper.

Evidence & scope

- **Program:** the 6 x 6 arrow matrix with Schur complement -phi_r, r = 3..8, with controls (supplement C24). `scripts/auditoria_S26/s26_comprobaciones.py`

Figure 6



The two-parameter plane of Remark 8.2; node (r,b) is the ring $\mathcal{U}_r \cap \Gamma'_b$. (a) Height: the number of strict inclusions proved along row r, starting from $b=1$. (b) Status of each step for $r, b \le 5$: rows $r \ge 2$ are strict (witnesses s^{-1} and f_{b+2}^{-1}) and row $r=1$ is constant; column $b=1$ is constant and the step $r=1 \to 2$ is strict for $b \ge 2$ (witness s^{-1}); the column steps $r-1 \to r$ with $r \ge 3$ are open for $b \in \{2,3\}$ (individually; in each such column one of the steps $2 \to 3$, $3 \to 4$ is strict, since $g_2^{-1} \in (\mathcal{U}_4 \cap \Gamma_2) \setminus \mathcal{U}_2$) and strict for $b \ge 4$ (witness φ_r^{-1}). The plane starts at $r=1$; the step from $\mathcal{U}_0 = E_1$ to $\mathcal{U}_1$ is strict in every column. Blue nodes are rings equal to $\mathcal{U}_1$. (c) Each witness placed by its exact matrix size sz (Remark 7.11) and by the least r with the witness in $\mathcal{U}_r$; all have inversion height 2. The rows $r=1$ (e_d, size d, Corollary 7.10(a)) and $r=2$ (f_k^{-1}, size $k-1$) and the column of size 5 (φ_r^{-1}, every $r \ge 3$) show that the two complexities are independent; ρ_d lies outside $\mathcal{U}_\infty$. The witnesses g_m^{-1} are not drawn: their least level is only known to lie between $m+1$ and $2m$.

Remark 8.3. (later strata) p. 36

Theorem 7.1 settles the steps $b_{kXX} = k[\pi]$. The band between $2[\pi]$ and $3[\pi]$ contains no stopping step: an element of a stratum $2[\pi]+x$ is $d_1^{-1}+d_2^{-1}+r$ with $r \in V_1$ (or has fewer pure summands), and if a Case 3 selected that stratum, the current ring would already contain V_1^{-1} (as $D_\sigma = V_1$ is an earlier stratum) and hence $(d_1^{-1}+d_2^{-1}+r)^{-1} = d_2(d_1 r d_2 + d_1 + d_2)^{-1} d_1$ (Lemma 7.4). With one pure summand, $(d_1^{-1}+r)^{-1} = (1+d_1 r)^{-1} d_1$ with $1+d_1 r \in V_1$, so π and the strata $[\pi]+x$ are not stopping steps either. The strata $(k-1)[\pi]+x$ with $x \neq 0$ and $k \ge 4$ are not decided, nor any step involving ordinals born after π (products of three or more binomial inverses, or the inverses adjoined at the stopping steps themselves); nor do we decide which strata before π (sums of $n \ge 3$ monomials, sums of elements of $D_{\omega+1}$) are stopping steps. On the free algebra P the harmonic matrices need not be atoms. For a related family, not used here (axes u^2, v^2, w^2 and conjugates by single letters with pairwise distinct signed outer letters), we certified atomicity for $k \le 9$ by computation. The $k=10$ extensions we tested use a conjugator of length two that repeats an outer letter; exactly over $\mathbb{Q}$, their linear

pencils have a one-dimensional composition factor of letter type, and so do the targets f_k of §7.6 already for $k = 4, 5$. Theorem 3.4, applied to one atomic factor, makes such factors harmless, because letters evaluate to units on the arc.

Cite: Remark 8.3, p. 36, doi:10.5281/zenodo.23481332

↳ Not used by another result of the paper.

