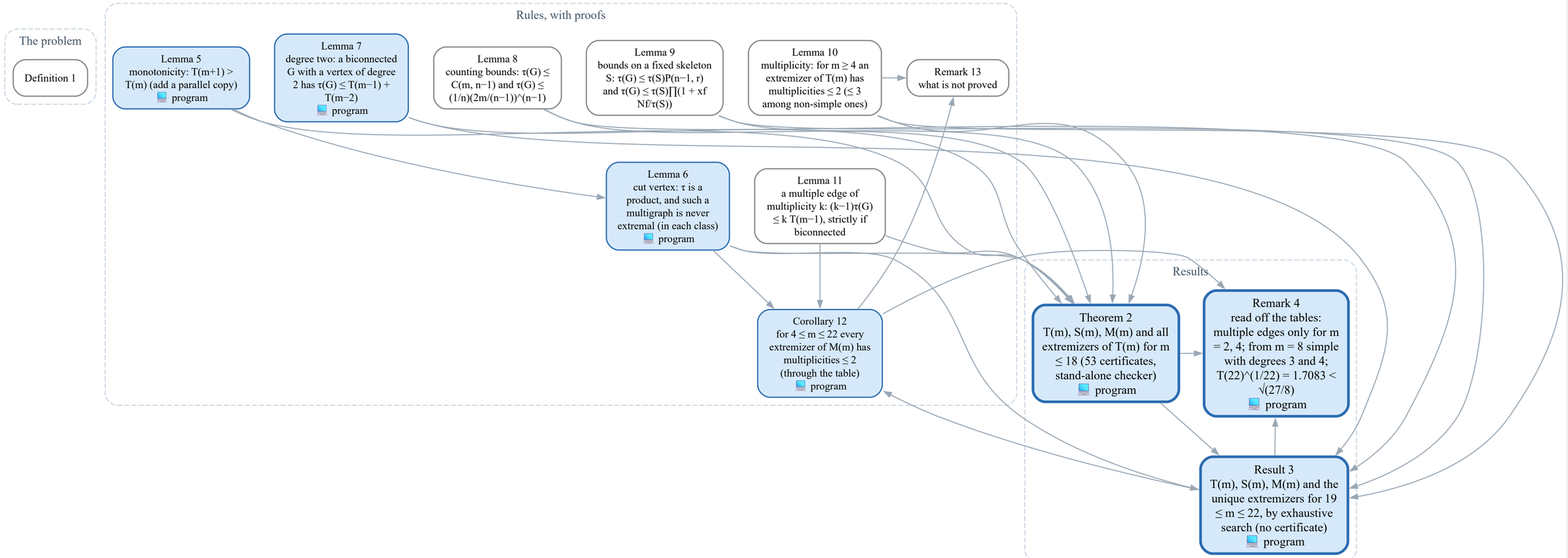
Map of *Maximum number of spanning trees of a multigraph with m edges: exact values for m ≤ 22*

Extremal graph theory · spanning trees of multigraphs · computer-assisted enumeration with certificates

13 results | **0** entirely in Lean | **0** partly in Lean | **7** checked by a program | written proof only | (a result can carry several tags)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



What the paper contributes

$\mathrm{T}(m)$ is the largest number of spanning trees of a connected loopless multigraph with m edges, the number of vertices being free: the first question of Problem 2 of Kenyon (1996). The paper is a data note:

- **Exact values and all extremizers for $m\le 22$**, with the simple maximum $S(m)$ and the non-simple maximum $M(m)$ (Table 1, Table 2). The sequence begins $1, 2, 3, 5, 8, 16, 24, 45, 81, 135, \dots$ and ends with $\mathrm{T}(22)=130691$, attained only by the circulant $C_{11}(1,3)$.
- **Certified for $m\le 18$** (Theorem 2, computer-assisted): 53 certificates accepted by a stand-alone checker that uses only the Python standard library and checks the completeness of every enumeration by its own counting, without nauty. **Search only for $19\le m\le 22$** (Computational result 3): the same proved rules, skeleton lists from nauty, no certificate.
- **The pruning rules, each with a written proof** (Lemmas 5–11, Corollary 12): monotonicity, cut vertices, vertices of degree two, counting bounds, bounds on a fixed skeleton, and two lemmas on multiple edges, from Kirchhoff's matrix-tree theorem, effective resistance and Rayleigh monotonicity.
- **Observations read off the tables** (Remark 4): from $m=8$ on every extremizer is simple with degrees 3 and 4; a multiple edge is needed only for $m=2,4$.

