Maximum number of spanning trees of a multigraph with m edges: exact values for m ≤ 22

Extremal graph theory · spanning trees of multigraphs · computer-assisted enumeration with certificates

Carles Marín

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At a glance

What the paper contributes

$\mathrm{T}(m)$ is the largest number of spanning trees of a connected loopless multigraph with m edges, the number of vertices being free: the first question of Problem 2 of Kenyon (1996). The paper is a data note:

- **Exact values and all extremizers for** $m \le 22$, with the simple maximum $S(m)$ and the non-simple maximum $M(m)$ (Table 1, Table 2). The sequence begins $1, 2, 3, 5, 8, 16, 24, 45, 81, 135, \ldots$ and ends with $\mathrm{T}(22) = 130691$, attained only by the circulant $C_{11}(1, 3)$.
- **Certified for** $m \le 18$ (Theorem 2, computer-assisted): 53 certificates accepted by a stand-alone checker that uses only the Python standard library and checks the completeness of every enumeration by its own counting, without nauty. **Search only for** $19 \le m \le 22$ (Computational result 3): the same proved rules, skeleton lists from nauty, no certificate.
- **The pruning rules, each with a written proof** (Lemmas 5–11, Corollary 12): monotonicity, cut vertices, vertices of degree two, counting bounds, bounds on a fixed skeleton, and two lemmas on multiple edges, from Kirchhoff's matrix-tree theorem, effective resistance and Rayleigh monotonicity.
- **Observations read off the tables** (Remark 4): from $m = 8$ on every extremizer is simple with degrees 3 and 4; a multiple edge is needed only for $m = 2, 4$.

The rules rest on three classical facts (Kirchhoff's matrix-tree theorem, effective resistance, Rayleigh monotonicity); the bound $\tau(G)^2\, 8^m < 27^m$ and the answer to Kenyon's second question are Majchrzak's (2026); the classical extremal problem with n fixed is Kelmans–Chelnokov's, Shier's, Cheng's and Petingi–Boesch–Suffel's. The paper reports that the sequence was not in the OEIS on 25 November 2026. The table below uses the paper's colour code.

m	T(m)	#ext.	n	S(m)	M(m)	status
1	1	1	2	1	–	C
2	**2**	1	2	1	2	C
3	**3**	2	2, 3	3	3	C
4	**5**	1	3	4	5	C
5	**8**	2	3, 4	8	8	C
6	16	1	4	16	13	C
7	**24**	2	4, 5	24	24	C
8	45	1	5	45	40	C
9	81	1	6	81	69	C

10	135	1	6	135	126	C
11	231	1	7	231	210	C
12	432	1	7	432	366	C
13	720	2	7, 8	720	648	C
14	1280	1	8	1280	1152	C
15	2304	1	8	2304	1992	C
16	4096	1	8	4096	3495	C
17	7168	1	9	7168	6272	C
18	12584	1	10	12584	11264	C
19	22720	1	10	22720	19712	**E**
20	40960	1	10	40960	34944	**E**
21	72657	1	11	72657	62296	**E**
22	130691	1	11	130691	112640	**E**

□□ certified (Theorem 2, status C) · □□ exhaustive search only (Computational result 3, status **E**, below the thick rule; the two shades alternate with m) · □ **T(m)** in bold: some extremizer has a multiple edge. Generated from `extremal_graphs.json` of the Zenodo record by `lab/hacer_datos.py`; this is the paper's Table 1.

🚀 **Start here: Theorem 2**: T(m), S(m), M(m) and all extremizers of T(m) for m ≤ 18 (53 certificates, stand-alone checker).

Main results:

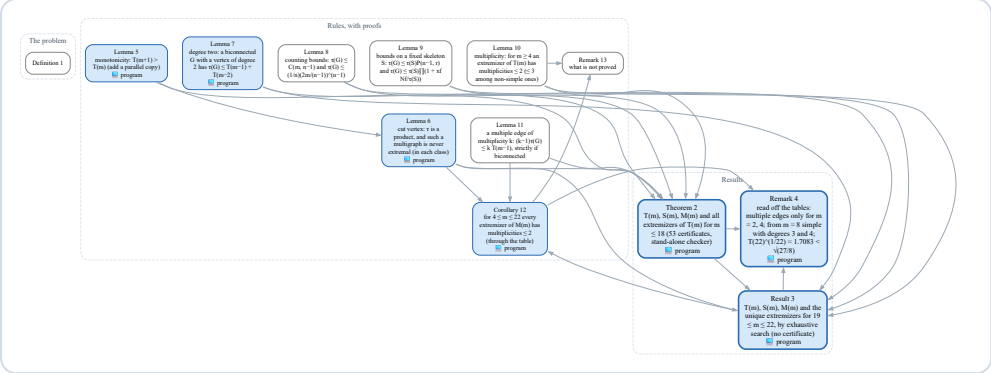
- **Theorem 2**: T(m), S(m), M(m) and all extremizers of T(m) for m ≤ 18 (53 certificates, stand-alone checker)
- **Result 3**: T(m), S(m), M(m) and the unique extremizers for 19 ≤ m ≤ 22, by exhaustive search (no certificate)
- **Remark 4**: read off the tables: multiple edges only for m = 2, 4; from m = 8 simple with degrees 3 and 4; T(22)^(1/22) = 1.7083 < √(27/8)

13 results | ✅ **0** entirely in Lean | ◐ **0** partly in Lean | 💻 **7** checked by a program | written proof only

(a result can carry several tags)

