Exact limits on writing a superconducting quantum memory: when can pulse shaping beat the resonant swap?

Carles Marín

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At a glance

🚀 **Start here: Theorem 1**: Relaxation only: no pulse transfers more population than the resonant swap (closed form).

Main results:

- **Theorem 1**: Relaxation only: no pulse transfers more population than the resonant swap (closed form)
- **Theorem 4**: With Markovian dephasing too: the resonant swap still bounds the transferred population
- **Theorem 7**: At a prescribed duration: a Lyapunov function bounds the population by the resonant pulse
- **Corollary 9**: The write channel: infidelity r from population and coherence, and the bounds on $r_o pt$

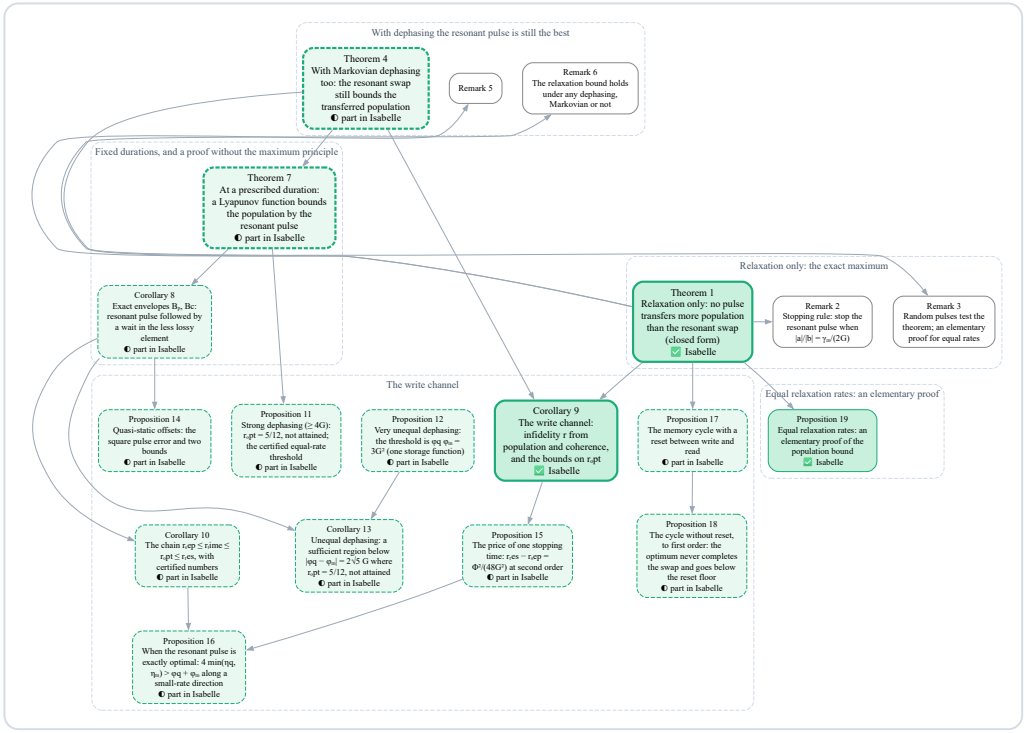
19 results | ✅ **0** entirely in Lean | ◐ **12** partly in Lean

✅ **3** in Isabelle | 💻 **0** checked by a program | written proof only

(a result can carry several tags)

🗺️ Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



📐 Setting (copied from the paper, §§1–2)

One excitation shared by a qubit (amplitude a) and a memory mode (amplitude b), each relaxing into the vacuum through its own channel at rates $\gamma_q, \gamma_m \geq 0$, and dephasing at rates $\phi_q, \phi_m \geq 0$; the control is a complex coupling $g(t)$ with $|g(t)| \leq G$ ($G>0$) and a detuning $D(t)$, and the duration T is free: $i\dot{a} = \big(D(t) - \frac{i}{2}\gamma_q\big)a + g(t)\,b,\ \ i\dot{b} = -\frac{i}{2}\gamma_m\, b + \overline{g(t)}\,a$, with $a(0)=1, b(0)=0$. p is the population transferred to the memory and c the coherence transferred with it; the write channel has average infidelity $r = 1 - F = \frac{1-p}{6} + \frac{1-c}{3}$, and r_{opt} is the infimum of r over all pulses and durations. The *resonant* pulse is $g \equiv G$ with no detuning, stopped at the best time; r_{res} is its infidelity.

2 Relaxation only: the exact maximum

Theorem 1. Isabelle p. 6

relaxation only (no dephasing); two states; |g| <= G; no relation between G and the rates
Memory_Access_All_Pulses.memory_access_all_pulses,
Memory_Access_Over.memory_access_over,
Memory_Access_Over.memory_access_critical,
Memory_Access_All_Pulses.bound_sharp,
Memory_Access_Floors.resonant_pulse_max_bound,
Memory_Access_Floors.resonant_pulse_max_attained,
Memory_Access_Floors.max_population_symmetric,
Memory_Access_Extras.folk_number_bound,
Memory_Access_Regimes.over_pulse_global_max,
Memory_Access_Regimes.over_pulse_max_attained,
Memory_Access_Regimes.critical_pulse_max,
Memory_Access_Regimes.log_derivative_identity,
Memory_Access_Regimes.folk_bound_over Let $\bar{\gamma} = (\gamma_q + \gamma_m)/2$ *and*

$$t_* = \begin{cases} \Omega^{-1}\arctan(2\Omega/\bar{\gamma}), & G>|\delta|/4,\ \ \Omega = \sqrt{G^2 - \delta^2/16}, \\ 2/\bar{\gamma}, & G = |\delta|/4, \\ \kappa^{-1}\operatorname{artanh}(2\kappa/\bar{\gamma}), & G<|\delta|/4,\ \ \kappa = \sqrt{\delta^2/16 - G^2}. \end{cases}$$

For every control in (1) and every $t \geq 0$,

$$|b(t)|^2 \ \leq\ P_*(\gamma_q, \gamma_m; G) \ =\ \frac{4G^2}{4G^2 + \gamma_q\gamma_m}\, e^{-\bar{\gamma} t_*}, \qquad (4)$$

with equality for the resonant pulse $g \equiv G$, $D \equiv 0$ *stopped at* t_* *(this optimal stopping time and value of the resonant pulse, in all three regimes, are Eqs. (16)–(17) of [16]). In the under-damped case* $P_* = \frac{4G^2}{4G^2+\gamma_q\gamma_m}\exp\big(-\frac{\bar{\gamma}}{\Omega}\arctan\frac{2\Omega}{\bar{\gamma}}\big)$*; in the over-damped case*

> $P_* = \frac{4G^2}{4G^2+\gamma_q\gamma_m}\left(\frac{\bar\gamma-2\kappa}{\bar\gamma+2\kappa}\right)^{\bar\gamma/(2\kappa)}$, *real because* $\bar\gamma^2 - 4\kappa^2 = 4G^2 + \gamma_q\gamma_m > 0$. *In particular* P_* *is symmetric in the two rates, and*
>
> $$P_* \;\geq\; \exp\bigl(-\pi(\gamma_q+\gamma_m)/(4G)\bigr), \qquad (5)$$
>
> *with* $\ln P_* = -\pi(\gamma_q+\gamma_m)/(4G) + ({\gamma_q}^2+{\gamma_m}^2)/(8G^2) + O(\gamma^3/G^3)$*: the familiar estimate of the loss of the standard pulse is a rigorous upper bound for the smallest loss. (If* $\gamma_q = \gamma_m = 0$*, read* $t_* = \pi/(2G)$ *and* $P_* = 1$*.)*

Isabelle · ours **show**

memory_access_all_pulses · Isabelle/MemoryAccessFloors/Memory_Access_All_Pulses.thy, line 594

```
theorem memory_access_all_pulses:
  fixes G gq gm T :: real and A B u :: "real \<Rightarrow> real"
  assumes G: "0 < G" and gq: "0 \<le> gq" and gm: "0 \<le> gm" and
Gd: "\<bar>gq - gm\<bar> / 4 < G" and T: "0 \<le> T"
    and dA: "\<And>t. t \<in> {0..T} \<Longrightarrow> (A
has_real_derivative (- gq * A t - 2 * u t)) (at t within {0..T})"
    and dB: "\<And>t. t \<in> {0..T} \<Longrightarrow> (B
has_real_derivative (- gm * B t + 2 * u t)) (at t within {0..T})"
    and A_nn: "\<And>t. t \<in> {0..T} \<Longrightarrow> 0 \<le> A
t" and B_nn: "\<And>t. t \<in> {0..T} \<Longrightarrow> 0 \<le> B
t"
    and uineq: "\<And>t. t \<in> {0..T} \<Longrightarrow> (u t)\
<^sup>2 \<le> G\<^sup>2 * A t * B t"
    and A0: "A 0 = 1" and B0: "B 0 = 0" and t: "t \<in> {0..T}"
  shows "B t \<le> max_population G gq gm"
    and "0 < gq + gm \<Longrightarrow> B t \<le> 4 * G\<^sup>2 /
(4 * G\<^sup>2 + gq * gm)
           * exp (- (((gq + gm) / 2) / sqrt (G\<^sup>2 - (gq -
gm)\<^sup>2 / 16)) * arctan (2 * sqrt (G\<^sup>2 - (gq - gm)\
<^sup>2 / 16) / ((gq + gm) / 2)))"
    and "gq + gm = 0 \<Longrightarrow> B t \<le> 1"
```

