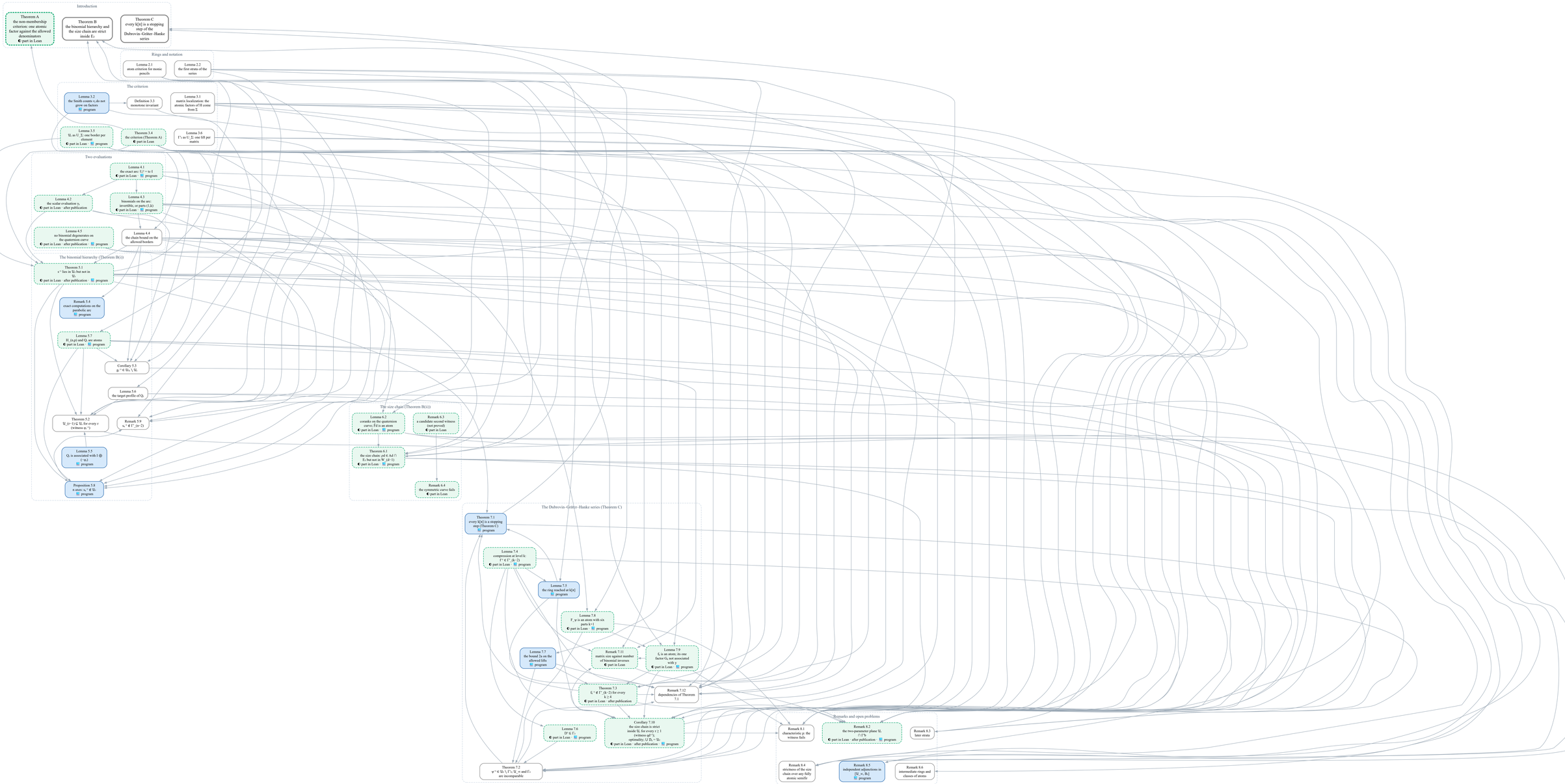
Blueprint

Map of *Inversion height two in the free field of the free group: strict hierarchies and a non-membership criterion*

Associative rings · division rings 1.0.0.42 1.0.0.43 1.0.1.1 1.0.2.34 1.0.2.52

47 results | ✅ 0 entirely in Lean | ◐ 21 partly in Lean | 🖥 22 checked by a program | written proof only | (a result can carry several tags)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



What the paper contributes

The free field D of $A = \mathbb{C}[F_3]$ is built in stages: E_1 inverts the nonzero elements of A, E_2 those of E_1, and so on. The stages never stabilise (Reutenauer; Herbera–Sánchez). The paper works **inside the single stage E_2**, where one restricts which elements are inverted.