Remark 8.4. (strictness of the size chain in general) p. 36

Theorem 6.1 is specific to $A = \mathbb{C}[F_3]$ and uses the quaternion curve. In general, let R be a fully atomic semifir (for instance a fir) with universal field D, and $R_b \subseteq D$ the subring generated by R and the entries of the inverses of full matrices of order $\leq b$. Then $R_{b-1} \subsetneq R_b$ exactly when some $b \times b$ atom is not stably associated with a matrix of smaller size. The direction "$\Leftarrow$" is Lemma 3.1, since the atomic factors of a full $n \times n$ matrix are square of order $\leq n$ [Coh06, Thms 3.2.3, 3.2.7]. Conversely, a full $b \times b$ matrix is a product of $b \times b$ matrices each stably associated with an atom of order $\leq b$ (R is fully atomic); if every $b \times b$ atom is stably associated with a matrix of smaller size, each factor has the entries of its inverse in R_{b-1} (stable association does not change this: converse part of Lemma 3.1), so $R_b = R_{b-1}$. A matrix with all its entries in a proper ideal $\mathfrak{a}$ with $R/\mathfrak{a}$ weakly finite is not stably associated with a matrix of smaller size [Coh85a, Lemma 3.4.11(i)], [Coh06, Lemma 3.3.11(i)]. This gives a second proof that the chain is strict at every b in $\mathbb{C}\langle x, y, z\rangle$; the first follows from [HS15, Thm 4.5] (§1). There $P_b = xI + y\,\mathrm{diag}(1, \ldots, b) + z\,\mathsf{C}_b$ is an atom: after $x \mapsto x + 1$ it is a monic pencil with invertible constant term whose coefficients generate $M_b(\mathbb{C})$ ($\mathrm{diag}(1, \ldots, b)$ has distinct eigenvalues and C_b permutes its eigenlines cyclically, as in the proof of Lemma 6.2(c)), so Lemma 2.1 applies. All its entries lie in (x, y, z). In an Ore domain, by contrast, $R_1 = D$. When $R/\mathfrak{a}$ is a field this criterion is the analogue over R of Theorem 3.4 with μ the corank at $R \to R/\mathfrak{a}$. It does not locate witnesses inside E_2; Theorem 6.1 does. The characterization by atoms is a short corollary of published lemmas.

Cite: Remark 8.4, p. 36, doi:10.5281/zenodo.23481332

↳ Not used by another result of the paper.

Remark 8.5. (independent adjunctions in $[\mathcal{U}_\infty, B_2]$) p. 37

For $\lambda \in \mathcal{T} = \mathbb{C} \setminus \{-1, 0, 1, 2\}$ let $\rho^{(\lambda)} = ((F^{(\lambda)})^{-1})_{11} = (q_\lambda - c\,q_{\lambda+1}^{-1}c)^{-1}$, and for $J \subseteq \mathcal{T}$ let $\mathcal{U}_J$ be the subring generated by $\mathcal{U}_\infty$ and $\{\rho^{(\lambda)} : \lambda \in J\}$. Then $\mathcal{U}_\infty \subseteq \mathcal{U}_J \subseteq B_2$, and $J \mapsto \mathcal{U}_J$ is an order embedding $\mathcal{P}(\mathcal{T}) \hookrightarrow [\mathcal{U}_\infty, B_2]$: $\mathcal{U}_J \subseteq \mathcal{U}_K$ if and only if $J \subseteq K$. It preserves finite joins. Restricted to $J \subseteq \{3, 4, 5, \ldots\}$ it gives $2^{\aleph_0}$ distinct rings generated over $\mathcal{U}_\infty$ by elements with integer coefficients; on all of $\mathcal{T}$ it gives $2^{\mathfrak{c}}$. $\mathcal{T}$ is the set where the atom criterion of Lemma 6.2(c) applies; it is sufficient, not maximal. Intersections $\mathcal{U}_J \cap \mathcal{U}_K$ are not determined.

Proof. $\rho^{(\lambda)} \in B_2$ because $F^{(\lambda)}$ is a full 2×2 matrix over A, whose inverse has entries in E_2 (pivot argument of §7.5) and in W_2. $J \subseteq K$ gives $\mathcal{U}_J \subseteq \mathcal{U}_K$, and $\mathcal{U}_{J\cup K}$ is generated by $\mathcal{U}_J \cup \mathcal{U}_K$, by definition; so it suffices to show $\rho^{(\lambda')} \in \mathcal{U}_J \Rightarrow \lambda' \in J$. Since $q_{\lambda+1}^{-1} \in E_1$, the entries of $(F^{(\lambda)})^{-1}$ lie in the ring generated by E_1 and $\rho^{(\lambda)}$ (block inverse, as in (6.2)); hence $\mathcal{U}_J = U_\Sigma$ with $\Sigma = \Sigma^{(1)} \cup \{F^{(\lambda)} : \lambda \in J\}$, and $\rho^{(\lambda')} \in \mathcal{U}_J$ gives $(F^{(\lambda')})^{-1} \in M_2(U_\Sigma)$. By Lemma 3.1 the atom $F^{(\lambda')}$ (Lemma 6.2(c)) is stably associated with an atomic factor of some $Q \in \Sigma$. It is not a factor of a member of $\Sigma^{(1)}$: at Θ every such Q has corank at most 2 (Lemma 6.2(a)), hence so do its atomic factors (Definition 3.3, Example (b)), while $F^{(\lambda')}$ has corank 4 (Lemma 6.2(b)). So $F^{(\lambda')}$ is stably associated with $F^{(\lambda)}$ for some $\lambda \in J$ (an atom, so its only atomic factor). Stable association over P survives abelianization, and the units of the commutative polynomial