memory_access_over · Isabelle/MemoryAccessFloors/Memory_Access_Over.thy, line 1007

```
theorem memory_access_over:
  fixes G gq gm T :: real and A B u :: "real \<Rightarrow> real"
  assumes G: "0 < G" and gq: "0 \<le> gq" and gm: "0 \<le> gm" and
Ov: "G < \<bar>gq - gm\<bar> / 4" and T: "0 \<le> T"
    and dA: "\<And>t. t \<in> {0..T} \<Longrightarrow> (A
has_real_derivative (- gq * A t - 2 * u t)) (at t within {0..T})"
    and dB: "\<And>t. t \<in> {0..T} \<Longrightarrow> (B
has_real_derivative (- gm * B t + 2 * u t)) (at t within {0..T})"
    and A_nn: "\<And>t. t \<in> {0..T} \<Longrightarrow> 0 \<le> A
t" and B_nn: "\<And>t. t \<in> {0..T} \<Longrightarrow> 0 \<le> B
t"
    and uineq: "\<And>t. t \<in> {0..T} \<Longrightarrow> (u t)\
<^sup>2 \<le> G\<^sup>2 * A t * B t"
    and A0: "A 0 = 1" and B0: "B 0 = 0" and t: "t \<in> {0..T}"
  shows "B t \<le> 4 * G\<^sup>2 / (4 * G\<^sup>2 + gq * gm)
     * exp (- ((gq + gm) / 2) * (artanh (2 * sqrt ((gq - gm)\
<^sup>2 / 16 - G\<^sup>2) / ((gq + gm) / 2))
          / sqrt ((gq - gm)\<^sup>2 / 16 - G\<^sup>2)))"
```

memory_access_critical · Isabelle/MemoryAccessFloors/Memory_Access_Over.thy, line 1024

```
theorem memory_access_critical:
  fixes G gq gm T :: real and A B u :: "real \<Rightarrow> real"
  assumes G: "0 < G" and gq: "0 \<le> gq" and gm: "0 \<le> gm" and
```

```
Cr: "G = \<bar>gq - gm\<bar> / 4" and T: "0 \<le> T"
    and dA: "\<And>t. t \<in> {0..T} \<Longrightarrow> (A
has_real_derivative (- gq * A t - 2 * u t)) (at t within {0..T})"
    and dB: "\<And>t. t \<in> {0..T} \<Longrightarrow> (B
has_real_derivative (- gm * B t + 2 * u t)) (at t within {0..T})"
    and A_nn: "\<And>t. t \<in> {0..T} \<Longrightarrow> 0 \<le> A
t" and B_nn: "\<And>t. t \<in> {0..T} \<Longrightarrow> 0 \<le> B
t"
    and uineq: "\<And>t. t \<in> {0..T} \<Longrightarrow> (u t)\
<^sup>2 \<le> G\<^sup>2 * A t * B t"
    and A0: "A 0 = 1" and B0: "B 0 = 0" and t: "t \<in> {0..T}"
  shows "B t \<le> 4 * G\<^sup>2 / (4 * G\<^sup>2 + gq * gm) * exp
(- 2)"
```

bound_sharp · Isabelle/MemoryAccessFloors/Memory_Access_All_Pulses.thy, line 706

```
theorem bound_sharp:
  fixes G gq gm :: real
  assumes "0 < G" and "0 \<le> gq" and "0 \<le> gm" and "\<bar>gq
- gm\<bar> / 4 < G" and "0 < gq + gm"
  shows "\<exists>T. 0 \<le> T \<and> pulse G gq gm T (res_A G gq
gm) (res_B G gq gm) (res_u G gq gm)
              \<and> res_B G gq gm T = max_population G gq gm"
```

resonant_pulse_max_bound · Isabelle/MemoryAccessFloors/Memory_Access_Floors.thy, line 356

```
theorem resonant_pulse_max_bound:
  fixes G gq gm gb d \<Omega> t :: real
  assumes "0 < G" and "0 \<le> gq" and "0 \<le> gm" and "gb = (gq
+ gm) / 2" and "0 < gb"
    and "d = gq - gm" and "G\<^sup>2 > d\<^sup>2 / 16" and "\
<Omega> = sqrt (G\<^sup>2 - d\<^sup>2 / 16)" and "0 \<le> t"
  shows "G\<^sup>2 / \<Omega>\<^sup>2 * exp (- gb * t) * (sin (\
<Omega> * t))\<^sup>2
         \<le> 4 * G\<^sup>2 / (4 * G\<^sup>2 + gq * gm) * exp (-
(gb / \<Omega>) * arctan (2 * \<Omega> / gb))"
```

resonant_pulse_max_attained · Isabelle/MemoryAccessFloors/Memory_Access_Floors.thy, line 377

```
theorem resonant_pulse_max_attained:
  fixes G gq gm gb d \<Omega> tstar :: real
  assumes "0 < G" and "0 \<le> gq" and "0 \<le> gm" and "gb = (gq
+ gm) / 2" and "0 < gb"
    and "d = gq - gm" and "G\<^sup>2 > d\<^sup>2 / 16" and "\
<Omega> = sqrt (G\<^sup>2 - d\<^sup>2 / 16)"
    and "tstar = arctan (2 * \<Omega> / gb) / \<Omega>"
  shows "G\<^sup>2 / \<Omega>\<^sup>2 * exp (- gb * tstar) * (sin
(\<Omega> * tstar))\<^sup>2
         = 4 * G\<^sup>2 / (4 * G\<^sup>2 + gq * gm) * exp (- (gb
/ \<Omega>) * arctan (2 * \<Omega> / gb))"
```

max_population_symmetric · Isabelle/MemoryAccessFloors/Memory_Access_Floors.thy, line 404

```
lemma max_population_symmetric:
  "max_population G gq gm = max_population G gm gq"
```

folk_number_bound · Isabelle/MemoryAccessFloors/Memory_Access_Extras.thy, line 91

```
theorem folk_number_bound:
  fixes G gq gm gb d \<Omega> :: real
  assumes "0 < G" and "0 \<le> gq" and "0 \<le> gm" and "gb = (gq
+ gm) / 2" and "0 < gb"
```

```
    and "d = gq - gm" and "G\<^sup>2 > d\<^sup>2 / 16" and "\
<Omega> = sqrt (G\<^sup>2 - d\<^sup>2 / 16)"
  shows "exp (- pi * (gq + gm) / (4 * G))
         \<le> 4 * G\<^sup>2 / (4 * G\<^sup>2 + gq * gm) * exp (-
(gb / \<Omega>) * arctan (2 * \<Omega> / gb))"
```

over_pulse_global_max · Isabelle/MemoryAccessFloors/Memory_Access_Regimes.thy, line 302

```
theorem over_pulse_global_max:
  fixes G gq gm gb d \<kappa> tstar t :: real
  assumes "0 < G" and "0 \<le> gq" and "0 \<le> gm" and "gb = (gq
+ gm) / 2" and "d = gq - gm"
    and "G < \<bar>d\<bar> / 4" and "\<kappa> = sqrt (d\<^sup>2 /
16 - G\<^sup>2)"
    and "tstar = artanh (2 * \<kappa> / gb) / \<kappa>" and "0 \
<le> t"
  shows "pulse_over G gb \<kappa> t \<le> pulse_over G gb \<kappa>
tstar"
```

over_pulse_max_attained · Isabelle/MemoryAccessFloors/Memory_Access_Regimes.thy, line 277

```
theorem over_pulse_max_attained:
  fixes G gq gm gb d \<kappa> tstar :: real
  assumes "0 < G" and "0 \<le> gq" and "0 \<le> gm" and "gb = (gq
+ gm) / 2" and "d = gq - gm"
    and "G < \<bar>d\<bar> / 4" and "\<kappa> = sqrt (d\<^sup>2 /
16 - G\<^sup>2)"
    and "tstar = artanh (2 * \<kappa> / gb) / \<kappa>"
  shows "pulse_over G gb \<kappa> tstar = 4 * G\<^sup>2 / (4 * G\
<^sup>2 + gq * gm) * exp (- gb * tstar)"
```

critical_pulse_max · Isabelle/MemoryAccessFloors/Memory_Access_Regimes.thy, line 363

```
theorem critical_pulse_max:
  fixes G gq gm gb d t :: real
  assumes "0 < G" and "0 \<le> gq" and "0 \<le> gm" and "gb = (gq
+ gm) / 2" and "d = gq - gm"
    and "G = \<bar>d\<bar> / 4"
  shows "0 < gb"
    and "4 * G\<^sup>2 / gb\<^sup>2 = 4 * G\<^sup>2 / (4 * G\
<^sup>2 + gq * gm)"
    and "0 \<le> t \<Longrightarrow> G\<^sup>2 * t\<^sup>2 * exp
(- gb * t) \<le> G\<^sup>2 * (2 / gb)\<^sup>2 * exp (- gb * (2 /
gb))"
    and "G\<^sup>2 * (2 / gb)\<^sup>2 * exp (- gb * (2 / gb)) = 4
* G\<^sup>2 / (4 * G\<^sup>2 + gq * gm) * exp (- 2)"
```

log_derivative_identity · Isabelle/MemoryAccessFloors/Memory_Access_Regimes.thy, line 40

```
lemma log_derivative_identity:
  fixes G gq gm \<theta> w L :: real
  assumes "0 < \<theta>" and "\<theta> < pi / 2"
    and "w = G + ((gq - gm) / 4) * sin (2 * \<theta>)"
    and "L = (gq * (cos \<theta>)\<^sup>2 + gm * (sin \<theta>)\
<^sup>2) / 2"
    and "w \<noteq> 0"
  shows "2 * cos \<theta> / sin \<theta> - 2 * L / w = (2 * G *
cos \<theta> / sin \<theta> - gm) / w"
```