- **A non-membership criterion** (Theorem A = Theorem 3.4). If H^{-1} has its entries in the ring U_Σ generated by the inverses of a family Σ of full matrices, each atomic factor of H is stably associated with an atomic factor of some $Q \in \Sigma$ (Lemma 3.1). To refute that, it suffices that one atomic factor of H be more singular than every Q at some evaluation, measured by an invariant that does not grow on factors: Smith counts along an arc, the corank on a curve of integral quaternions, the corank at a character. The allowed denominators may themselves be singular there.
- **Two hierarchies inside E_2, strict at every step** (Theorem B). The rings $\mathcal{U}_r$ generated by inverses of sums of products of at most r inverse binomials: $\mathcal{U}_{r-1} \subsetneq \mathcal{U}_r$, with witness $(u^{-2}+v^{-2}+w^{-2})^{-1}$ for $r = 2$; their union is a proper subring of E_2. The matrix-size chain $A_{d-1} \cap E_2 \subsetneq A_d \cap E_2$ for every $d \geq 2$. The two hierarchies are independent, and the size chain is already strict inside $\mathcal{U}_2$ (Corollary 7.10).
- **Infinitely many stopping steps of the Dubrovin–Gräter–Hanke series inside E_2** (Theorem C): for every $k \geq 3$ the stratum of sums of k elements of D_π, with explicit witnesses f_k. That there are infinitely many stopping steps follows from Herbera–Sánchez, but no infinite family had been located within a fixed stage. The rings reached at these steps form a strict chain whose union is $\mathcal{U}_2$.

Main tools: Cohn's theory of firs, atoms and stable association, and of rational closures; Smith normal forms over $\mathbb{C}[[t]]$ along an exact arc of 2×2 matrices; the free group generated by $1+2\mathbf{i}$, $1+2\mathbf{j}$, $1+2\mathbf{k}$; a grading argument that gives an atom for every k.

🚀 **Start here:** **Theorem A**: the non-membership criterion: one atomic factor against the allowed denominators.

Main results:

- **Theorem A**: the non-membership criterion: one atomic factor against the allowed denominators
- **Theorem B**: the binomial hierarchy and the size chain are strict inside E_2
- **Theorem C**: every $k[\pi]$ is a stopping step of the Dubrovin–Gräter–Hanke series

Fields: where the paper lives and what it connects

Tools taken from other branches

- **Theory of firs (Cohn)** → complete factorization into atoms, stable association, torsion modules and rational closures — Lemma 2.1, Lemma 3.1, Theorem 3.4
- **Linear algebra over power series** → Smith normal forms over $\mathbb{C}[[t]]$ along an exact arc of 2×2 matrices — Lemma 3.2, Lemma 4.1, Lemma 4.4, Lemma 7.7
- **Arithmetic of quaternions** → the free group generated by 1+2i, 1+2j, 1+2k (Lubotzky–Phillips–Sarnak), residues modulo 5 — Lemma 4.5, Theorem 6.1
- **Ordered groups** → a grading argument and a bi-invariant order on the free group, to obtain an atom for every k — Lemma 7.9, Theorem 7.3
- **Formal verification** → Lean 4 + Mathlib: the finite identities and certificates of the appendix — Lemma 4.1, Lemma 5.7, Lemma 7.8, Corollary 7.10, 8.2

What the results serve

- **Inversion height and complexity of rational expressions** ← strict hierarchies of subrings inside one stage of the rational closure; stopping steps of the Dubrovin–Gräter–Hanke series — Theorem B, Theorem C
- **L^2-invariants of groups** ← the stages studied here are those of the division ring that carries the L^2-invariants of F_3; the paper says nothing new about the Atiyah conjecture
- **Universal localization** ← Malcolmson's chain A_b inside E_2, and already inside $\mathcal{U}_1$ (Corollary 7.10(a)) — Theorem 6.1, Corollary 7.10