ring are the scalars, so $\det F^{(\lambda')} = a \det F^{(\lambda)}$ for some $a \in \mathbb{C}^\times$. At $(\xi, \eta, \zeta) = (\xi, 0, 1)$, $c = 0$ and $\det F^{(\lambda)} = (\lambda - \xi^2)(\lambda + 1 - \xi^2)$, so $\lambda' = \lambda \in J$. □

Cite: Remark 8.5, p. 37, doi:10.5281/zenodo.23481332

↳ Not used by another result of the paper.

Evidence & scope

- **Program:** the excluded values of F^(lambda) (supplement C13). `scripts/auditoria_S11/s11_a2_a4.py`

Remark 8.6. (intermediate rings and classes of atoms) p. 38

The proof of Lemma 3.1 shows that, when H^{-1} has entries in U_Σ, the cokernel of every atomic factor of H is a composition factor, in the category of torsion modules, of the cokernel of a matrix built from Σ (Claims A and B). That universal localizations of a hereditary ring correspond to classes of modules is known in general: Schofield gives a bijection between universal localizations and well-placed subcategories of finitely presented modules, and between the localizations into which the ring embeds and Hom-perpendicular sets of finitely presented bound modules [Sch07, Thms 2.3, 2.5]; Krause and Šťovíček give, for a hereditary ring, a bijection between extension-closed abelian subcategories of finitely presented modules and universal localizations [KS10, Thm B]; and for a factor-complete set Σ over a fir, Cohn shows that every matrix atom either becomes a unit or remains an atom in R_Σ, and describes the divisor group of R_Σ by the Σ-relevant primes [Coh06, Thm 7.11.9, Prop. 7.11.10, Thm 7.11.11]. The rings P and A are firs, hence hereditary. We do not claim this correspondence; Lemma 3.1 uses only the instance above. Describing the intermediate rings of this paper ($\mathcal{U}_r$, $A_b \cap E_2$, Γ'_b, D') by the classes of atoms they invert is a programme, not a result of this paper.

Cite: Remark 8.6, p. 38, doi:10.5281/zenodo.23481332

NEW **Classical and new:** the correspondence is known [KS10], [Sch07], [Coh06]; not claimed.

↳ Not used by another result of the paper.

8.7 What is open

1. *A uniform atom criterion.* For five-chain borders the condition R1 ∧ R2* (no chain whose boundary coefficient equals its boundary letter; no two chains with the same last letter and the same ω_τ, or the same first letter and the same α_τ) is necessary for atomicity (a violation gives a nonzero vector in some $\mathcal{K}_i^R$ or $\mathcal{K}_i^L$, and the converse after Claim 3 in the proof of Lemma 5.7 splits off a letter). Sufficiency is a conjecture: all 317 random designs satisfying it are atoms, and none of the 140 violating it is. The proof of Lemma 5.7(2) gives sufficiency only under the degree hypothesis (exactly one constant ω_τ).
2. *The column steps $r - 1 \to r$, $r \geq 3$, $b \in \{2, 3\}$, of the two-parameter plane* (Remark 8.2), *the later strata* (Remark 8.3), *the separation $\mathcal{U}_1 \subsetneq \mathcal{U}_2$ in characteristic p* (Remark 8.1).

8.8 Further related work

This section collects the comparison with the literature in more detail than the introduction.

Where the stages come from. For a torsion-free group G satisfying the strong Atiyah conjecture, the L^2-Betti numbers of matrices over $\mathbb{Q}[G]$ are ranks over Linnell's division ring, the division closure of $\mathbb{C}[G]$ in the algebra $\mathcal{U}(G)$ of affiliated operators; for free groups the conjecture holds [Lin93]. When G is locally indicable (free groups are) and $\mathbb{C}[G]$ has a universal epic division ring, that division ring is isomorphic to the division closure [JZLA19, Cor. 8.2]; for a free group it is Cohn's free field. So the stages $E_1 \subset E_2 \subset \cdots$ studied here are the stages of the rational closure inside the division ring that carries the L^2-invariants of F_3. Our results describe that structure; they say nothing new about the Atiyah conjecture.