folk_bound_over · Isabelle/MemoryAccessFloors/Memory_Access_Regimes.thy, line 634

```
theorem folk_bound_over:
  fixes G gq gm gb d \<kappa> t0 :: real
  assumes "0 < G" and "0 \<le> gq" and "0 \<le> gm" and "gb = (gq
+ gm) / 2" and "d = gq - gm"
    and "G < \<bar>d\<bar> / 4" and "\<kappa> = sqrt (d\<^sup>2 /
16 - G\<^sup>2)" and "t0 = pi / (2 * G)"
  shows "exp (- pi * (gq + gm) / (4 * G)) \<le> pulse_over G gb \
<kappa> t0"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Theorem 1, p. 6, doi:10.5281/zenodo.24250501

↳ Used in: 2 (p. 8), 5 (p. 10), 6 (p. 11), Corollary 9 (p. 17), Proposition 17 (p. 36), Proposition 19 (p. 41)

Proof

The resonant pulse. With $g = G$, $D = 0$ the solution is $b(t) = -iG\,e^{-\bar\gamma t/2}\sin(\Omega t)/\Omega$, so $|b|^2 = (G^2/\Omega^2)e^{-\bar\gamma t}\sin^2\Omega t$ (with $\sin(\Omega t)/\Omega$ read as t in the critical case and as $\sinh(\kappa t)/\kappa$ in the over-damped one). Under damping its first maximum is at $\tan\Omega t_* = 2\Omega/\bar\gamma$, where $\sin^2\Omega t_* = 4\Omega^2/(4\Omega^2+\bar\gamma^2)$, and $4\Omega^2+\bar\gamma^2 = 4G^2+\gamma_q\gamma_m$; in the other two cases the same computation with $\tanh\kappa t_* = 2\kappa/\bar\gamma$, or $t_* = 2/\bar\gamma$, gives the same prefactor. This is the right-hand side of (3). For (4) stop instead at $t_0 = \pi/(2G)$. Under damping, with $x = \Omega/G \in (0,1]$, $|b(t_0)|^2 = e^{-\pi(\gamma_q+\gamma_m)/(4G)}\,[\sin(\pi x/2)/x]^2$, and $\sin(\pi x/2) \ge x$ on $[0,1]$ by concavity; otherwise $|b(t_0)|^2 = e^{-\pi(\gamma_q+\gamma_m)/(4G)}(Gt_0)^2\,[\sinh(\kappa t_0)/(\kappa t_0)]^2 \ge e^{-\pi(\gamma_q+\gamma_m)/(4G)}\pi^2/4$.

Polar form. From (1), $\frac{d}{dt}|a|^2 = -\gamma_q|a|^2 - 2u$ and $\frac{d}{dt}|b|^2 = -\gamma_m|b|^2 + 2u$ with $u = -\operatorname{Im}(g\,\bar a\,b)$; the detuning does not appear, and $|u| \le G|a||b|$. Write $|a| = r\cos\theta$, $|b| = r\sin\theta$ with $\theta \in [0,\pi/2]$ and $u = Gs\,r^2\sin\theta\cos\theta$, $s \in [-1,1]$. Then

$$\frac{d}{dt}\ln r = -L(\theta), \qquad L(\theta) = \tfrac12\big(\gamma_q\cos^2\theta + \gamma_m\sin^2\theta\big), \qquad \dot\theta = Gs + \tfrac{\delta}{4}\sin 2\theta.$$

Comparison. Let $\omega(\theta) = G + \frac{\delta}{4}\sin 2\theta$. It is positive on $[0,\pi/2]$ unless $\delta \le -4G$, and then its first zero is $\theta_b = \frac12\arcsin(4G/|\delta|) \le \pi/4$; since $\dot\theta \le \omega(\theta)$ and $\theta(0) = 0$, the angle never reaches θ_b. On the interval J where $\omega>0$ put $R(\theta) = -\int_0^\theta L/\omega$ and $W = \ln r - R(\theta)$. Then

$$\dot W = -L + \frac{L}{\omega}\Big(Gs + \frac{\delta}{4}\sin 2\theta\Big) = -\frac{L\,G\,(1-s)}{\omega} \le 0,$$

and $W(0) = 0$, so $\ln r(t) \le R(\theta(t))$ and $|b(t)|^2 \le F(\theta(t))$ with $F(\theta) = e^{2R(\theta)}\sin^2\theta$. A direct computation gives

$$\frac{d}{d\theta}\ln F = \frac{2G\cot\theta - \gamma_m}{\omega(\theta)},$$

so F increases up to $\theta_* = \arctan(2G/\gamma_m)$ and decreases after it on J. When θ_b exists, $\gamma_m \geq 4G$, so $\theta_* < \pi/4$ and $\omega(\theta_*) = G(4G^2 + \gamma_q\gamma_m)/(4G^2 + {\gamma_m}^2) > 0$: the maximum of F lies inside J. With $s \equiv 1$, which is the resonant pulse, $W \equiv 0$ and $\dot\theta = \omega > 0$, so θ reaches θ_* and the maximum is attained; by the first step its value is (3).

The end points. The polar angle is singular at $\theta = 0, \pi/2$. The functions $|a|^2$, $|b|^2$ and $r^2 > 0$ are absolutely continuous, hence so is $c = \cos 2\theta$, with $\dot c = -\sin 2\theta\,[\frac{\delta}{2}\sin 2\theta + 2Gs]$ and therefore $|\dot c| \leq K\sqrt{1-c^2}$, $K = |\delta|/2 + 2G$. On the open set where $c \neq \pm 1$ the angle $\theta = \frac{1}{2}\arccos c$ is $K/2$-Lipschitz; it is continuous, and off that set it takes only the values 0 and $\pi/2$, between which it can pass only along a stretch of length at least π/K inside the set. So θ is Lipschitz on the whole interval, W is absolutely continuous, and $\dot W \leq 0$ almost everywhere (where $\theta \in \{0, \pi/2\}$, $\dot\theta = 0$ a.e. and $\dot W = -L \leq 0$).

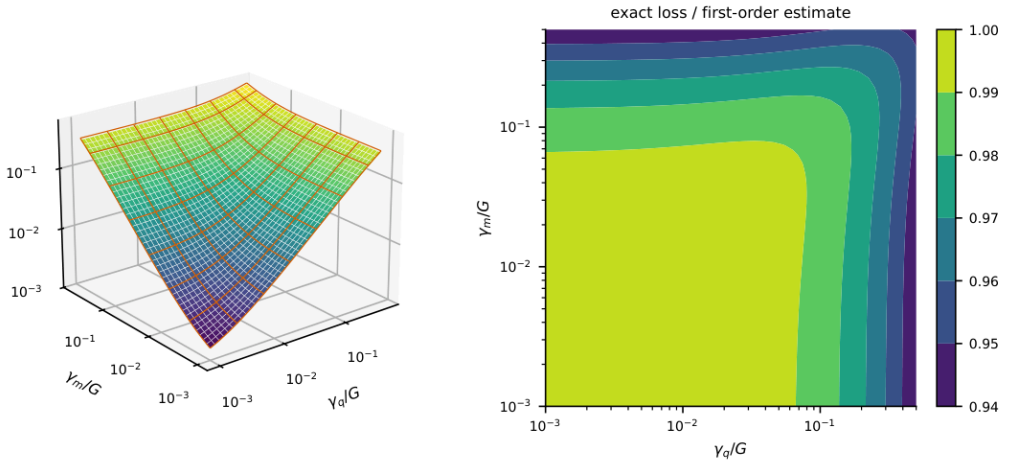
Qubit and memory only lose energy (no dephasing), and the coupling is any pulse of strength at most G, with any phase and detuning. However it is shaped, the population that reaches the memory never exceeds what the resonant pulse delivers when stopped at the right time, and that maximum has a closed form: shaping the pulse cannot beat the resonant swap for the population.

Remark 2. p. 7

The symmetry also follows without the formula: if M is the matrix of the resonant pulse and M' the one with the rates exchanged, then $M' = (\Pi M \Pi)^{\mathsf{T}}$ for the transposition Π, so $(e^{M't})_{21} = (e^{Mt})_{21}$. No relation between G and the rates is needed: the critical and over-damped cases follow from the same comparison, because the maximum of F comes before the zero of ω. (For $G = 1$, $\gamma_q = 5$, $\gamma_m = 0$: $t_* = \frac{4}{3}\ln 2$ and $P_* = 2^{-10/3} \approx 0.0992$.) Figure 1 shows $1 - P_*$ for both rates from $10^{-3}G$ to $G/2$. The Isabelle/HOL formalisation of the bound over all pulses covers the three regimes, on the reduced population equations. The over-damped case uses the eigen-coordinates of the resonant matrix and a logarithmic Lyapunov function, and the critical case its Jordan coordinates (no polar angle); the delicate case $\delta \leq -4G$ appears as the positivity of one coordinate.

↳ Not used by another result of the paper.

Figure 1



Theorem 1 for both rates from $10^{-3}G$ to $G/2$. Left: the smallest loss $1-P_*$ that any pulse can reach (surface, vertical axis $1-P_*$, logarithmic), and the familiar estimate $1-e^{-\pi(\gamma_q+\gamma_m)/(4G)}$ (orange mesh), which lies above it everywhere, as (4) says. The surface is symmetric in the two rates. Right: the ratio of the two; the estimate is within $1\,\%$ while both rates are below about $0.05\,G$ and $5.3\,\%$ off at $\gamma_q=\gamma_m=G/2$.

Remark 3. (stopping rule) p. 7

Since the optimum is at $\tan\theta_*=2G/\gamma_m$, the resonant pulse should be stopped when $|a(t)|/|b(t)|=\gamma_m/(2G)$, in all three regimes. With $\gamma_m=0$ the rule is complete transfer, $a=0$. The rule needs only the ratio of the amplitudes, not t_*.

↳ Not used by another result of the paper.

3 With dephasing the resonant pulse is still the best

Theorem 4. p. 7

MarBprop:costcomparesMarBthm:robustsame-setting Memory_Access_Floors.alignment_lemma Let $\gamma_q,\gamma_m,\phi_q,\phi_m\geq 0$ and $G>0$. For every measurable control with $|g|\leq G$, every detuning and every duration, the solution of (2) with $p_a(0)=1$ satisfies

$$p_b(T)\;\leq\;P_\phi:=\sup_{t\geq 0}p_{\mathrm{res}}(t),$$

where p_{res} is the population transferred by the resonant pulse $g\equiv G$, $D\equiv 0$:

$$p_{\mathrm{res}}(t) = \left(e^{Mt}\right)_{21}, \qquad M = \begin{pmatrix} -\gamma_q & 0 & -2G \\ 0 & -\gamma_m & 2G \\ G & -G & -\Gamma \end{pmatrix} \text{ in the variables } (p_a, p_b, \operatorname{Im}\rho_{ab}). \tag{6}$$

No relation between G and the rates is assumed.

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

`alignment_lemma` · `Isabelle/MemoryAccessFloors/Memory_Access_Floors.thy`, line 93

```
lemma alignment_lemma:
  fixes z :: complex
  assumes "\<forall>\<phi>::real. Re (exp (- (\<i> * of_real \
<phi>)) * z) \<le> Re z"
  shows "Im z = 0" and "Re z = norm z" and "0 \<le> Re z"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Theorem 4, p. 7, doi:10.5281/zenodo.24250501

↳ Used in: 3 (p. 9), Theorem 7 (p. 12), Corollary 9 (p. 17)

Proof

This follows from Theorem 7 of the next section: there a Lyapunov function shows that no pulse of duration at most T transfers more than the best resonant pulse of duration at most T, so that $p_b(T) \le \max_{s \le T} p_{\mathrm{res}}(s) \le P_\phi$ for every finite T. A second proof, by the maximum principle, is in Appendix B.