🗺️ Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



📐 Setting (copied from the paper: §1, Definition 1, and §3)

All multigraphs are finite and loopless; parallel edges are distinct edges. For a multigraph G let $\tau(G)$ be its number of spanning trees, so $\tau(G) = 0$ if G is disconnected. (§1)

For $m \geq 1$ let $\mathrm{T}(m) = \max \tau(G)$ over connected loopless multigraphs with m edges and any number of vertices; $S(m)$ is the same maximum over simple graphs and $M(m)$ over multigraphs with at least one edge of multiplicity ≥ 2 ($M(1)$ is undefined). An *extremizer* of $\mathrm{T}(m)$ is a multigraph attaining it. (Definition 1)

Notation: $R_{uv}(G)$ is the effective resistance between u and v with unit resistance on every copy. We use three classical facts (Kirchhoff): (K1) the matrix-tree theorem, valid for multigraphs; (K2) the fraction of spanning trees using a given copy $f = uv$ is $R_{uv}(G)$, so $\tau(G - f) = \tau(G)(1 - R_{uv})$ and $\tau(G + uv) = \tau(G)(1 + R_{uv})$; (K3) Rayleigh monotonicity: adding edges does not increase effective resistances. (§3)

1 The problem

Definition 1. p. 1

> For $m \geq 1$ let $\mathrm{T}(m) = \max \tau(G)$ over connected loopless multigraphs with m edges and any number of vertices; $S(m)$ is the same maximum over simple graphs and $M(m)$ over multigraphs with at least one edge of multiplicity ≥ 2 ($M(1)$ is undefined). An *extremizer* of $\mathrm{T}(m)$ is a multigraph attaining it.

↳ Not used by another result of the paper.

2 Results

Theorem 2. (computer-assisted) p. 2

> *For $1 \le m \le 18$ the values $\mathrm{T}(m)$, $S(m)$ and, for $m \ge 2$, $M(m)$ in Table 1 are exact, and the extremizers of $\mathrm{T}(m)$ are exactly the multigraphs of Table 2, up to isomorphism. The extremizers of $S(m)$ and $M(m)$ are listed in the data files.*

Trust base: verified external checker (a verdict, not a theorem). The acceptance of 53 certificates (three classes for each m ≤ 18, none for M(1)) by the stand-alone checker code/comprobar.py (Python standard library only, imports nothing from the search): it recomputes every bound in integers, every determinant (Bareiss) and the completeness of every list of skeletons by its own counting (method A; method C for n ≤ 6), without nauty; it rejects deliberately corrupted certificates. Trust base: the checker's code and the rules of §3

Proof: in this note · **Covers:** not formalized · **Cite:** Theorem 2, p. 2, doi:10.5281/zenodo.24561280

Classical and new: the paper: it gives exact data for the first question of Kenyon's Problem 2 (1996), which Majchrzak (2026) notes is not answered by his bound; the sequence was not in the OEIS when the note was written (searched on 25 November 2026).

↳ Used in: 3 (p. 6), 4 (p. 7)

In plain words (commentary, not part of the paper)

Take m edges and build any connected multigraph you like, with as many vertices as you want: how many spanning trees can it have at most? For $m \le 18$ the paper gives the exact answer, $\mathrm{T}(m) = 1, 2, 3, 5, 8, 16, 24, 45, \ldots, 12584$, and every multigraph that attains it.

Why it is true: the proof is a computation that can be re-run. For each m and each of the three classes (all, simple, non-simple) a certificate splits the class into pieces and either rules a piece out with a bound proved in §3 or lists its graphs; a checker written with the Python standard library only recomputes every bound, every determinant and the completeness of every list by its own counting, without nauty. It accepted all 53 certificates.

What the paper claims: the values and the complete lists of extremizers for $m \le 18$; the trust base is the checker's code and the rules of §3. The lab below recomputes τ of every listed extremizer in your browser.

Evidence & scope

- **Program:** the stand-alone checker: 53 certificates ACCEPTED, from the copies in the published record (m = 18: 659, 462 and 235 s for the three classes). `code/comprobar.py` in `code.zip` of Zenodo record 10.5281/zenodo.24561280, `logs/standalone_check_m1_18_zenodo_package.log` in `logs.zip` of Zenodo record 10.5281/zenodo.24561280
- **Program:** independent re-check of extremal_graphs.json (own Fraction determinant): DISCREPANCIES: 0. `comprobar_paquete.py` in `code.zip` of Zenodo record 10.5281/zenodo.24561280, `logs/extremal_graphs_recheck_zenodo_package.log` in `logs.zip` of Zenodo record 10.5281/zenodo.24561280
- **Program:** this site: τ of all 70 listed graphs (26 extremizers of T, 23 of S, 21 of M) recomputed by exact rational elimination equals the file and the table; extremal_graphs.json and T_m_values.csv agree; control: m = 22 with one edge removed gives $\tau = 70523 \neq T(22)$. `lab/comprueba_lab.py`
- **Program:** the lab's JavaScript (Kirchhoff with BigInt, Bareiss) on the same 70 graphs: 0 disagreements; five corruptions (a multiplicity changed, an edge removed, a Laplacian without multiplicities, a wrong cofactor, degrees without multiplicities) are all caught. `labjs/control_js.js`, `labjs/nucleo.js`
- **Program:** Controls 1 and 6 of the paper: brute force over all 41 207 connected simple skeletons with $n \leq 13$ and at most 12 edges (same T(m) and extremizers for $m \leq 12$); Sage recomputation of every extremizer of T and M (47 of 47). `logs/bruteforce_control_m1_12.log`, `logs/sage_recheck.log` in `logs.zip` of Zenodo record 10.5281/zenodo.24561280

Result 3. p. 2

For $19 \leq m \leq 22$ the exhaustive search of Section 4 gives the values in Table 1 and the unique extremizers of Table 2. These rows rest on the same proved rules, on the completeness of the lists produced by nauty [7] , and on the correctness of the search code; there is no certificate.