Stages, depth and rank. Reutenauer [Reu96] and Herbera–Sánchez [HS15] separate the whole stages $E(n) \subsetneq E(n+1)$. They invert every nonzero element, and [HS15, §§7–8] contains no subrings inside $E(2)$. Cohn's depth [Coh06, §7.7] and the rank of Cohn–Reutenauer are numerical invariants of elements, with height $\leq$ depth $\leq$ rank [Reu96, p. 95]. E_1 already contains elements of arbitrary depth [Coh06, Prop. 7.7.8], so depth does not separate the rings $\mathcal{U}_r$. Over a fully atomic semifir the depth does not increase under specialization to an epic field, for elements in its minimal domain [Coh06, Thm 7.7.5]; this comparison of an element with its image at an evaluation is the closest technique we found to ours, which compares one atomic factor with the allowed denominators at the same evaluation. Counting inverses without restricting *what* is inverted collapses: $(x_1 \cdots x_n)^{-1}$ yields every x_j^{-1} with one inversion [HW15, Lemma 5.1].

The rings A_b are the images in D of the one-step localizations of A at all full $b \times b$ matrices. Malcolmson [Mal84a, Thm 3] proved, for any semifir, that these localizations are semifirs, that $A_b \to A_{b+1}$ is an embedding and that the union is the universal field of fractions, building on Gerasimov [Ger82, Cor. 1–2], whose Cor. 2 answers a question of Bergman [Ber74, p. 72]; see [Coh85a, §7.11]. Malcolmson also showed that the bound in the n-fir property of these localizations is sharp [Mal84b, Cor. B]. That is a separation of the property, not of the rings. In a free algebra on at least two generators the chain A_b is strict at every step: an entry of the inverse of a full $(b-1) \times (b-1)$ matrix has inversion height at most $b-1$ (Gaussian elimination), while [HS15, Thm 4.5] gives $b \times b$ polynomial matrices whose inverses have entries of height exactly b (for infinitely many generators already [Reu96, Thm 2.1]). Over the free group algebra [HS15, Thm 4.6] gives the $b \times b$ matrices $M_b = (z_{i+jb})_{0 \leq i,j < b}$, $z_i = x^i y x^{-i}$, all of whose inverse entries have height $b-1$ relative to A; these witnesses lie outside E_2 for $b \geq 4$ (for $b = 3$ the height is 2). Since $z_0 = y$ is a unit, one pivot gives $M_b \sim 1 \oplus S$ with S a full $(b-1) \times (b-1)$ matrix over A and S^{-1} a block of M_b^{-1}; as $A_a \subseteq E_a$ (Gaussian elimination), every step $A_{b-2} \subsetneq A_{b-1}$ is strict. The entries $z_{i+jb} - z_i z_0^{-1} z_{jb}$ of S are units of A times binomials, so a second pivot leaves a full $(b-2) \times (b-2)$ matrix over V_1 whose inverse has entries of height $b-1$. For $b = 3$ this gives $E_1 \subsetneq \mathcal{U}_1$ (the case $r = 1$ of Theorem 5.2), and in general $\Gamma'_a \not\subseteq E_a$ (in particular $\Gamma'_b \not\subseteq E_2$ for $b \geq 2$). Since $\Gamma_a \subseteq E_a$ for $a \geq 2$ ($\mathcal{U}_1 \subseteq E_2$ and $A_a \subseteq E_a$), this gives $\Gamma_a \subsetneq \Gamma'_a$ for every $a \geq 2$, by an element of height $a+1$ (Remark 8.2 gives $\Gamma_4 \subsetneq \Gamma'_4$ by elements of height two), while $\Gamma'_a \subseteq E_{a+1}$ by elimination over $V_1 \subseteq E_1$. These consequences of [HS15] are elementary; we did not find them stated. To our knowledge, the strictness of $A_{d-1} \cap E_2 \subsetneq A_d \cap E_2$ for $d \geq 3$ (for $d = 2$ it follows from the case $b = 3$ above), its refinement inside $\mathcal{U}_1$ (Corollary 7.10) and the rings $\mathcal{U}_r$ have not been studied. For the iterated localization with fixed size see [Coh85a, Ex. 7.10.3] (Schofield), and for its cofinality [Coh06, Ex. 7.11.3]. For the same reason depth does not separate the chain A_b: $a_1^{-1} + \cdots + a_n^{-1}$ has depth n under mild conditions [Coh85a, Prop. 7.7], [Coh85b] and lies in A_1. Remark 8.4 gives a second proof of the strictness in a free algebra, from Lemma 3.1 and a lemma of Cohn; Theorem 6.1 gives witnesses inside E_2.