The same ceiling survives Markovian dephasing of both elements: the resonant swap still bounds the transferred population. Only the alignment step of the proof is machine-checked; the rest is a written proof, audited step by step.

Evidence & scope

- **Literature:** Companion paper: this statement is compared with paper B (Robust qubit–memory transfer against quasi-static frequency noise costs time and exposure), Proposition 3 (silent link, not printed in the paper). `https://karlesmarin.github.io/blueprints/robust-offset/`
- **Literature:** Companion paper: this statement shares the setting of paper B (Robust qubit–memory transfer against quasi-static frequency noise costs time and exposure), Theorem 1 (silent link, not printed in the paper). `https://karlesmarin.github.io/blueprints/robust-offset/`

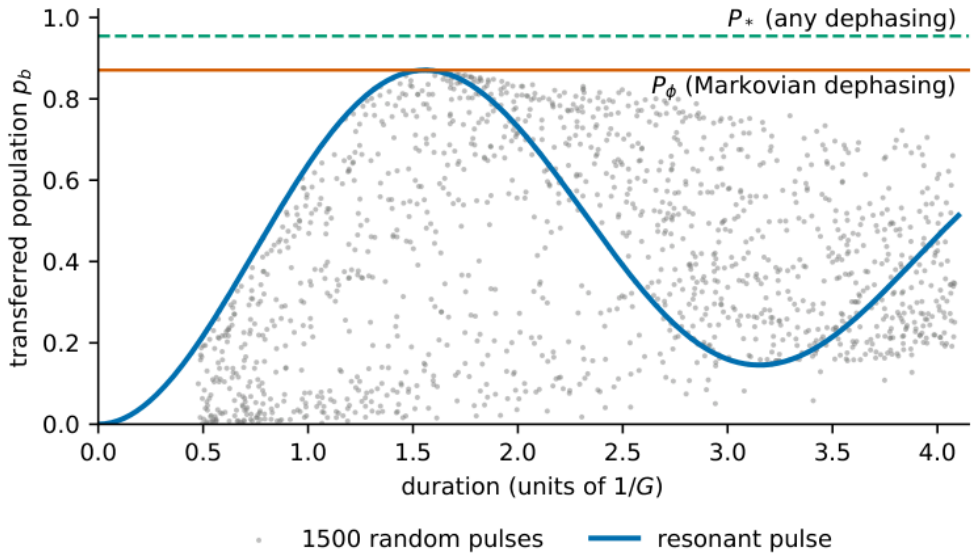
Remark 5. p. 9

Figure 2 tests the theorem against 1500 random pulses in a strongly dephased case. When the two relaxation rates are equal there is a proof in

the style of Theorem 1, with no optimal control theory; it is in Appendix A and also gives the optimality of the resonant pulse at each fixed short duration. When there is no relaxation and $\Phi<4G$ the supremum is explicit: $P_\phi = \frac{1}{2}\left(1 + e^{-\pi\Phi/(2\omega_p)}\right)$ with $\omega_p = \sqrt{4G^2 - \Phi^2/4}$; for $\Phi \geq 4G$ it is $1/2$ and is not attained. For the same reason the infimum in Corollary 9 need not be attained (with $G = 1$, no relaxation and $\phi_q = \phi_m = 10$ the resonant pulse gives $r(t)>5/12$ for every t and $r(t) \to 5/12$; Proposition 11 shows that no pulse does better). With positive relaxation rates the error tends to $1/2$ as the duration grows, so the resonant family attains its own infimum at a finite duration; existence of an optimal control over all pulses (with unbounded detuning) is not claimed.

↳ Not used by another result of the paper.

Figure 2



Theorem 4 in a strongly dephased case ($G = 1$; $T_1 = 20$, $T_2 = 4$ for the qubit and $T_1 = 100$, $T_2 = 40$ for the mode). Each grey dot is a random piecewise-constant pulse with complex coupling $|g| \leq G$ and detuning, placed at its duration. None passes the first maximum $P_\phi = 0.871$ of the resonant pulse; the bound for arbitrary dephasing is $P_* = 0.954$.

Remark 6. p. 9

The bound of Theorem 1 holds for the population under *any* dephasing, Markovian or not: for a state of the one-excitation block the populations obey the first line of (2) and positivity gives $|\rho_{ab}| \leq \sqrt{p_a p_b}$, which is all the proof used. So $p_b \leq P_*(\gamma_q, \gamma_m; G)$ always, and $P_\phi \leq P_*$ is the sharper value when the dephasing is Markovian with known rates.

Cite: Remark 6, p. 9, doi:10.5281/zenodo.24250501

↳ Not used by another result of the paper.

4 Fixed durations, and a proof without the maximum principle

Theorem 7. p. 9

> *Fixed_Duration_Core.H_max, Fixed_Duration_Core.h0_negative, Fixed_Duration_Core.H_aligned_cases, Fixed_Duration_Core.case_i, Fixed_Duration_Core.case_ii, Fixed_Duration_Core.case_iii, Fixed_Duration_Core.case_iv, Fixed_Duration_Core.epsilon_limit, Fixed_Duration_Core.lyapunov_conclusion, Pop_Lyapunov_Step.dini_danskin, Pop_Lyapunov_Step.dini_monotone, Pop_Lyapunov_Step.v_antitone, Pop_Lyapunov_Step.pb_bound relaxation rates and pure dephasing rates non-negative and constant; Markovian dephasing; two states; |g| <= G Let* $\gamma_q, \gamma_m, \phi_q, \phi_m \geq 0$. *Along every solution of (2) with* $|g| \leq G$, *from any positive initial block, the function* $t \mapsto V_{T-t}(R(t))$ *is non-increasing on* $[0, T]$. *Hence, for* $p_a(0) = 1$ *and every* T,
>
> $$p_b(T) \leq \max_{0 \leq s \leq T} p_{\mathrm{res}}(s),$$
>
> *and in particular* $p_b(T) \leq p_{\mathrm{res}}(T)$ *for every* T *up to the first maximum of* p_{res}. *Since the right-hand side is at most* P_ϕ *for every finite* T, *this proves Theorem 4 without the existence of an optimal control.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

H_max · Isabelle/MemoryAccessFloors/Fixed_Duration_Core.thy, line 92

```
theorem H_max:
  assumes "0 \<le> G"
  shows "(\<forall>w. norm w \<le> G \<longrightarrow> Hfun h0 K w
\<le> h0 + 2 * G * norm K)
         \<and> (\<exists>w. norm w \<le> G \<and> Hfun h0 K w = h0
+ 2 * G * norm K)"
```

h0_negative · Isabelle/MemoryAccessFloors/Fixed_Duration_Core.thy, line 121

```
corollary h0_negative:
  assumes "0 < gmin" "gmin \<le> gq" "gmin \<le> gm" "gmin \<le>
Gam" "0 \<le> A" "0 \<le> B" "0 \<le> a" "0 \<le> b"
    and "0 \<le> Re (cnj Eab * rho)" and "0 < ffun A B Eab a b rho"
  shows "h0f gq gm Gam A B Eab a b rho < 0"
```

H_aligned_cases · Isabelle/MemoryAccessFloors/Fixed_Duration_Core.thy, line 152

```
theorem H_aligned_cases:
  assumes "K = of_real c * Eab_pulse z th" and "0 \<le> G"
```

```
  shows "Hfun h0 K (g_th G th) = h0 + 2 * G * norm K
       \<or> Hfun h0 K (g_th G th) = h0 - 2 * G * norm K"
```

case_i · Isabelle/MemoryAccessFloors/Fixed_Duration_Core.thy, line 200

```
theorem case_i:
  assumes "K = of_real c * Eab_pulse z th" and "0 \<le> G" and "h0
< 0"
    and "Hfun h0 K (g_th G th) = 0" and "norm w \<le> G"
  shows "h0 + 2 * G * norm K = 0"
    and "Hfun h0 K w \<le> 0"
```

case_ii · Isabelle/MemoryAccessFloors/Fixed_Duration_Core.thy, line 184

```
theorem case_ii:
  assumes "K = of_real c * Eab_pulse z th" and "0 \<le> G" and "h0
< 0"
    and "0 \<le> Hfun h0 K (g_th G th)" and "norm w \<le> G"
  shows "Hfun h0 K (g_th G th) = h0 + 2 * G * norm K"
    and "Hfun h0 K w \<le> Hfun h0 K (g_th G th)"
    and "Hfun h0 K w - Hfun h0 K (g_th G th) \<le> 0"
```

case_iii · Isabelle/MemoryAccessFloors/Fixed_Duration_Core.thy, line 241

```
theorem case_iii:
  assumes "h0 < 0"
  shows "\<exists>th\<in>{0..2 * pi}. Hfun h0 K (g_th G th) < 0"
```

case_iv · Isabelle/MemoryAccessFloors/Fixed_Duration_Core.thy, line 254

```
theorem case_iv:
  assumes "norm w \<le> G"
  shows "Hfun (h0f gq gm Gam 0 1 0 a b rho) (Kf 0 1 0 a b rho) w \
<le> - gm * b + 2 * G * norm rho"
```