Trust base: computation (not a formal proof). An exhaustive search (code/T_m_ampliado.py) with the proved pruning rules, skeleton lists from nauty geng/multig (each batch counted against nauty's own -u count), τ modulo 2^31 − 1 with every extremizer recomputed by an exact determinant; no certificate, so nauty's completeness and the search code are trusted. m = 22 took 2639 s

Proof: in this note · **Covers:** not formalized · **Cite:** Result 3, p. 2,

doi:10.5281/zenodo.24561280

> 🆕 **Classical and new:** the paper: these rows «rest on the same proved rules, on the completeness of the lists produced by nauty, and on the correctness of the search code; there is no certificate».

↳ Used in: 4 (p. 7), Corollary 12 (p. 16)

💡 **In plain words (commentary, not part of the paper)**

For $19 \le m \le 22$ the values 22720, 40960, 72657, 130691 come from the same search with the same proved rules, but without certificates: they rest also on nauty producing complete lists of skeletons and on the search code being correct. Each of these four rows has a unique extremizer, simple, with degrees 3 and 4; for $m = 20$ and $m = 22$ it is a circulant, $C_{10}(1, 4)$ and $C_{11}(1, 3)$.

The paper keeps this status visible everywhere: in its tables these rows are below a rule and in vermilion, and so they are on this site.

🛡️ **Evidence & scope**

- **Program:** the search logs for m ≤ 21 and for m = 22. `code/T_m_ampliado.py` in `code.zip` of Zenodo record 10.5281/zenodo.24561280, `logs/search_m1_21.log`, `logs/search_m22.log` in `logs.zip` of Zenodo record 10.5281/zenodo.24561280
- **Program:** this site: the four rows have one extremizer each, simple with degrees 3 and 4; those for m = 20 and 22 are isomorphic to $C_{10}(1,4)$ and $C_{11}(1,3)$ (networkx); control: the one for m = 22 is not $C_{11}(1,2)$. `lab/comprueba_lab.py`

Remark 4. (read off the tables, checked by script) 📕 p. 2

1. A multiple edge is needed to attain $\mathrm{T}(m)$ only for $m = 2$ and $m = 4$; for $m = 3, 5, 7$ a simple and a non-simple extremizer tie. Hence $S(m) = \mathrm{T}(m)$ for every $m \le 22$ except $m = 2, 4$.
2. For $8 \le m \le 22$ every extremizer is simple with all degrees 3 or 4, and it is unique except at $m = 13$ (two). Identified up to isomorphism: K_4 ($m = 6$), $K_{3,3}$ (9), $K_{3,4}$ (12), $K_{4,4}$ minus an edge (15), $K_{4,4} \cong C_8(1, 3)$ (16), the circulants $C_{10}(1, 4)$ (20) and $C_{11}(1, 3)$ (22). The extremizer for $m = 14$ is bipartite; those for $m = 13, 17, 18, 19, 21$ are not.
3. T is strictly increasing, $\mathrm{T}(m)^{1/m}$ decreases only from $m = 6$ to $m = 7$ ($1.5874 \to 1.5746$), and $\mathrm{T}(22)^{1/22} = 1.7083$, below $\sqrt{27/8} = 1.8371$.
4. $M(m)/S(m) \in [0.81, 0.94]$ for $6 \le m \le 22$, $m \ne 7$ (at $m = 7$ they tie at 24), and every extremizer of $M(m)$ with

$4 \le m \le 22$ has maximal multiplicity exactly 2; this follows from the table by Corollary 12.

Majchrzak [5 , § 9] reports, from an exhaustive computation over multigraphs with at most 14 edges, that the maximum of $\tau^{1/m}$ is 1.667031, at $\tau = 1280$, $m = 14$; this agrees with $\mathrm{T}(14) = 1280$ and with Table 1, where $\mathrm{T}(m)^{1/m}$ is largest at $m = 14$ among $m \le 14$.

Trust base: 🧮 computation (not a formal proof). Finite checks on the published tables (rows m ≤ 18 certified, 19 ≤ m ≤ 22 search only), re-done by lab/comprueba_lab.py with networkx isomorphism tests for the named graphs

Cite: Remark 4, p. 2, doi:10.5281/zenodo.24561280

🆕 **Classical and new:** the paper: observations «read off the tables, checked by script»; for m ≥ 19 they rest on the search-only rows. The agreement with Majchrzak's exhaustive computation for m ≤ 14 (maximum of τ^(1/m) = 1.667031 at τ = 1280, m = 14) is noted in the paper.

↳ Not used by another result of the paper.

💡 In plain words (commentary, not part of the paper)

What the tables show: multiple edges help only at the start. A multiple edge is needed only for $m = 2$ and $m = 4$; from $m = 8$ on every extremizer is a simple graph in which every vertex has degree 3 or 4, and many are familiar graphs (K_4, $K_{3,3}$, $K_{3,4}$, $K_{4,4}$, circulants). The growth rate $\mathrm{T}(m)^{1/m}$ reaches 1.7083 at $m = 22$, still below Majchrzak's optimal base $\sqrt{27/8} = 1.8371\ldots$

These are observations read off the data, not theorems for all m; each one is re-checked by `lab/comprueba_lab.py` on the published files (the identifications by a graph-isomorphism test).