The correspondence between universal localizations and classes of modules, or of atoms, is known [KS10, Thm B], [Sch07, Thm 2.5], [Coh06, Thm 7.11.9, Prop. 7.11.10, Thm 7.11.11]; for an independent set Σ the necessary direction also follows from Malcolmson's invertibility criterion [Coh85a, Ex. 7.11.13]. We use only its first half, in the form of Lemma 3.1, and do not claim it (Remark 8.6).

Theorem 3.4 is then a short consequence of two further facts. Stable association is preserved by every ring homomorphism [Coh06, §0.5], and the dimension of the kernel of an evaluation, $\dim \ker F(X) = \dim \operatorname{Hom}(\operatorname{coker} F, V)$, cannot increase when F is replaced by a factor. Shamovich and Volčič use these kernel dimensions to separate stable association classes of full matrices over the free algebra [SV26, Lemma 3.2, Thm 3.4]. The corank at a representation is the nullity of the Sylvester rank function it induces [Mal80], [Sch85, Ch. 7]. When the representation inverts every allowed denominator, the criterion reduces to the universal property of the localization: a matrix that becomes invertible over R_Σ is full for every rank function pulled back from R_Σ [SP25, Lemma 2.13]. What we add is the case in which the allowed denominators are themselves singular at the evaluation: it suffices that one atomic factor of H be more singular than each of them.

Our second invariant also comes from coranks. Truncating an arc $P \to M_m(\mathbb{C}[[t]])$ modulo t^j gives a representation Θ_j of dimension mj, and $\nu_j = \operatorname{corank} F^{\Theta_j} - \operatorname{corank} F^{\Theta_{j-1}}$ (Definition 3.3);

summed over j it gives the t-order of $\det F^{\gamma}$. Valuations of Dieudonné determinants over the free field with coefficients in $K(t)$ are computed by Hirai [Hir19] and Oki [Oki23]. Ramani bounds the degree over $\mathbb{Q}(t)$ along a one-parameter family of 2×2 evaluations to show that an inverse is not rational [Ram26]. Cohn's N-values [Coh06, §§7.10–7.11] record which atoms a factor-complete localization inverts: the simple N-value of a matrix prime extends to the universal field [Coh06, Thm 7.10.2, Thm 7.10.6, pp. 494–496], and for a factor-complete Σ over a fully atomic semifir the sum of the Σ-relevant ones shows that an atom not inverted by Σ stays a non-unit in R_{Σ} [Coh06, Thm 7.11.9, p. 508]. Lemma 3.1 is the analogue for the subring U_{Σ} of D, without factor-completeness; what the invariants of Definition 3.3 add is a way to certify that an atom is absent. In the literature we read we found no use of Smith profiles along an arc, nor of coranks of atoms at a representation at which the allowed denominators are singular, to prove non-membership in an intermediate ring. The freeness of the group generated by $1 + 2\mathbf{i}, 1 + 2\mathbf{j}, 1 + 2\mathbf{k}$ behind §4.2 is classical [LPS88]. The atom criterion is Cohn's [Coh06, Thm 5.8.8] in the form [HKV22, Rem. 2.6].