epsilon_limit · Isabelle/MemoryAccessFloors/Fixed_Duration_Core.thy, line 270

```
theorem epsilon_limit:
  fixes v :: "real \<Rightarrow> real" and y :: "real \<Rightarrow>
real \<Rightarrow> real"
  assumes lohi: "\<And>e t. 0 < e \<Longrightarrow> t \<in> {0..T}
\<Longrightarrow> exp (- e * T) * v t \<le> y e t \<and> y e t \
<le> v t"
    and mono: "\<And>e s t. 0 < e \<Longrightarrow> s \<in> {0..T}
\<Longrightarrow> t \<in> {0..T} \<Longrightarrow> s \<le> t \
<Longrightarrow> y e t \<le> y e s"
    and s: "s \<in> {0..T}" and t: "t \<in> {0..T}" and st: "s \
<le> t"
  shows "v t \<le> v s"
```

lyapunov_conclusion · Isabelle/MemoryAccessFloors/Fixed_Duration_Core.thy, line 294

```
theorem lyapunov_conclusion:
  fixes V :: "real \<Rightarrow> 'a \<Rightarrow> real" and R ::
"real \<Rightarrow> 'a" and pb M :: real
  assumes mono: "\<And>s t. 0 \<le> s \<Longrightarrow> s \<le> t \
<Longrightarrow> t \<le> T \<Longrightarrow> V (T - t) (R t) \<le>
V (T - s) (R s)"
    and "0 \<le> T" and fin: "pb = V 0 (R T)" and start: "V T (R 0)
```

```
= M"
  shows "pb \<le> M"
```

dini_danskin · Isabelle/MemoryAccessFloors/Pop_Lyapunov_Step.thy, line 59

```
lemma dini_danskin:
  fixes F :: "'k::metric_space \<Rightarrow> real \<Rightarrow>
real" and H :: "'k \<Rightarrow> real \<Rightarrow> real" and v ::
"real \<Rightarrow> real"
  assumes cK: "compact K"
    and Fc: "continuous_on (K \<times> {a..b}) (\<lambda>p. F (fst
p) (snd p))"
    and Hc: "continuous_on (K \<times> {a..b}) (\<lambda>p. H (fst
p) (snd p))"
    and der: "\<And>k x. k \<in> K \<Longrightarrow> x \<in> {a..b}
\<Longrightarrow> (F k has_real_derivative H k x) (at x within
{a..b})"
    and vmax: "\<And>t. t \<in> {a..b} \<Longrightarrow> (\
<forall>k\<in>K. F k t \<le> v t) \<and> (\<exists>k\<in>K. F k t =
v t)"
    and t: "t \<in> {a..b}"
    and c: "\<And>k. k \<in> K \<Longrightarrow> F k t = v t \
<Longrightarrow> H k t \<le> c"
    and eps: "0 < eps"
  shows "\<exists>delta>0. \<forall>s. t < s \<longrightarrow> s <
t + delta \<longrightarrow> s \<le> b \<longrightarrow> v s \<le> v
t + (c + eps) * (s - t)"
```

dini_monotone · Isabelle/MemoryAccessFloors/Pop_Lyapunov_Step.thy, line 233

```
lemma dini_monotone:
  fixes v :: "real \<Rightarrow> real"
  assumes vc: "continuous_on {a..b} v"
    and dini: "\<And>t eps. a \<le> t \<Longrightarrow> t < b \
<Longrightarrow> 0 < eps \<Longrightarrow>
       \<exists>delta>0. \<forall>s. t < s \<longrightarrow> s < t
+ delta \<longrightarrow> s \<le> b \<longrightarrow> v s \<le> v t
+ eps * (s - t)"
    and x: "a \<le> x" and xy: "x \<le> y" and y: "y \<le> b"
  shows "v y \<le> v x"
```

v_antitone · Isabelle/MemoryAccessFloors/Pop_Lyapunov_Step.thy, line 780

```
theorem v_antitone:
  assumes "0 \<le> s" and "s \<le> t" and "t \<le> T"
  shows "vf t \<le> vf s"
```

pb_bound · Isabelle/MemoryAccessFloors/Pop_Lyapunov_Step.thy, line 786

```
theorem pb_bound: "b T \<le> vf 0"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Theorem 7, p. 9, doi:10.5281/zenodo.24250501

↳ Used in: Corollary 8 (p. 16), Proposition 11 (p. 20)

Proof

Assume first $\gamma_{\min}>0$ and $G>0$; then $0 \le E_{\tau,\vartheta} \le e^{-\gamma_{\min}\tau}$, the supremum is attained, and $V_h(R)>0$ for $R \neq 0$ and $h>0$ (for $R=0$ there is nothing to prove, and the end point $t=T$ follows by continuity). To have a fixed set of indices write $\tau = uh$ with $h = T-t$ and $u \in [0,1]$: $V_h(R) = \max_{(u,\vartheta)\in[0,1]\times S^1} F_{u,\vartheta}(h,R)$, $F_{u,\vartheta}(h,R) = \mathrm{Tr}(E_{uh,\vartheta}R)$. The index set is compact and the $F_{u,\vartheta}$ have uniformly bounded derivatives in (h,R), so $v(t) = V_{T-t}(R(t))$ is Lipschitz and, at almost every t, its derivative is bounded by the largest derivative along the trajectory of an *active* functional $f = F_{u,\vartheta}$, one that attains the maximum (Danskin). That derivative is $H_{g(t)} - u\,H_{g_\vartheta}$ in the notation that follows, which gives the cases below: u interior, $u=1$ and $u=0$. For an active functional and a control value w write
$H_w = \mathrm{Tr}(E\,\mathcal{L}_w R) = h_0 + 2\,\mathrm{Im}(\bar{w}K)$, with

$$h_0 = -\gamma_q A a - \gamma_m B b - 2\Gamma\ \mathrm{Re}\big(\overline{E_{ab}}\rho\big), \qquad K = [R,E]_{ab} = E_{ab}(a -$$

where now A, B, E_{ab} are the entries of E; so $\max_{|w|\le G} H_w = h_0 + 2G|K|$. Since ϑ is free in the maximum, $\overline{E_{ab}}\rho \ge 0$, and therefore $h_0 \le -\gamma_{\min} f < 0$. Also $\partial_\tau f = H_{g_\vartheta}$ with $g_\vartheta = Ge^{i\vartheta}$, the coupling that generates E.

(i) $0<\tau<T-t$ and $z(\tau)\neq 0$. Alignment makes K a real multiple of $E_{ab} = ize^{i\vartheta}$, so $H_{g_\vartheta} = h_0 \pm 2G|K|$. Stationarity in τ gives $H_{g_\vartheta}=0$, and $h_0<0$ forces the plus sign: $\max_{|w|\le G} H_w = h_0 + 2G|K| = 0$. The derivative of f along the trajectory is $H_{g(t)} \le 0$.

(ii) $\tau = T-t>0$ and $z(\tau)\neq 0$. The admissible τ shrink at unit rate, so the derivative is $H_{g(t)} - H_{g_\vartheta}$; one-sided optimality gives $H_{g_\vartheta} \ge 0 > h_0$, the plus sign again, so $H_{g_\vartheta} = \max_w H_w \ge H_{g(t)}$.

(iii) $\tau>0$ and $z(\tau)=0$ cannot be active: every phase would be, each with $H_{g_\vartheta}\ge 0$, while their average over ϑ is $h_0<0$.

(iv) $\tau = 0$: E is the projector on b, $f=b$, and optimality to the right, maximised over the phase, reads $-\gamma_m b + 2G|\rho| \le 0$; but $H_{g(t)} \le -\gamma_m b + 2G|\rho|$.

So $\dot v \le 0$ almost everywhere. Then $p_b(T) = V_0(R(T)) \le V_T(R(0))$, which for $R(0)$ the projector on a is $\max_{s\le T} p_{\mathrm{res}}(s)$. Vanishing rates: add $\varepsilon>0$ to both relaxation rates. Every solution and every $E_{\tau,\vartheta}$ is multiplied by $e^{-\varepsilon t}$, respectively $e^{-\varepsilon\tau}$, so the function of the shifted system satisfies $e^{-\varepsilon T}V_h(R(t)) \le V_h^{(\varepsilon)}\big(e^{-\varepsilon t}R(t)\big) \le V_h(R(t))$ for $t\le T$; it is non-increasing in t by what was proved, and letting $\varepsilon\to 0$ at fixed T so is $V_{T-t}(R(t))$.

If the duration is imposed, the best pulse is the resonant one followed by a wait in the element that loses less. The proof builds a function of the state that can only decrease along any admissible pulse; its algebraic core is formalised in Isabelle, the derivative of a maximum over a family (Danskin) is written.

Corollary 8. p. 10

> *Robust_Cost_Branches.fixed_duration_envelope, Robust_Cost_Branches.envelope_ge_resonant Let* $\mu = \min(\gamma_q, \gamma_m)$ *and* $\nu = \min(\eta_q, \eta_m)$*. For every prescribed duration* T *the largest transferred population and the largest transferred coherence over all pulses are*
>
> $$B_p(T) = e^{-\mu T} \max_{0 \le s \le T} e^{\mu s} p_{\mathrm{res}}(s), \qquad B_c(T) = e^{-\nu T} \max_{0 \le s \le T} e^{\nu s} c_{\mathrm{res}}(s).$$
>
> *Each is attained by a resonant pulse of the maximising duration* s *and a wait of* $T - s$ *in the element with the smaller rate* γ *(for* B_c*: the smaller rate* η*), before the pulse if that element is the qubit and after it if it is the mode.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

`fixed_duration_envelope` · `Isabelle/MemoryAccessFloors/Robust_Cost_Branches.thy`, line 69

```
theorem fixed_duration_envelope:
  fixes mu T M p :: real
  assumes "exp (mu * T) * p \<le> M"
  shows "p \<le> exp (- (mu * T)) * M"
```

`envelope_ge_resonant` · `Isabelle/MemoryAccessFloors/Robust_Cost_Branches.thy`, line 82