🛡️ Evidence & scope

- **Program:** every item re-checked on the published files: (1) multiple edge needed for m = 2, 4, ties at 3, 5, 7, S = T except m = 2, 4; (2) for 8 ≤ m ≤ 22 simple with degrees in 3, 4, unique except m = 13 (two); K_4, $K_{3,3}$, $K_{3,4}$, $K_{4,4} - e$, $K_{4,4} \cong C_8(1,3)$, $C_{10}(1,4)$, $C_{11}(1,3)$ by isomorphism; bipartite for m = 14, not for 13 (both), 17, 18, 19, 21; (3) T strictly increasing, T(m)^(1/m) decreases only from 6 to 7 (1.5874 → 1.5746), T(22)^(1/22) = 1.7083 < 1.8371, largest for m ≤ 14 at m = 14 (1.667031); (4) M/S in [0.8125, 0.9333] for 6 ≤ m ≤ 22, m ≠ 7, M(7) = S(7) = 24, every extremizer of M with m ≥ 4 has multiplicity exactly 2. Control: the isomorphism test rejects $C_{11}(1,2)$. `lab/comprueba_lab.py`

m	$T(m)$	#ext.	n	$S(m)$	$M(m)$	status

1	1	1	2	1	–	C
2	**2**	1	2	1	2	C
3	**3**	2	2, 3	3	3	C
4	**5**	1	3	4	5	C
5	**8**	2	3, 4	8	8	C
6	16	1	4	16	13	C
7	**24**	2	4, 5	24	24	C
8	45	1	5	45	40	C
9	81	1	6	81	69	C
10	135	1	6	135	126	C
11	231	1	7	231	210	C
12	432	1	7	432	366	C
13	720	2	7, 8	720	648	C
14	1280	1	8	1280	1152	C
15	2304	1	8	2304	1992	C
16	4096	1	8	4096	3495	C
17	7168	1	9	7168	6272	C
18	12584	1	10	12584	11264	C
19	22720	1	10	22720	19712	**E**
20	40960	1	10	40960	34944	**E**
21	72657	1	11	72657	62296	**E**
22	130691	1	11	130691	112640	**E**

Table 1: $\mathrm{T}(m)$, the number of extremizers up to isomorphism, their numbers of vertices, the simple maximum $S(m)$ and the non-simple maximum $M(m)$. Status: C = certified (Theorem 2); E = exhaustive search only (Result 3). Colours: ▢▢ certified rows, ▢▢ search-only rows (below the rule; the two shades alternate with m); ▢ $\mathrm{T}(m)$ in bold: some extremizer has a multiple edge.

Colour: the paper's own colours (see the caption).

m	τ	n	degrees	graph6 (skeleton)	multiplicities ≥ 2
1	1	2	1^2	A_	–
2	2	2	2^2	A_	**0–1×2**
3	3	3	2^3	Bw	–
3	3	2	3^2	A_	**0–1×3**

4	5	3	$3^2\,2$	Bw	**0–1×2**
5	8	4	$3^2\,2^2$	C^	–
5	8	3	$4\,3^2$	Bw	**0–1×2, 0–2×2**
6	16	4	3^4	C~	–
7	24	5	$3^4\,2$	D]w	–
7	24	4	$4^2\,3^2$	C~	**0–1×2**
8	45	5	$4\,3^4$	D]{	–
9	81	6	3^6	EFz_	–
10	135	6	$4^2\,3^4$	EFzo	–
11	231	7	$4\,3^6$	FCxv_	–
12	432	7	$4^3\,3^4$	F?~v_	–
13	720	7	$4^5\,3^2$	FFzf_	–
13	720	8	$4^2\,3^6$	G?o~f_	–
14	1280	8	$4^4\,3^4$	G?zVf_	–
15	2304	8	$4^6\,3^2$	G?zvf_	–
16	4096	8	4^8	G?~vf_	–
17	7168	9	$4^7\,3^2$	H?o~fbo	–
18	12584	10	$4^6\,3^4$	I?qbF_]{?	–
19	22720	10	$4^8\,3^2$	I?bFxw{?	–
20	40960	10	4^{10}	I?rFf_{N?	–
21	72657	11	$4^9\,3^2$	J?F?|wlF_?	–
22	130691	11	4^{11}	J?bFFwN?{?	–

Table 2: All extremizers of $\mathrm{T}(m)$. The graph6 string [7] encodes the underlying simple graph (skeleton) on vertices $0, \ldots, n-1$; the last column lists the edges u–v of multiplicity $k \geq 2$ in that labelling. Degrees count multiplicities; $4^2\,3^4$ means two vertices of degree 4 and four of degree 3. Colours as in Table 1: certified, search only (below the rule), rows of the same m share a shade; multiplicities in bold: the extremizer is not simple.

Colour: the paper's own colours (see the caption).

3 Rules, with proofs

Lemma 5. (monotonicity) p. 4

> $\mathrm{T}(m+1) > \mathrm{T}(m)$.

Trust base: ⚠️ not checked. Written proof (one line, from (K2)); not machine-checked. On the table: T strictly increasing for m ≤ 22

Proof: in this note · **Covers:** not formalized · **Cite:** Lemma 5, p. 4, doi:10.5281/zenodo.24561280

> 🆕 **Classical and new:** the paper gives a one-line proof from (K2); no novelty claimed.