The rules rest on three classical facts (Kirchhoff's matrix-tree theorem, effective resistance, Rayleigh monotonicity); the bound $\tau(G)^2\,8^m<27^m$ and the answer to Kenyon's second question are Majchrzak's (2026); the classical extremal problem with n fixed is Kelmans–Chelnokov's, Shier's, Cheng's and Petingi–Boesch–Suffel's. The paper reports that the sequence was not in the OEIS on 25 November 2026. The table below uses the paper's colour code.

m	T(m)	#ext.	n	S(m)	M(m)	status
1	1	1	2	1	–	C
2	**2**	1	2	1	2	C
3	**3**	2	2, 3	3	3	C
4	**5**	1	3	4	5	C
5	**8**	2	3, 4	8	8	C
6	16	1	4	16	13	C
7	**24**	2	4, 5	24	24	C
8	45	1	5	45	40	C
9	81	1	6	81	69	C
10	135	1	6	135	126	C
11	231	1	7	231	210	C
12	432	1	7	432	366	C
13	720	2	7, 8	720	648	C
14	1280	1	8	1280	1152	C
15	2304	1	8	2304	1992	C
16	4096	1	8	4096	3495	C
17	7168	1	9	7168	6272	C
18	12584	1	10	12584	11264	C
19	22720	1	10	22720	19712	**E**

20	40960	1	10	40960	34944	**E**
21	72657	1	11	72657	62296	**E**
22	130691	1	11	130691	112640	**E**

□□ certified (Theorem 2, status C) · □□ exhaustive search only (Computational result 3, status **E**, below the thick rule; the two shades alternate with m) · □ **T(m)** in bold: some extremizer has a multiple edge. Generated from `extremal_graphs.json` of the Zenodo record by `lab/hacer_datos.py`; this is the paper's Table 1.

🚀 **Start here: Theorem 2**: T(m), S(m), M(m) and all extremizers of T(m) for m ≤ 18 (53 certificates, stand-alone checker).

Main results:

- **Theorem 2**: T(m), S(m), M(m) and all extremizers of T(m) for m ≤ 18 (53 certificates, stand-alone checker)
- **Result 3**: T(m), S(m), M(m) and the unique extremizers for 19 ≤ m ≤ 22, by exhaustive search (no certificate)
- **Remark 4**: read off the tables: multiple edges only for m = 2, 4; from m = 8 simple with degrees 3 and 4; T(22)^(1/22) = 1.7083 < √(27/8)

🧭 Fields: where the paper lives and what it connects

Tools taken from other branches

- **Electrical networks and the matrix-tree theorem (Kirchhoff; Rayleigh)** → (K1) the matrix-tree theorem for multigraphs, (K2) the share of spanning trees through a copy is the effective resistance of its ends, (K3) Rayleigh monotonicity: every pruning rule is proved from these three facts — Lemma 5, Lemma 6, Lemma 9, Lemma 10, Lemma 11
- **Graph enumeration (nauty; generating functions for labelled graphs)** → the search lists skeletons with nauty's geng and multig; the checker proves each list complete by orbit sums Σ n!/|Aut| against labelled counts from generating functions (OEIS A001187, A013922), without nauty — Theorem 2, 3
- **Kenyon's Problem 2 and Majchrzak's sharp bound** → the question (Kenyon 1996) and the bound τ(G)²8^m < 27^m with optimal base √(27/8) (Majchrzak 2026), against which the growth of T(m) is compared — 4

What the results serve

- **Exact values of T(m), S(m), M(m) and the extremal multigraphs** ← a new integer sequence for m ≤ 22 with its 26 extremizers (the paper reports it was not in the OEIS), certified data for m ≤ 18, and observations (degrees 3 and 4, circulants) that a general answer to Kenyon's first question would have to explain — Theorem 2, 3, 4, Corollary 12