A Formal verification and computations

A.1 The exact arc

The finite identities behind Sections 4.1 and 5 are formally verified in Lean 4 / Mathlib, module `Formalizations.RingTheory.FreeField.ThreeAxes.ExactArc`: zero-trace 2×2 matrices square to $-\det \cdot I$; the exact arc satisfies $U^2 = t\varepsilon I$ and $U^2 = t^{k+1} I$ at $\varepsilon = t^k$; the interleaving identity $a_0(cb_1)\cdots(cb_r) = c^r(a_0 b_1 \cdots b_r)$ for a central scalar c; the evaluation of H_0 when $u^2 = v^2 = w^2 = c \cdot I$; the two-axes identity $(a^{-1} + b^{-1})^{-1} = b(a+b)^{-1}a$; and the characteristic-p identities of Remark 8.1. Module `ThreeAxes.InverseIdentities` formalizes the chain inverse of §3.4, namely $(C^{-1})_{1,q} = \ell_1^{-1} a_1 \ell_2^{-1} \cdots \ell_q^{-1}$ over any division ring, and the Woodbury identity of §5.1. Module `ThreeAxes.ParabolicFamily` formalizes the conjugate form $U_i(t,\varepsilon) = K^i U_0 K^{-i}$ of §4.1 (and recovers $U_i^2 = t\varepsilon I$ from it), the parabolic family $N_j = K^j N K^{-j} = t\, c_j r_j$ with $r_a c_b = (b-a)t$ and $N_j^2 = 0$, the cyclic trace formula $\operatorname{tr}(N_{j_1}\cdots N_{j_q}) = t^{2q}\prod_l (j_{l+1} - j_l)$ for every nonempty list of indices (the core of Lemma 4.3), the fact that a central square survives conjugation, and $\det(I_n + J_n) = n + 1$. Module `ThreeAxes.ArrowMatrix` gives, over any division ring, an explicit two-sided inverse of $\begin{pmatrix} \operatorname{diag}(d) & \boldsymbol{\omega} \\ \boldsymbol{\alpha} & 0 \end{pmatrix}$ with corner $-x^{-1}$, $x = \sum_\tau \alpha_\tau d_\tau^{-1} \omega_\tau$. It applies to s^{-1}, f_k^{-1} and e_d, and to φ_r^{-1} before the compression of Remark 8.2 (a 6×6 arrow matrix), each minus a corner of the inverse of an arrow matrix; the instances are not stated in Lean. Module `ThreeAxes.BinomialSizeFamily` gives, over any division ring, the two-sided inverse of $\operatorname{diag}(\Delta) + uJ$ (`rankOne_mul_inv`, `rankOneInv_mul`, from the Woodbury identity of Mathlib), $q^{-1} = u - u\mathbf{1}^{\mathsf{T}} H^{-1} \mathbf{1} u$ for $q = u^{-1} + \sum_i \Delta_i^{-1}$ (`inv_q_eq`), and the polynomial entries of L in Corollary 7.10(a) (`conj_add_mul_pow`); for every $d \geq 1$, $\det(\operatorname{diag}(0,1,\ldots,1) + J_d) = 1$ over any commutative ring (`det_scalar_point`), and the invertibility of L at the idempotent point over a field (`idemBlock_isUnit`). Since u and u^2 evaluate alike at these two points, the last two are also the two evaluations in the proof of Lemma 7.9(c). The atomicity of q_d and f_k and the corank argument are not formalized. The compression matrix of Lemma 7.4, whose relevant Schur complement is $d_1 f d_2 \neq 0$ (its corner Y may vanish), is covered by `InverseIdentities` (A.2), and its case $r = 0$, used in Remark 7.11(iii), by `ThreeAxes.PhiCompression` (`harmonic_compression_inv`). The same module gives, over any division ring, the compressions of Remark 8.2 for arbitrary data $d_\tau, \alpha_\tau, \omega_\tau, d_2, w$: the explicit two-sided inverse of M_r with $\varphi_r^{-1} = d_2 (M_r^{-1})_{55}$ (`phi_inv_eq_corner`) and the formula with S_r (`phi_inv_woodbury`), and the identity $m_0 \ell^{-1} m_1 = (m_0 \ell m_0^{-1})^{-1} m_0 m_1$ of Corollary 7.10(c) (`inv_mul_conj`); the stable association of M_r with Q_r and $M_r^{\chi} = 0$ are not formalized. Module `ThreeAxes.QuaternionCurve` gives the identities of §4.2 in the quaternion algebra $\mathbb{H}[R]$ over a commutative ring: $(t\mathbf{i})^2 = (t\mathbf{j})^2 = (t\mathbf{k})^2 = -t^2$ and pairwise anticommutation; the 2×2 complex representation Θ is not formalized. The step $\operatorname{Schur}(B_f) = -f$ for several chains in direct sum is not formalized. Not formalized either: the Smith calculus (Lemma 3.2), the localization lemma, the full

trace law of Lemma 4.3 (only the trace of products of the N_j), the atom proofs, and the non-membership theorems themselves.