↳ Used in: Theorem 2 (p. 5), Lemma 6 (p. 11)

Proof

> Add a parallel copy of an edge uv of an extremizer; by (K2) τ is multiplied by $1 + R_{uv} > 1$.

🛡️ **Evidence & scope**

- **Program:** T strictly increasing on the published table, m = 1..22. `lab/comprueba_lab.py`

Lemma 6. (cut vertex) p. 4

> *If G has a cut vertex, $G = G_1 \cup_v G_2$ with $m_1 + m_2 = m$, $m_i \geq 1$, and $\tau(G) = \tau(G_1)\tau(G_2) \leq \mathrm{T}(m_1)\mathrm{T}(m_2)$. Moreover, for $m \geq 2$ (multigraphs), $m \geq 3$ (simple graphs) and in the class of non-simple multigraphs such a G is never extremal: there is G' in the same class with m edges and $\tau(G') > \tau(G)$.*

Trust base: ⚠️ not checked. Written proof; not machine-checked. Control 7 of the paper checks the construction on all 229 089 non-simple multigraphs with a cut vertex (n ≤ 6, m ≤ 8); Control 5 on every multigraph with m ≤ 12, n ≤ 7

Proof: in this note · **Covers:** not formalized · **Cite:** Lemma 6, p. 4, doi:10.5281/zenodo.24561280

> 🆕 **Classical and new:** the paper proves it from (K2) and the series law for effective resistance; no novelty claimed.

↳ Used in: Theorem 2 (p. 5), 3 (p. 6), Corollary 12 (p. 16)

Proof

> A spanning tree of G is the union of one of G_1 and one of G_2. For the second claim: if G has a bridge b, contract it (τ unchanged, $m - 1$ edges, no loop) and either duplicate an edge (multigraphs) or subdivide an edge of a cycle, which adds $\tau(G/b - f) \geq 1$ trees (simple graphs; if there is no cycle, G is

> a tree and C_m is better). If G has no bridge and v is a cut vertex, let u be a neighbour of v on one side, e a copy of uv, $w \neq v$ a vertex on the other side and $H = G - e$, which is connected. Every u–w path in H passes through v, so $R_{uw}(H) = R_{uv}(H) + R_{vw}(H) > R_{uv}(H)$, and by (K2) $\tau(H + uw) > \tau(H + uv) = \tau(G)$; $H + uw$ is simple if G is. Non-simple class: contracting a bridge keeps every multiple edge and the added copy is one more; without bridges, every edge lies inside one side of v, and if G has a single pair of multiplicity ≥ 2, of multiplicity exactly 2, we take u on the side that does not contain that pair, so that e is a simple edge and $H + uw$ keeps the pair (otherwise removing one copy leaves a multiple edge anyway).

💡 **In plain words (commentary, not part of the paper)**

A multigraph that falls apart into two pieces at a single vertex counts its spanning trees as a product, and it is never the best one: either contract a bridge and spend the saved edge elsewhere, or move one edge so that it jumps across the cut vertex, which by (K2) gains trees because the effective resistance across a cut vertex is a sum. So the search may assume two vertices joined by parallel edges, or a biconnected multigraph.

The class of non-simple multigraphs needs care (the move must keep a multiple edge), and the proof treats it separately; the paper checks the construction on all 229 089 non-simple multigraphs with a cut vertex with $n \leq 6$, $m \leq 8$ (Control 7).

🛡️ **Evidence & scope**

- **Audit:** Controls 5 and 7 of the paper (brute force: no counterexample); their log is in the record, the script that produced it is not. `logs/extra_controls.log` in `logs.zip` of Zenodo record 10.5281/zenodo.24561280
- **Program:** the product bound used by the certificates of M: max_2≤a<m M(a)T(m−a) < M(m) for 3 ≤ m ≤ 22 (largest ratio 0.8000). `lab/comprueba_lab.py`

Lemma 7. (degree two) p. 4

> *If G is biconnected with $n \geq 3$ and has a vertex of degree 2, then $\tau(G) \leq \mathrm{T}(m-1) + \mathrm{T}(m-2)$, with $\mathrm{T}(0) = 1$.*

Trust base: ⚠️ not checked. Written proof (deletion–contraction); not machine-checked. The condition T(m−1) + T(m−2) < T(m) is checked on the table

Proof: in this note · **Covers:** not formalized · **Cite:** Lemma 7, p. 4, doi:10.5281/zenodo.24561280

🆕 **Classical and new:** the paper: deletion–contraction; no novelty claimed. It is applied only where T(m−1) + T(m−2) < T(m) on the computed values, «never assumed».

↳ Used in: Theorem 2 (p. 5), 3 (p. 6)

Proof

The vertex x has two distinct neighbours u, w (otherwise its neighbour is a cut vertex), and $H = G - x$ is connected with $m - 2$ edges. Either x is a leaf ($2\tau(H)$ trees) or it uses both edges ($\tau(H/uw)$ trees), so $\tau(G) = \tau(H) + \tau(H + uw)$ by deletion–contraction.

🛡️ **Evidence & scope**

- **Program:** T(m−1) + T(m−2) < T(m) exactly for m = 6 and 8 ≤ m ≤ 22; control: at m = 7 it fails (T(6) + T(5) = 24 = T(7)), so the rule is not applied there, as the paper says. `lab/comprueba_lab.py`

Lemma 8. (counting bounds) p. 4

For connected G on n vertices, $\tau(G) \le \binom{m}{n-1}$ and $\tau(G) \le \frac{1}{n}\left(\frac{2m}{n-1}\right)^{n-1}$.