```
theorem envelope_ge_resonant:
  fixes mu T :: real and pres :: "real \<Rightarrow> real"
  assumes "0 \<le> T" "\<forall>s \<in> {0..T}. exp (mu * s) * pres
s \<le> Mx"
  shows "pres T \<le> exp (- (mu * T)) * Mx"
```

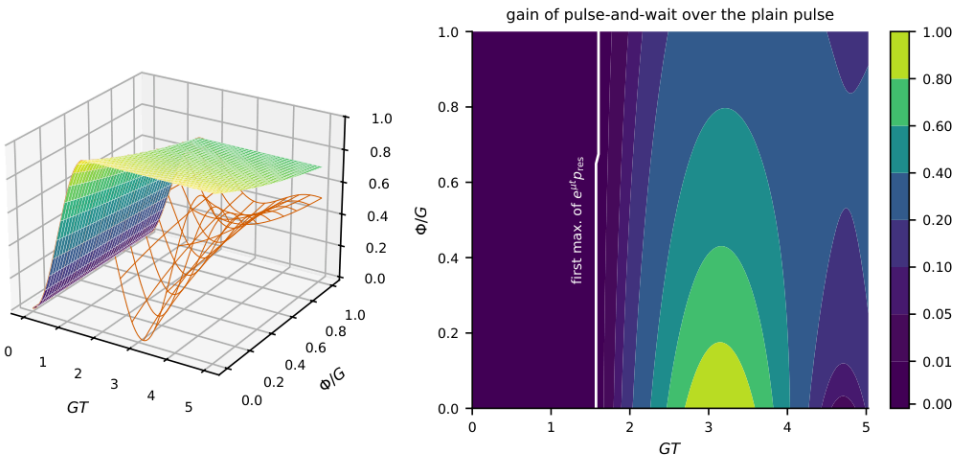
Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Corollary 8, p. 10, doi:10.5281/zenodo.24250501

↳ Used in: Corollary 10 (p. 19), Corollary 13 (p. 27), Proposition 14 (p. 29)

Proof

> Subtracting μ from both relaxation rates multiplies every population by $e^{\mu t}$ and keeps the rates non-negative; apply Theorem 7 to the shifted system. The coherence obeys the amplitude equations, that is, the population equations without dephasing and with rates $2\eta_q, 2\eta_m$ for c^2. The protocol described attains the bound by direct computation.

Figure 3



Corollary 8 as a surface, for $G = 1$, $\gamma_q = 0.02$, $\gamma_m = 0.002$ and dephasing Φ up to G. Left: the largest population any pulse can transfer in a prescribed time T, $B_p(T)$ (surface), and what the plain resonant pulse of that duration transfers (orange mesh). Right: their difference, the gain of stopping the resonant pulse at its best moment and waiting in the mode. It is exactly zero up to the first maximum τ_p of the weighted curve $e^{\mu t}p_{\mathrm{res}}(t)$, $\mu = \min(\gamma_q, \gamma_m)$ (white line), and reaches 0.98 near $GT = \pi$ without dephasing, where the plain pulse has returned the excitation to the qubit.

5 The write channel

Corollary 9. Isabelle **(identities)** p. 11

MarBprop:slow2used-byMarBprop:noiselossused-by Memory_Access_Floors.write_channel_infidelity, Memory_Access_Floors.worst_case_case_one, Memory_Access_Floors.worst_case_case_two Let $c_* = \sqrt{P_*(2\eta_q, 2\eta_m; G)}$ *(assuming* $G > |\eta_q - \eta_m|/2$*) and let* r_{opt} *be the infimum of* r *over all pulses and durations. Then*

$$r_{\mathrm{sep}} := \frac{1 - P_\phi}{6} + \frac{1 - c_*}{3} \;\le\; r_{\mathrm{opt}} \;\le\; r_{\mathrm{res}} := \inf_{t \ge 0} \left[\frac{1 - p_{\mathrm{res}}(t)}{6} + \frac{1 - c_1}{3} \right.$$

where $c_{\mathrm{res}}(t) = G\, e^{-(\eta_q+\eta_m)t/2}\, |\sin \omega_c t| / \omega_c$, $\omega_c^2 = G^2 - (\eta_q - \eta_m)^2/4$. *The same pulses are optimal for* F *and for* F_e.

Isabelle · ours **show**

`write_channel_infidelity` · `Isabelle/MemoryAccessFloors/Memory_Access_Floors.thy`, line 16

```
lemma write_channel_infidelity:
  fixes p c :: real
  defines "Fe \<equiv> (1 + p + 2 * c) / 4"
  defines "F \<equiv> (2 * Fe + 1) / 3"
```

```
  shows "1 - F = (1 - p) / 6 + (1 - c) / 3"
    and "1 - Fe = (3 / 2) * (1 - F)"
```

worst_case_case_one · Isabelle/MemoryAccessFloors/Memory_Access_Floors.thy, line 35

```
lemma worst_case_case_one:
  fixes p c x :: real
  assumes "0 \<le> p" and "p \<le> 1" and "0 \<le> c" and "c\
<^sup>2 \<le> p"
    and "3 * p - 2 * c \<le> 1" and "0 \<le> x" and "x \<le> 1"
  shows "p \<le> wc_poly p c x" and "wc_poly p c 1 = p"
```

worst_case_case_two · Isabelle/MemoryAccessFloors/Memory_Access_Floors.thy, line 61

```
lemma worst_case_case_two:
  fixes p c x :: real
  assumes "0 \<le> p" and "p \<le> 1" and "0 \<le> c" and "c\
<^sup>2 \<le> p"
    and "3 * p - 2 * c > 1"
  defines "q \<equiv> p - c" and "b \<equiv> 1 + p - 2 * c"
  shows "c < p"
    and "0 \<le> b / (4 * q)" and "b / (4 * q) \<le> 1"
    and "wc_poly p c x = 1 - b\<^sup>2 / (8 * q) + 2 * q * (x - b
/ (4 * q))\<^sup>2"
    and "0 \<le> x \<Longrightarrow> x \<le> 1 \<Longrightarrow> 1
- b\<^sup>2 / (8 * q) \<le> wc_poly p c x"
    and "wc_poly p c (b / (4 * q)) = 1 - b\<^sup>2 / (8 * q)"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Corollary 9, p. 11, doi:10.5281/zenodo.24250501

↳ Used in: Proposition 15 (p. 32)

Proof

> $p \le P_\phi$ by Theorem 4; $c^2 \le P_*(2\eta_q, 2\eta_m; G)$ by Theorem 1 applied to the coherence equations; the upper bound is the value of one admissible pulse.

For a memory what matters is the channel, not only the population: its average infidelity is $r = (1-p)/6 + (1-c)/3$, with p the transferred population and c the transferred coherence. The population and coherence bounds then become lower bounds on the best write error r_{opt}, between the separate bound r_{sep} and the resonant value r_{res}.

Evidence & scope

- **Literature:** Companion paper: this statement is used in paper B (Robust qubit–memory transfer against quasi-static frequency noise costs time and exposure), Proposition 5 (silent link, not printed in the paper). https://karlesmarin.github.io/blueprints/robust-offset/
- **Literature:** Companion paper: this statement is used in paper B (Robust qubit–memory transfer against quasi-static frequency noise costs time and exposure), Proposition 8 (silent link, not printed in the paper). https://karlesmarin.github.io/blueprints/robust-offset/

Corollary 10. (numbers) p. 12

> *Memory_Time_Bound.time_bound, Memory_Time_Bound.time_equality, Memory_Time_Bound.time_limit With B_p, B_c of Corollary 8,*
>
> $$r_{\text{sep}} \;\le\; r_{\text{time}} := \frac{1}{2} - \frac{1}{6}\sup_{T\ge 0}\big[B_p(T) + 2B_c(T)\big] \;\le\; r_{\text{opt}} \;\le\; r_{\text{res}}.$$
>
> *If the supremum is attained at a duration T_0 at which $B_p(T_0) = p_{\text{res}}(T_0)$ and $B_c(T_0) = c_{\text{res}}(T_0)$, then $r_{\text{opt}} = r_{\text{res}}$: the resonant pulse is exactly optimal for the average fidelity. The same holds if the supremum is approached along durations T_n with $[B_p(T_n) - p_{\text{res}}(T_n)] + 2[B_c(T_n) - c_{\text{res}}(T_n)] \to 0$.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is certified by computation

`time_bound` · `Isabelle/MemoryAccessFloors/Memory_Time_Bound.thy`, line 44

```
lemma time_bound:
  fixes Bp Bc :: "real \<Rightarrow> real" and dur pp cc :: "'a \
<Rightarrow> real"
  assumes bdd: "bdd_above {Bp T + 2 * Bc T | T. 0 \<le> T}"
    and pulses: "\<And>x. 0 \<le> dur x \<and> pp x \<le> Bp (dur
x) \<and> cc x \<le> Bc (dur x)"
  shows "r_time Bp Bc \<le> infidelity (pp x) (cc x)"
```

`time_equality` · `Isabelle/MemoryAccessFloors/Memory_Time_Bound.thy`, line 62

```
lemma time_equality:
  fixes Bp Bc pres cres :: "real \<Rightarrow> real"
  assumes bdd: "bdd_above {Bp T + 2 * Bc T | T. 0 \<le> T}"
    and T0: "0 \<le> T0"
    and att: "Ssup Bp Bc = Bp T0 + 2 * Bc T0"
    and p0: "pres T0 = Bp T0" and c0: "cres T0 = Bc T0"
  shows "infidelity (pres T0) (cres T0) = r_time Bp Bc"
```

`time_limit` · `Isabelle/MemoryAccessFloors/Memory_Time_Bound.thy`, line 72

```
lemma time_limit:
  fixes Bp Bc pres cres :: "real \<Rightarrow> real" and T :: "nat
\<Rightarrow> real"
  assumes gap: "(\<lambda>n. (Bp (T n) - pres (T n)) + 2 * (Bc (T
n) - cres (T n))) \<longlonglongrightarrow> 0"
    and approach: "(\<lambda>n. Bp (T n) + 2 * Bc (T n)) \
<longlonglongrightarrow> Ssup Bp Bc"
  shows "(\<lambda>n. infidelity (pres (T n)) (cres (T n))) \
<longlonglongrightarrow> r_time Bp Bc"
```