A.2 Weighted trinomials, the atom certificate and the quaternion residues

The identities $I + X_i + \theta X_{i+1} = N_i$ and $N_i^2 = 0$ of Remark 6.3, the commutator identity of Remark 6.4 and, for $d = 2$ over an arbitrary ring, the fact that the block matrix in (6.2) is a right inverse of $\hat{F}_2$ provided p_2 and σ are invertible, are formalized in Lean 4 / Mathlib, module `Formalizations.RingTheory.FreeField.ThreeAxes.WeightedTrinomials`. The integer data of the atom certificate of Lemma 7.8(a) are formalized in `ThreeAxes.AtomCertificate`: $Q(2,2,2)K_i = 16Q_i$, $\det Q(2,2,2) = 256$, identity (7.1), and the two determinants (7.2), the latter through a triangular factorization certificate. The finite part of Lemma 4.5 is formalized in `ThreeAxes.QuaternionResidues`: over $\mathbb{Z}/5$ the six residues have rank one, a product of two residues vanishes exactly for an inverse pair, and products along reduced words are nonzero. The inverse identities over an arbitrary division ring are formalized in `ThreeAxes.InverseIdentities`: the identities used in the proofs of Lemmas 7.4 and 7.6, and the cycle inverse (6.1)–(6.2) for every $d \geq 2$. The identities of these two lemmas are the three-term and one-term inverses, the Woodbury corners of the harmonic matrix and the Schur compression (the module also contains the intermediate inverse, the case $n' = 3$ of Lemma 7.4 up to a permutation). Fullness in the free field is an assumption there, not a theorem. Module `ThreeAxes.ChainPencilConditions` checks by `decide`, from integer certificates, the no-common-kernel conditions behind Lemma 5.7(2) for Q_3 and Q_4; `ThreeAxes.SizeChainPencil` formalizes the linear algebra behind Lemma 6.2(c): $D = \mathrm{diag}(3, \dots, d+2)$ and the cyclic permutation C generate M_d, and D and $D - I - C^2$ are invertible (the latter by Gershgorin's theorem), with the analogue for $F^{(\lambda)}$; `ThreeAxes.Criterion` states Definition 3.3, proves Theorem 3.4 with the conclusion of Lemma 3.1 (together with the complete factorization of Q and [Coh06, Cor. 0.5.5]) as a named hypothesis, and proves Example (b) for a commutative field. Every result in the repository uses only the standard axioms. Not formalized: the step from these data to "the pencil is an atom" (Cohn's criterion), the norm argument of Lemma 4.5, the corank or Smith obstructions, and the theorems.

A.3 Computations

The Smith-profile and atom computations use exact rational or modular arithmetic: truncated power series and Toeplitz rank counts for Smith profiles (two independent implementations), and coefficient-algebra dimensions modulo two primes, with Python integers; the inverse identities were also checked in floating point with random non-commuting matrices. Since Lemmas 5.7 and 6.2(c) are proved in the text, the atom computations are corroboration. The Lean files are in the public repository karlesmarin/formalizations. The supplement lists every computation cited in the paper with the script that performs it and its saved output (28 entries), with SHA-256 hashes, and contains a driver that re-runs the scripts and compares their output with the saved one.

The Smith profiles of Section 7 use an independent counter over $\mathbb{F}_p[[t]]/t^N$; the identities (6.1), (6.2) and Lemma 7.4 with $n' = 3$ were also checked numerically with random non-commuting 3×3 matrices, with controls (wrong orientation, wrong sign) that fail as they should.

A.4 The companion site

This paper has a companion website, its *blueprint*, at https://karlesmarin.github.io/blueprints/inversion-height-two/. It shows the paper as a whole — what it proves, how its results depend on each other, and what exactly has been checked — and lets the reader go from any statement to its proof, its dependencies and its checks. There one finds:

- *The statements.* Every numbered statement of the paper, copied verbatim with its number (10 theorems, 1 proposition, 21 lemmas, 2 corollaries, 1 definition and 12 remarks, in 8 sections), each with its proof, where it has one, a click away.
- *The dependencies.* For each statement, what it uses and where it is used, as a list and as a graph of the whole paper: one can follow a theorem down to the lemmas it rests on, or a lemma up to

what it serves.