Trust base: ⚠️ not checked. Written proof; not machine-checked. The checker recomputes every use of it in integers

Proof: in this note · **Covers:** not formalized · **Cite:** Lemma 8, p. 4, doi:10.5281/zenodo.24561280

🆕 **Classical and new:** the paper: a tree is a set of n − 1 edges; the second bound from (K1) and AM–GM; no novelty claimed.

↳ Used in: Theorem 2 (p. 5), 3 (p. 6)

Proof

A tree is a set of $n - 1$ edges; and $\tau = \frac{1}{n}\prod_{i\ge 2}\lambda_i$ with $\sum \lambda_i = 2m$ by (K1), then AM–GM.

Lemma 9. (bounds on a fixed skeleton) p. 4

Let S be a simple graph with e edges, G a multigraph on it with multiplicities $m_f = 1 + x_f$ and $r = m - e = \sum x_f$, and N_f the number of spanning trees of S through f. Then

$$\tau(G) \le \tau(S)\, P(n-1, r) \quad \text{and} \quad \tau(G) \le \tau(S) \prod_f \left(1 + x_f \frac{N_f}{\tau(S)}\right.$$

where $P(k, r)$ is the largest product of k integers ≥ 1 with sum $k + r$: writing $k + r = qk + s$ with $0 \le s < k$, $P(k, r) = q^{k-s}(q+1)^s$.

Trust base: ⚠️ not checked. Written proof; not machine-checked. Control 4 of the paper compares the second bound with all biconnected non-simple multigraphs with n ≤ 7 for m = 6, 8, 10: never violated, attained on 3, 11 and 46 skeletons

Proof: in this note · **Covers:** not formalized · **Cite:** Lemma 9, p. 4, doi:10.5281/zenodo.24561280

🆕 **Classical and new:** the paper proves both bounds from (K2) and (K3) (the second telescopes); no novelty claimed.

↳ Used in: Theorem 2 (p. 5), 3 (p. 6)

Proof

First: $\tau(G) = \sum_T \prod_{f \in T} m_f$ over trees T of S, and each product has $n - 1$ factors whose excesses add up to at most r. Second: add the extra copies of f one by one. By (K2) the j-th one multiplies τ by $1 + R$, where R is the current resistance of f's endpoints; the network contains S (conductance $\tau(S)/N_f$ between them) in parallel with $j - 1$ extra copies, so by (K3) $R \le 1/(\tau(S)/N_f + j - 1)$, and the product telescopes. The maximum of the right-hand side over $\sum x_f = r$ is attained by the greedy choice (each copy goes to an edge maximising $N_f/(\tau(S) + x_f N_f)$, the marginal gain), since each factor is log-concave in x_f; with a cap c on multiplicities, edges with $x_f = c - 1$ are skipped.

💡 **In plain words (commentary, not part of the paper)**

Fix the underlying simple graph S (the skeleton) and ask how much spreading r extra parallel copies over its edges can raise the number of spanning trees. Two ceilings: each tree of S gains at most the largest product of $n - 1$ multiplicities with the given total; and each extra copy of an edge multiplies τ by at most 1 plus the current resistance between its ends, which Rayleigh monotonicity bounds by what S alone gives. If even the ceiling is below the best value known, the whole skeleton is discarded without expanding it.

This is the rule that keeps the search over multigraphs finite in practice; the paper compares the second bound with all biconnected non-simple

multigraphs with $n \le 7$ for $m = 6, 8, 10$ (Control 4).

Evidence & scope

- **Audit:** Control 4 of the paper (skeleton bound P6); its log is in the record, the script that produced it is not. `logs/skeleton_bound_P6_test.log` in `logs.zip` of Zenodo record 10.5281/zenodo.24561280

Lemma 10. (multiplicity) p. 5

> *If $m \ge 4$ and G is biconnected with $n \ge 3$ and an edge uv of multiplicity $k \ge 3$, then removing one copy of uv and subdividing another gives G' with m edges and $\tau(G') > \tau(G)$. So the extremizers of $\mathrm{T}(m)$, $m \ge 4$, have multiplicities ≤ 2 (the bundle with $n = 2$ has $\tau = m$, while the triangle with multiplicities $m-2, 1, 1$ has $\tau = 2m - 3 > m$). Among non-simple multigraphs the same holds for $k \ge 4$.*

Trust base: ⚠️ not checked. Written proof; not machine-checked. Control 5 of the paper: no counterexample on every multigraph with m ≤ 12, n ≤ 7

Proof: in this note · **Covers:** not formalized · **Cite:** Lemma 10, p. 5, doi:10.5281/zenodo.24561280

🆕 **Classical and new:** the paper: from (K2); no novelty claimed. The remark «what is not proved» explains why the step k = 3 fails in the non-simple class.

↳ Used in: Theorem 2 (p. 5), 3 (p. 6), 13 (p. 17)

Proof

> With $H = G$ minus one copy and $R = R_{uv}(H)$, $\tau(G) = \tau(H)(1 + R)$ and $\tau(G') = \tau(H)(2 - R)$, so G' is better if and only if $R < 1/2$. In H the $k - 1 \ge 2$ copies of uv are in parallel with the rest, which has positive conductance between u and v when $k = 3$ (else u or v is a cut vertex); hence $R < 1/2$. For $k \ge 4$, $R \le 1/3$ and G' keeps a multiple edge.