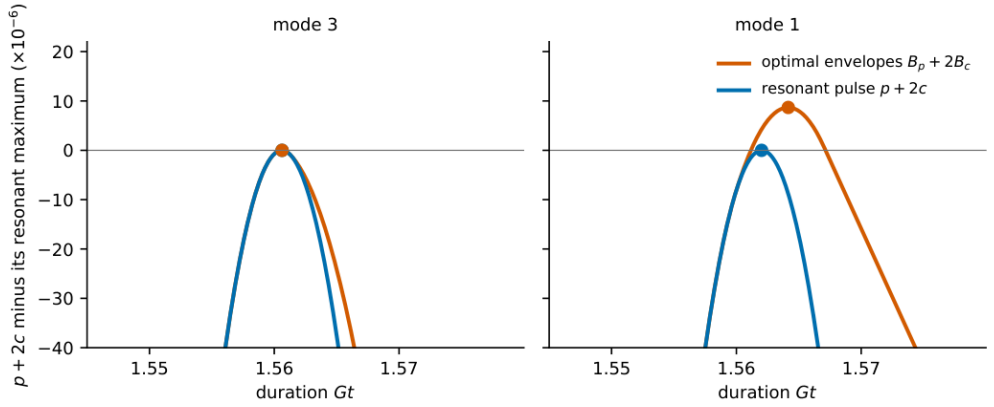
Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is certified by computation · **Cite:** Corollary 10, p. 12, doi:10.5281/zenodo.24250501

↳ Used in: Proposition 16 (p. 34)

Proof

A pulse of duration T has $p \le B_p(T)$ and $c \le B_c(T)$, so $r \ge \frac{1}{2} - \frac{1}{6}[B_p(T) + 2B_c(T)]$. In the case of equality the resonant pulse stopped at T_0 (or at T_n) attains the bound (in the limit).

Figure 4



Corollary 10 near the optimum, for two modes of [47] . Blue: $p + 2c$ for the resonant pulse stopped at duration t. Orange: the same quantity for the exact optimal population and the exact optimal coherence at that duration, which no pulse exceeds. For mode 3 the orange curve peaks where it still coincides with the blue one, so the resonant pulse is optimal. For mode 1 it peaks later, $8.7 \cdot 10^{-6}$ higher: that is the $1.45 \cdot 10^{-6}$ of infidelity not yet decided.

Proposition 11. p. 13

Oscillator_Comparison.strong_comparison, Oscillator_Comparison.zz_lower, Oscillator_Comparison.coherence_integral_bound, Oscillator_Comparison.strong_conclusion, Oscillator_Comparison.strong_unequal_chain Memory_Access_Quasistatic.strong_z_ge, Memory_Access_Quasistatic.strong_z_antimono, Memory_Access_Quasistatic.strong_gap, Memory_Access_Quasistatic.strong_limit Without relaxation and with dephasing rates $\phi_q, \phi_m \ge 4G$, $r_{\mathrm{opt}} = r_{\mathrm{res}} = 5/12$, and no pulse of finite duration attains it. For equal rates $\phi_q = \phi_m = \varphi$ there is a unique φ_*, certified in $2.0941221534\,G < \varphi_* < 2.0941221542\,G$ (equivalently $\phi_q + \phi_m \approx 4.18824\,G$): for $\varphi > \varphi_*$ the same holds; at $\varphi = \varphi_*$, $r_{\mathrm{opt}} = 5/12$ and the resonant pulse attains it at one finite duration; for $\varphi < \varphi_*$ the resonant pulse itself reaches $r < 5/12$ at a finite duration.

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

`strong_comparison` · `Isabelle/MemoryAccessFloors/Oscillator_Comparison.thy`, line 233

```
theorem strong_comparison:
  fixes G m phibar T :: real
  assumes "0 < G" and "2 * G < m" and "m \<le> phibar" and "0 \<le>
T"
  shows "exp (- m * T) * (1 + m * T) \<le> zz G m T" and "zz G m T
\<le> zz G phibar T"
```

zz_lower · Isabelle/MemoryAccessFloors/Oscillator_Comparison.thy, line 142

```
theorem zz_lower:
  fixes G phi T :: real
  assumes "0 < G" and "2 * G < phi" and "0 \<le> T"
  shows "exp (- phi * T) * (1 + phi * T) \<le> zz G phi T"
```

coherence_integral_bound · Isabelle/MemoryAccessFloors/Oscillator_Comparison.thy, line 241

```
theorem coherence_integral_bound:
  fixes G m T c :: real and v :: "real \<Rightarrow> real"
  assumes "0 \<le> G" and "0 \<le> T" and "continuous_on {0..T} v"
    and "\<forall>s\<in>{0..T}. \<bar>v s\<bar> \<le> exp (- m *
s)"
    and "c \<le> integral {0..T} (\<lambda>s. G * exp (- m * (T -
s)) * \<bar>v s\<bar>)"
  shows "c \<le> G * T * exp (- m * T)"
```

strong_conclusion · Isabelle/MemoryAccessFloors/Oscillator_Comparison.thy, line 271

```
theorem strong_conclusion:
  fixes G m T p c Z :: real
  assumes "0 < G" and "4 * G \<le> m" and "0 \<le> T"
    and "p \<le> (1 - Z) / 2" and "exp (- m * T) * (1 + m * T) \
<le> Z"
    and "c \<le> G * T * exp (- m * T)"
  shows "p + 2 * c \<le> 1 / 2 - exp (- m * T) * (1 + m * T - 4 * G
* T) / 2"
    and "p + 2 * c < 1 / 2"
```

strong_unequal_chain · Isabelle/MemoryAccessFloors/Oscillator_Comparison.thy, line 298

```
theorem strong_unequal_chain:
  fixes G m phibar T p c :: real
  assumes "0 < G" and "4 * G \<le> m" and "m \<le> phibar" and "0 \
<le> T"
    and "p \<le> (1 - zz G phibar T) / 2" and "c \<le> G * T * exp
(- m * T)"
  shows "p + 2 * c < 1 / 2"
```

strong_z_ge · Isabelle/MemoryAccessFloors/Memory_Access_Quasistatic.thy, line 194

```
theorem strong_z_ge:
  fixes G phi T :: real
  assumes "0 < G" and "4 * G \<le> phi" and "0 \<le> T"
  shows "exp (- phi * T) * (1 + phi * T) \<le> strong_z G phi T"
```

strong_z_antimono · Isabelle/MemoryAccessFloors/Memory_Access_Quasistatic.thy, line 238

```
theorem strong_z_antimono:
  fixes G phi T1 T2 :: real
  assumes "0 < G" and "4 * G \<le> phi" and "0 \<le> T1" and "T1 \
```

```
<le> T2"
  shows "strong_z G phi T2 \<le> strong_z G phi T1"
```

strong_gap · Isabelle/MemoryAccessFloors/Memory_Access_Quasistatic.thy, line 254

```
theorem strong_gap:
  fixes G phi T s :: real
  assumes "0 < G" and "4 * G \<le> phi" and "0 < T" and "0 \<le> s"
and "s \<le> T"
  shows "(1 - strong_z G phi T) / 2 + 2 * (exp (- phi * T) * \
<bar>sin (G * s)\<bar>) < 1 / 2"
```

strong_limit · Isabelle/MemoryAccessFloors/Memory_Access_Quasistatic.thy, line 311

```
theorem strong_limit:
  fixes G phi :: real
  assumes "0 < G" and "4 * G \<le> phi"
  shows "((\<lambda>T. (1 - strong_z G phi T) / 2) \
<longlongrightarrow> 1 / 2) at_top"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Proposition 11, p. 13, doi:10.5281/zenodo.24250501

↳ Not used by another result of the paper.

Proof

Equal rates first. With $\kappa=\sqrt{\varphi^2-4G^2}$ the resonant pulse has $p_{\mathrm{res}}=(1-z)/2$, $z(T)=e^{-\varphi T}[\cosh\kappa T+(\varphi/\kappa)\sinh\kappa T]$, and $\dot z=-4G^2e^{-\varphi T}\sinh(\kappa T)/\kappa<0$, so $B_p=p_{\mathrm{res}}$. Also $z(T)\ge e^{-\varphi T}(1+\varphi T)$ and $B_c(T)=e^{-\varphi T}\max_{s\le T}|\sin Gs|\le GT\,e^{-\varphi T}$, hence $z>4B_c$ and $B_p+2B_c=\frac12-\frac12(z-4B_c)<\frac12$ for every finite $T>0$, while both B_p+2B_c and $p_{\mathrm{res}}+2c_{\mathrm{res}}$ tend to $\frac12$. So $r_{\mathrm{time}}=\frac12-\frac{1}{12}=r_{\mathrm{res}}$.

Unequal rates. Let $m=\min(\phi_q,\phi_m)\ge 4G$ and $\bar\varphi=(\phi_q+\phi_m)/2$. The coherence ρ_{ab} decays at $\Gamma=2\bar\varphi$, so the resonant pulse has $p_{\mathrm{res}}=(1-z_{\bar\varphi})/2$ with z_φ the function above, which solves $\ddot z+2\varphi\dot z+4G^2z=0$, $z(0)=1$, $\dot z(0)=0$. The difference $d=z_{\bar\varphi}-z_m$ solves the same equation with rate $\bar\varphi$, zero data and forcing $-2(\bar\varphi-m)\dot z_m\ge 0$; the Green function of an over-damped oscillator is non-negative, so $z_{\bar\varphi}\ge z_m\ge e^{-mT}(1+mT)$. The resonant population increases, so Theorem 7 gives $p\le p_{\mathrm{res}}(T)$ for every pulse. The coherences with the vacuum, $\mathbf v=(v_a,v_b)$ with $\mathbf v(0)=(1,0)$, decay at ϕ_q and ϕ_m, so $|\mathbf v(t)|\le e^{-mt}$, and variation of constants gives $c=|v_b(T)|\le\int_0^T G\,e^{-m(T-s)}e^{-ms}\,ds=GT\,e^{-mT}$. Hence $p+2c\le\frac12-\frac12e^{-mT}(1+mT-4GT)<\frac12$ at every finite T, while the resonant family tends to $\frac12$. The norm inequality holds almost everywhere for absolutely continuous solutions, so Gronwall's inequality and the variation-of-constants formula apply to every bounded measurable coupling and every locally integrable detuning of the model. (Checked step by step and on 300 random controls for each

of five rate pairs, four unequal and the equal-rate boundary case, `desfase_desigual.py`; with $\phi = G$ the conclusion fails, as it should. The condition $4G$ is not sharp: the exact threshold for equal rates and the criteria and Proposition 12 for unequal ones follow.)

Proposition 12. (the threshold for very unequal rates) p. 15

> *Memory_Floors_Pieces.omega_le_kappa_iff, Memory_Floors_Pieces.vertex_ineq, Memory_Floors_Pieces.delta_sq_over_omega_lt_a, Memory_Floors_Pieces.final_bound, Storage_Function.reduce_yz, Storage_Function.reduce_uh, Storage_Function.cone_yz, Storage_Function.cone_uh, Storage_Function.log_step, Metzler_Cone.ratio_nonincreasing, Storage_Function.unequal_storage_bound, Unequal_Dephasing_Threshold.threshold_3G2, Unequal_Dephasing_Threshold.eventually_four_u_gt_w, Unequal_Dephasing_Threshold.p_plus_2c_iff, Unequal_Dephasing_Threshold.resonant_below_five_twelfths* No relaxation; pulses of fixed phase and no detuning, $0 \le g(t) \le G$; $\delta = |\phi_q - \phi_m|/2 \ge \sqrt{5}\,G$ *and* $\Omega = \sqrt{\delta^2 - G^2}$. *Then* $r_{\mathrm{opt}} = 5/12$, *not attained, if and only if* $\phi_q\phi_m \ge 3G^2$. *More precisely:*
>
> 1. *if* $\phi_q\phi_m \ge 3G^2$, *every pulse has* $c(T) \le \frac{G}{2\Omega}\big(1 - 2p(T)\big)$ *and* $p(T) < \frac{1}{2}$ *at every duration, hence* $p + 2c < \frac{1}{2}$;
> 2. *if* $\phi_q\phi_m < 3G^2$, *the resonant pulse has* $p + 2c > \frac{1}{2}$ *at all long enough durations.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

`omega_le_kappa_iff` · `Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy`, line 229

