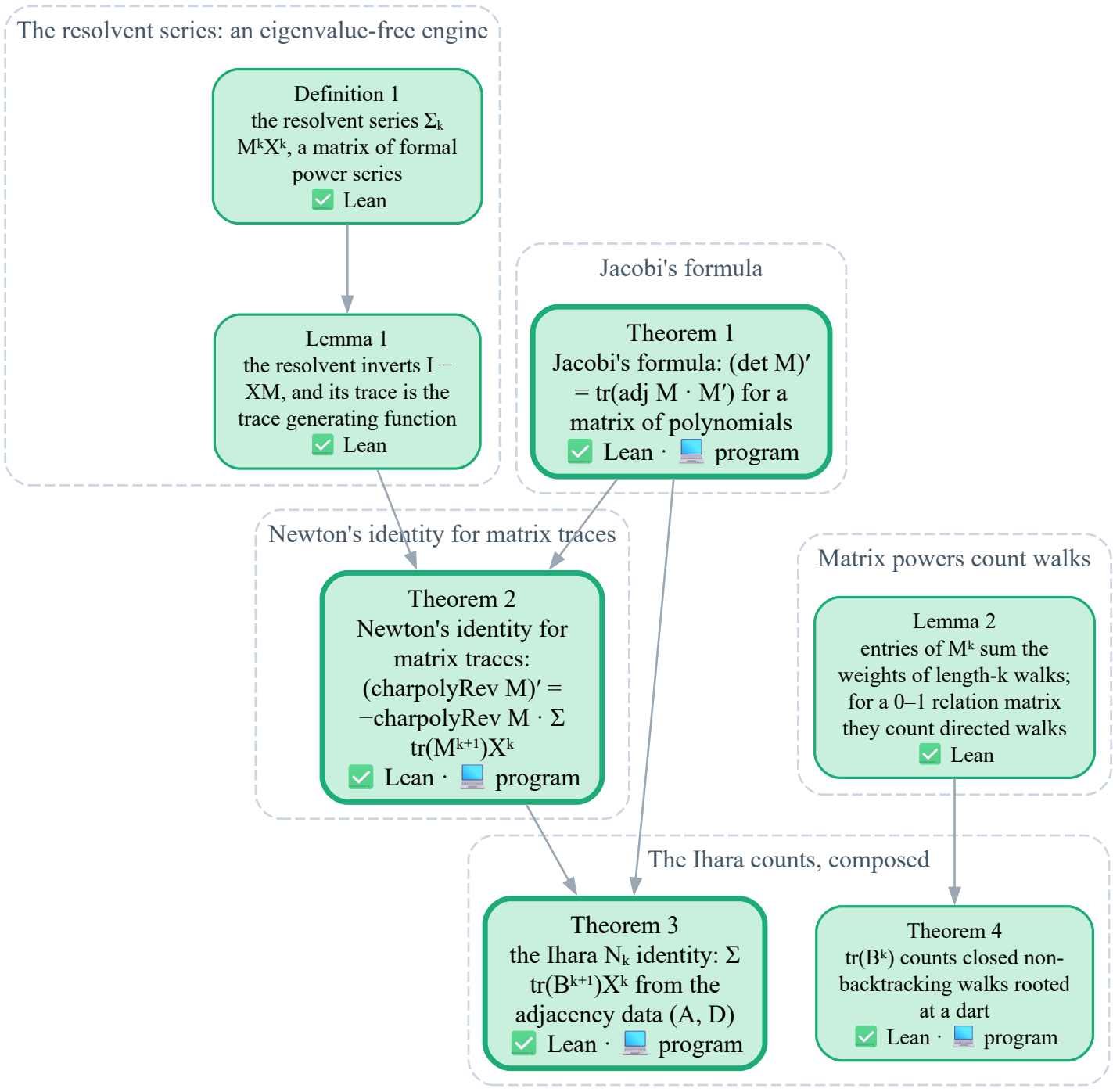
Map of *What a Determinant's Derivative Knows: Jacobi's Formula and Newton's Identity for Matrix Traces in Lean 4*

Linear algebra · characteristic polynomials (formalized)

7 results | 7 entirely in Lean | 0 partly in Lean | 4 checked by a program | written proof only | (a result can carry several tags)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



What the paper contributes

A machine-checked formalization, in Lean 4 / Mathlib, of Jacobi's formula and Newton's identity for matrix traces, composed with Bass's formula. The mathematics is classical; the contribution is the formalization.

- **Jacobi's formula** in its algebraic form, $(\det M)' = \operatorname{tr}(\operatorname{adj} M \cdot M')$ for a matrix of polynomials over a commutative ring (Theorem 1), for which the paper found no prior proof in any proof assistant. Axioms: `propext`, `Classical.choice`, `Quot.sound`.
- **Newton's identity for matrix traces** to all orders, eigenvalue-free: $(\text{charpolyRev } M)' = -\text{charpolyRev } M \cdot \Sigma_k \operatorname{tr}(M^{k+1})\, X^k$ (Theorem 2), through the resolvent series $(I - XM)^{-1} = \Sigma_k M^k X^k$ over formal power series (Lemma 1).
- **Matrix powers count walks**, also for directed graphs (Lemma 2), with the trace rewritten over cyclic walks and its weight invariant under rotation.
- **Composed with Bass's formula**: the generating function of the non-backtracking counts $N_k = \operatorname{tr}(B^k)$ from the adjacency data (A, D) (Theorem 3), and $\operatorname{tr}(B^k)$ counts rooted closed non-backtracking walks (Theorem 4).
- **The boundary**, as the paper states it: rooted walks, not geodesic classes; the prime-geodesic Euler product is mapped, not built.

🚀 **Start here:** **Theorem 1**: Jacobi's formula: $(\det M)' = \operatorname{tr}(\operatorname{adj} M \cdot M')$ for a matrix of polynomials.

Main results:

- **Theorem 1**: Jacobi's formula: $(\det M)' = \operatorname{tr}(\operatorname{adj} M \cdot M')$ for a matrix of polynomials
- **Theorem 2**: Newton's identity for matrix traces: $(\text{charpolyRev } M)' = -\text{charpolyRev } M \cdot \Sigma \operatorname{tr}(M^{k+1})X^k$
- **Theorem 3**: the Ihara N_k identity: $\Sigma \operatorname{tr}(B^{k+1})X^k$ from the adjacency data (A, D)

🧭 Fields: where the paper lives and what it connects

Tools taken from other branches

- **Linear algebra** → the Leibniz formula in product form, Cramer's rule read off the adjugate, $\operatorname{adj}(I - XM) = \det(I - XM)\cdot(I - XM)^{-1}$ — Theorem 1, Theorem 2
- **Formal power series** → the resolvent series $\Sigma M^k X^k$ as the inverse of $I - XM$, from the fixed-point equation $R = I + XM\cdot R$; units of constant term 1 — 1, Lemma 1, Theorem 2, Theorem 3

- **Graph combinatorics** → walks of a decidable relation, cyclic walks and rotation of the basepoint; darts and the non-backtracking relation — Lemma 2, Theorem 4

What the results serve

- **Ihara zeta function of graphs** ← the non-backtracking counts $N_k = \mathrm{tr}(B^k)$ from the adjacency data through Bass's formula; rooted walks, not prime geodesics — Theorem 3, Theorem 4
- **Faddeev–LeVerrier and Liouville** ← the identities the paper names as resting on Jacobi and Newton; not formalized here — Theorem 1, Theorem 2