Evidence & scope

- **Audit:** Control 5 of the paper (brute force); its log is in the record, the script that produced it is not. `logs/extra_controls.log` in `logs.zip` of Zenodo record 10.5281/zenodo.24561280

Lemma 11. (a multiple edge) p. 5

> *If G has an edge uv of multiplicity $k \geq 2$, then $(k-1)\,\tau(G) \leq k\,\mathrm{T}(m-1)$, strictly if G is biconnected with $n \geq 3$.*

Trust base: ⚠️ not checked. Written proof; not machine-checked. Control 7 of the paper: the inequality holds for all 384 650 multiple edges of labelled connected multigraphs with n ≤ 6, m ≤ 8, strictly in the biconnected case (while τ ≤ T(m−1) fails 10 107 times)

Proof: in this note · **Covers:** not formalized · **Cite:** Lemma 11, p. 5, doi:10.5281/zenodo.24561280

🆕 **Classical and new:** the paper: from (K2) and (K3); no novelty claimed.

↳ Used in: Theorem 2 (p. 5), Corollary 12 (p. 16)

Proof

> Let H be G minus one copy of uv. In H the $k-1$ remaining copies join u and v, so by (K3) $R_{uv}(H) \leq 1/(k-1)$, and by (K2) $\tau(G) = \tau(H)(1+R_{uv}(H)) \leq \frac{k}{k-1}\mathrm{T}(m-1)$. If G is biconnected with $n \geq 3$, the rest of H has positive conductance between u and v and the inequality for R is strict.

🛡️ **Evidence & scope**

- **Audit:** Control 7 of the paper (brute force, n ≤ 6, m ≤ 8); its log is in the record, the script that produced it is not. `logs/extra_controls.log` in `logs.zip` of Zenodo record 10.5281/zenodo.24561280

Corollary 12. p. 5

> *For $4 \leq m \leq 22$, every extremizer of $M(m)$ has multiplicities ≤ 2.*

Trust base: ⚠️ not checked. Written proof combining Lemmas 6 and 11 with the table (certified values for m ≤ 18, search-only values for 19 ≤ m ≤ 22); the table inequalities are re-checked

Proof: in this note · **Covers:** not formalized · **Cite:** Corollary 12, p. 5, doi:10.5281/zenodo.24561280

🆕 **Classical and new:** the paper: proved only for 4 ≤ m ≤ 22, through the table, not for all m (remark «what is not proved»); for 19 ≤ m ≤ 22 it uses the values of Computational result 3.

↳ Used in: 4 (p. 7), 13 (p. 17)

Proof

> The bundle has $\tau = m < 2m - 3 \le M(m)$; a non-simple G with a cut vertex has $\tau(G) \le \max_{2 \le a < m} M(a)\mathrm{T}(m-a) < M(m)$ (Lemma 6 and the table); a biconnected one with an edge of multiplicity $k \ge 3$ has $\tau(G) < \frac{3}{2}\mathrm{T}(m-1) \le M(m)$ by Lemma 11 and the table (equality in the last step at $m = 7$ and $m = 13$). For $m \le 18$ the values used are certified; for $19 \le m \le 22$ they are those of Result 3.

💡 **In plain words (commentary, not part of the paper)**

Among multigraphs that must have a multiple edge, the best one never uses a triple edge, for $4 \le m \le 22$: a double edge is always enough. The proof combines the lemmas with the numbers of the table, so it is proved only in that range (the paper's remark «what is not proved» says so), and for $19 \le m \le 22$ it uses the search-only values.

The table inequalities it needs ($m < 2m - 3 \le M(m)$, the cut-vertex products below $M(m)$, and $\frac{3}{2}\mathrm{T}(m-1) \le M(m)$ with equality at $m = 7$ and $m = 13$) are re-checked by `lab/comprueba_lab.py`.

🛡️ **Evidence & scope**

- **Program:** on the published table: m < 2m − 3 ≤ M(m) for 4 ≤ m ≤ 22; max_2≤a<m M(a)T(m−a) < M(m) for 3 ≤ m ≤ 22; (3/2)T(m−1) ≤ M(m) with equality exactly at m = 7 and m = 13, as the proof says; every listed extremizer of M with m ≥ 4 has maximal multiplicity 2. Control: the bound T(m−1) ≤ M(m)/2 is false on the table. `lab/comprueba_lab.py`

Remark 13. (what is not proved) p. 5

> For the class of non-simple multigraphs the step $k = 3$ of Lemma 10 fails when uv is the only multiple edge (G' is simple, outside the class); so the certificates of $M(m)$ use the cap 3 of Lemma 10, and "the best non-simple multigraph has multiplicities ≤ 2" is proved here only for $4 \le m \le 22$, through the table (Corollary 12), not for all m. Two further bounds that were considered (a reduction to a degree-≥ 3 core and a degree-product bound) were not proved and are not used.

↳ Not used by another result of the paper.

4 Method

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper, p. 5.

5 Certificates and the independent checker

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper, p. 6.

6 Controls

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper, p. 7.

7 Data

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper (PDF).

Acknowledgement

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper (PDF).

📊 Tables

Every table of the paper, in its order: click one to unfold it, with where the paper cites it, how its numbers were checked, and its data as CSV. The typeset table is one click away, on its page of the PDF.