- *The map.* One page that places the main results, the tools they rest on (the branches of mathematics they borrow from) and the fields they feed.
- *What was checked, and how.* Besides the written proof, the site says for each statement what else was checked: 18 of them have a part verified in Lean 4 (none of them in full), linked by name (43 declarations) to the public repository karlesmarin/formalizations; 22 statements are corroborated by programs of the supplementary material, each named with what it computes.
- *Labs.* Next to Lemma 4.1, Lemma 4.5 and Theorem 5.1, a form runs the paper's own computations in the reader's browser (Python, nothing to install) for values the reader chooses, and shows the exact results.
- *Figures in motion.* Figures 2, 3 and 4 are drawn live by the same computations: in 3D, which one can turn, where the figure is three-dimensional, and with its parameter on a slider; the lab's checks run before drawing, and the static figure of the paper is one click away.
- *The paper itself.* Its PDF, opened at the right page from every statement.

To read it, one can start from the map, or from a main theorem and follow its dependencies down; the side panel lists the sections of the paper, and every statement has its own link. The site is a reading aid, not a part of the proofs: the statements and proofs are those of this paper, whose text prevails, and a formal or computational check is reported for exactly the part it covers. It is generated from the source of this version of the paper; a later version gets its own site.

📊 Tables

Every table of the paper, in its order: click one to unfold it, with where the paper cites it, how its numbers were checked, and its data as CSV. The typeset table is one click away, on its page of the PDF.

Table without a number, in the text of §3.3 Monotone invariants and the criterion

Result	Witness	In	Not in	Invariant (Def. 3.3)	Target	Bound	N
Thm 5.2, $r=1$	S_3^{-1}, $S_3=u^{-1}+v^{-1}+w^{-1}$	$\mathcal{U}_1$	$E_1=\mathcal{U}_0$	corank at χ (target $H_{3,1}$)	2	≤ 1	+
Thm 5.1	s^{-1}, $s=u^{-2}+v^{-2}+w^{-2}$	$\mathcal{U}_2$ (also D', Γ_2)	$\mathcal{U}_1$	ν_{k+1} on γ_k (target H_0)	4	≤ 2	+
Thm 5.2, $r\ge 3$	φ_r^{-1}, (5.1)	$\mathcal{U}_r$	$\mathcal{U}_{r-1}$	$\nu_{(r-1)k+1}$ on γ_k (target Q_r)	≥ 3	≤ 2	≥
Cor 5.3	g_r^{-1}, $g_r=u^{-2r}+v^{-2r}+w^{-2r}$	$\mathcal{U}_{2r}$	$\mathcal{U}_r$	ν_{rk+1} on γ_k (target $H_{3,2r}$)	4	≤ 2	+
Thm 6.1	$\rho_d=(\hat{F}_d^{-1})_{11}$	$A_d\cap E_2$, B_d	W_{d-1} ($\supseteq A_{d-1}$, B_{d-1}; for $d=2$: $\mathcal{U}_\infty$)	corank at Θ over $\mathbb{C}(t)$ (target $\hat{F}_d$)	$2d$	$\le 2(d-1)$	+

Thm 7.2	ψ^{-1}, $\psi = u^{-2}+v^{-2}+w^{-2}+yu^{-2}y^{-1}$	$\mathcal{U}_2$	Γ'_2 ($\supseteq \Gamma_2 \supseteq D'$)	ν_{k+1} on γ_k (target F_ψ)	6	≤ 4	+
Thm 7.3, $k \geq 4$	f_k^{-1}	$\mathcal{U}_2 \cap \Gamma'_{k-1}$, $\mathcal{U}_2 \cap A_{k-1}$ (Rem 7.11)	Γ'_{k-2}	$\nu_{\kappa+1}$ on γ_κ (atomic factor G_k of F_k)	$2(k-1)$	$\leq 2(k-2)$	+
Cor 7.10(a), $d \geq 2$	$e_d = q_d^{-1}$, $q_d = w^{-1} + \sum_{i<d} y^i u^{-1} y^{-i}$	$\mathcal{U}_1 \cap A_d$	A_{d-1}	corank at χ (atomic factor G of L)	d	$\leq d-1$	+
Rem 7.11, $k \geq 4$	f_k^{-1}	$\mathcal{U}_2 \cap A_{k-1}$	A_{k-2}	corank at χ (atomic factor G_k of F_k)	$k-1$	$\leq k-2$	+
Rem 8.2, $r \geq 3$	φ_r^{-1}	$\mathcal{U}_r \cap A_5 \cap \Gamma'_4$	Γ_4 ($\supseteq A_4$)	corank at χ (lifts over A); $\nu_{(r-1)k+1}$ on γ_k (borders of $\mathcal{U}_1$); target M_r	5; ≥ 3	≤ 4; ≤ 2	≥

Colour: blue shade = the number's size within its column.

⬇ CSV

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