Table 1. $\mathrm{T}(m)$, the number of extremizers up to isomorphism, their numbers of vertices, the simple maximum $S(m)$ and the non-simple maximum $M(m)$. Status: C = certified (Theorem 2); E = exhaustive search only (Result 3). Colours: catC catCb certified rows, catS catSb search-only rows (below the rule; the two shades alternate with m); catM $\mathrm{T}(m)$ in bold: some extremizer has a multiple edge.

m	$T(m)$	#ext.	n	$S(m)$	$M(m)$	status
1	1	1	2	1	–	C
2	**2**	1	2	1	2	C
3	**3**	2	2, 3	3	3	C
4	**5**	1	3	4	5	C
5	**8**	2	3, 4	8	8	C
6	16	1	4	16	13	C
7	**24**	2	4, 5	24	24	C
8	45	1	5	45	40	C
9	81	1	6	81	69	C
10	135	1	6	135	126	C
11	231	1	7	231	210	C
12	432	1	7	432	366	C
13	720	2	7, 8	720	648	C
14	1280	1	8	1280	1152	C
15	2304	1	8	2304	1992	C
16	4096	1	8	4096	3495	C
17	7168	1	9	7168	6272	C
18	12584	1	10	12584	11264	C
19	22720	1	10	22720	19712	E
20	40960	1	10	40960	34944	E
21	72657	1	11	72657	62296	E
22	130691	1	11	130691	112640	E

Colour: the paper's own colours (see the caption).

p. 2 in the PDF | ⬇ CSV | cited in Theorem 2 | cited in Result 3 | cited in §2 Results | cited in Remark 4 | cited in §4 Method

Table 2. All extremizers of $\mathrm{T}(m)$. The graph6 string [McKayPiperno2014] encodes the underlying simple graph (skeleton) on vertices $0, \dots, n-1$; the last column lists the edges u–v of multiplicity $k \geq 2$ in that labelling. Degrees count multiplicities; $4^2\,3^4$ means two vertices of degree 4 and four of degree 3. Colours as in Table 1: catC catCb certified, catS catSb search only (below the rule), rows of the same m share a shade; catM multiplicities in bold: the extremizer is not simple.

m	τ	n	degrees	graph6 (skeleton)	multiplicities ≥ 2
1	1	2	1^2	A_	–
2	2	2	2^2	A_	**0–1**×2
3	3	3	2^3	Bw	–
3	3	2	3^2	A_	**0–1**×3
4	5	3	$3^2\,2$	Bw	**0–1**×2
5	8	4	$3^2\,2^2$	C	–
5	8	3	$4\,3^2$	Bw	**0–1**×2, 0–2×2
6	16	4	3^4	C	–
7	24	5	$3^4\,2$	D]w	–
7	24	4	$4^2\,3^2$	C	**0–1**×2
8	45	5	$4\,3^4$	D]\	–
9	81	6	3^6	EFz_	–
10	135	6	$4^2\,3^4$	EFzo	–
11	231	7	$4\,3^6$	FCxv_	–
12	432	7	$4^3\,3^4$	F?v_	–
13	720	7	$4^5\,3^2$	FFzf_	–
13	720	8	$4^2\,3^6$	G?of_	–
14	1280	8	$4^4\,3^4$	G?zVf_	–
15	2304	8	$4^6\,3^2$	G?zvf_	–
16	4096	8	4^8	G?vf_	–
17	7168	9	$4^7\,3^2$	H?ofbo	–

18	12584	10	$4^6\,3^4$	`I?qbF_]\?`	–
19	22720	10	$4^8\,3^2$	`I?bFxw\?`	–
20	40960	10	4^{10}	`I?rFf_\N?`	–
21	72657	11	$4^9\,3^2$	`J?F?wlF_?`	–
22	130691	11	4^{11}	`J?bFFwN?\?`	–

Colour: the paper's own colours (see the caption).

p. 3 in the PDF | ⬇ CSV | cited in Theorem 2 | cited in Result 3

References

1 C.-S. Cheng, *Maximizing the total number of spanning trees in a graph: two related problems in graph theory and optimum design theory*, J. Combin. Theory Ser. B **31** (1981), 240–248. 🔎 find it

2 D. J. Gross, J. T. Saccoman and C. L. Suffel, *Spanning Tree Results for Graphs and Multigraphs: A Matrix-Theoretic Approach*, World Scientific, 2014, doi:10.1142/8963.

3 A. K. Kelmans and V. M. Chelnokov, *A certain polynomial of a graph and graphs with an extremal number of trees*, J. Combin. Theory Ser. B **16** (1974), 197–214. 🔎 find it

4 R. Kenyon, *Tiling a rectangle with the fewest squares*, J. Combin. Theory Ser. A **76** (1996), 272–291, doi:10.1006/jcta.1996.0104.

5 F. Majchrzak, *Spanning trees per edge: a sharp bound and a question of Kenyon, with a Lean 4 formalisation of the bound and of its sharpness*, preprint, version 2.0.0, Zenodo (2026), doi:10.5281/zenodo.24523144.

6 B. D. McKay, *Spanning trees in regular graphs*, European J. Combin. **4** (1983), 149–160. 🔎 find it

7

B. D. McKay and A. Piperno, *Practical graph isomorphism, II*, J. Symbolic Comput. **60** (2014), 94–112. 🔎 find it

8

L. Petingi, F. Boesch and C. Suffel, *On the characterization of graphs with maximum number of spanning trees*, Discrete Math. **179** (1998), 155–166. 🔎 find it

9

D. R. Shier, *Maximizing the number of spanning trees in a graph with n nodes and m edges*, J. Res. Nat. Bur. Standards Sect. B **78B** (1974), 193–196. 🔎 find it