```
theorem omega_le_kappa_iff:
  fixes delta a P :: real
  assumes "1 < delta" and "2 < a" and "a\<^sup>2 = delta\<^sup>2 +
P"
  shows "sqrt (delta\<^sup>2 - 1) \<le> sqrt (a\<^sup>2 - 4) \
<longleftrightarrow> 3 \<le> P"
```

`vertex_ineq` · `Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy`, line 259

```
theorem vertex_ineq:
  fixes delta a P :: real
  assumes "1 < delta" and "3 \<le> P" and "a\<^sup>2 = delta\
<^sup>2 + P"
  shows "2 * delta\<^sup>2 - 3 \<le> a\<^sup>2 * (sqrt (delta\
<^sup>2 - 1))\<^sup>2 - delta ^ 4"
```

`delta_sq_over_omega_lt_a` · `Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy`, line 272

```
theorem delta_sq_over_omega_lt_a:
  fixes delta a P :: real
  assumes "1 < delta" and "3 \<le> P" and "a\<^sup>2 = delta\
<^sup>2 + P" and "0 < a" and "3 / 2 < delta\<^sup>2"
  shows "delta\<^sup>2 / sqrt (delta\<^sup>2 - 1) < a"
```

final_bound · Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy, line 291

```
theorem final_bound:
  fixes u w h Om :: real
  assumes "0 < u" and "0 < w" and "0 < h" and "0 < Om" and "h + 2 *
Om * u \<le> w"
  shows "u < w / (2 * Om)"
    and "2 \<le> Om \<Longrightarrow> 2 * u - w / 2 < 0"
```

reduce_yz · Isabelle/MemoryAccessFloors/Storage_Function.thy, line 22

```
lemma reduce_yz:
  fixes w y g :: "real \<Rightarrow> real" and a th t :: real
  assumes thp: "0 < th" and rel: "4 * th\<^sup>2 - 2 * a * th + 1 =
0"
    and dw: "(w has_real_derivative - 4 * g t * y t) (at t)"
    and dy: "(y has_real_derivative g t * w t - 2 * a * y t) (at
t)"
  shows "(y has_real_derivative (g t / th - 2 * a) * y t + g t / th
* (th * w t - y t)) (at t)"
    and "((\<lambda>s. th * w s - y s) has_real_derivative
          (1 - g t) * 2 * a * y t + (- (g t / th)) * (th * w t - y
t)) (at t)"
```

reduce_uh · Isabelle/MemoryAccessFloors/Storage_Function.thy, line 61

```
lemma reduce_uh:
  fixes va u g :: "real \<Rightarrow> real" and q m chi t :: real
  assumes chi: "chi\<^sup>2 + (q - m) * chi + 1 = 0"
    and dva: "(va has_real_derivative - q * va t - g t * u t) (at
t)"
    and du: "(u has_real_derivative g t * va t - m * u t) (at t)"
  shows "(u has_real_derivative (g t * chi - m) * u t + g t * (va t
- chi * u t)) (at t)"
    and "((\<lambda>s. va s - chi * u s) has_real_derivative
          (1 - g t) * (1 + chi\<^sup>2) * u t + (- (q + g t * chi))
* (va t - chi * u t)) (at t)"
```

cone_yz · Isabelle/MemoryAccessFloors/Storage_Function.thy, line 87

```
theorem cone_yz:
  fixes g y z :: "real \<Rightarrow> real" and th a T :: real
  assumes T0: "0 \<le> T" and gc: "continuous_on {0..T} g" and g01:
"\<forall>t\<in>{0..T}. 0 \<le> g t \<and> g t \<le> 1"
    and thp: "0 < th" and a0: "0 \<le> a"
    and dy: "\<forall>t\<in>{0..T}. (y has_real_derivative (g t /
th - 2 * a) * y t + g t / th * z t) (at t)"
    and dz: "\<forall>t\<in>{0..T}. (z has_real_derivative (1 - g
t) * 2 * a * y t + (- (g t / th)) * z t) (at t)"
    and y0: "0 \<le> y 0" and z0: "0 \<le> z 0"
  shows "\<forall>t\<in>{0..T}. 0 \<le> y t \<and> 0 \<le> z t"
```

cone_uh · Isabelle/MemoryAccessFloors/Storage_Function.thy, line 113

```
theorem cone_uh:
  fixes g u h :: "real \<Rightarrow> real" and q m chi T :: real
  assumes T0: "0 \<le> T" and gc: "continuous_on {0..T} g" and g01:
"\<forall>t\<in>{0..T}. 0 \<le> g t \<and> g t \<le> 1"
    and du: "\<forall>t\<in>{0..T}. (u has_real_derivative (g t *
chi - m) * u t + g t * h t) (at t)"
```

```
    and dh: "\<forall>t\<in>{0..T}. (h has_real_derivative (1 - g
t) * (1 + chi\<^sup>2) * u t + (- (q + g t * chi)) * h t) (at t)"
    and u0: "0 \<le> u 0" and h0: "0 \<le> h 0"
  shows "\<forall>t\<in>{0..T}. 0 \<le> u t \<and> 0 \<le> h t"
```

log_step · Isabelle/MemoryAccessFloors/Storage_Function.thy, line 173

```
lemma log_step:
  fixes a d Om kap g va u w y :: real
  assumes a0: "0 < a" and Om0: "0 < Om" and OmE: "Om\<^sup>2 = d\
<^sup>2 - 1" and k0: "0 < kap" and kE: "kap\<^sup>2 = a\<^sup>2 -
4"
    and a3: "d\<^sup>2 + 3 \<le> a\<^sup>2" and d5: "5 \<le> d\
<^sup>2"
    and g0: "0 \<le> g" and g1: "g \<le> 1"
    and h0: "0 \<le> va - (- d + Om) * u" and u0: "0 \<le> u" and
y0: "0 \<le> y"
    and N0: "0 < va - (- d - Om) * u" and D0: "0 < (a + kap) / 4 *
w - y"
  shows "((- (a + d) * va - g * u) - (- d - Om) * (g * va - (a - d)
* u)) * ((a + kap) / 4 * w - y)
      \<le> (va - (- d - Om) * u) * ((a + kap) / 4 * (- 4 * g * y)
- (g * w - 2 * a * y))"
```

ratio_nonincreasing · Isabelle/MemoryAccessFloors/Metzler_Cone.thy, line 233

```
theorem ratio_nonincreasing:
  fixes N D N' D' :: "real \<Rightarrow> real" and T :: real
  assumes T0: "0 \<le> T"
    and dN: "\<forall>t\<in>{0..T}. (N has_real_derivative N' t)
(at t)"
    and dD: "\<forall>t\<in>{0..T}. (D has_real_derivative D' t)
(at t)"
    and Dp: "\<forall>t\<in>{0..T}. 0 < D t"
    and lg: "\<forall>t\<in>{0..T}. N' t * D t \<le> N t * D' t"
  shows "N T / D T \<le> N 0 / D 0"
```

unequal_storage_bound · Isabelle/MemoryAccessFloors/Storage_Function.thy, line 270

```
theorem unequal_storage_bound:
  fixes a d Om kap :: real and g va u w y :: "real \<Rightarrow>
real" and T :: real
  assumes a0: "0 < a" and Om0: "0 < Om" and OmE: "Om\<^sup>2 = d\
<^sup>2 - 1" and k0: "0 < kap" and kE: "kap\<^sup>2 = a\<^sup>2 -
4"
    and a3: "d\<^sup>2 + 3 \<le> a\<^sup>2" and d5: "5 \<le> d\
<^sup>2"
    and T0: "0 \<le> T" and gc: "continuous_on {0..T} g" and g01:
"\<forall>t\<in>{0..T}. 0 \<le> g t \<and> g t \<le> 1"
    and dva: "\<forall>t\<in>{0..T}. (va has_real_derivative - (a +
d) * va t - g t * u t) (at t)"
    and du: "\<forall>t\<in>{0..T}. (u has_real_derivative g t * va
t - (a - d) * u t) (at t)"
    and dw: "\<forall>t\<in>{0..T}. (w has_real_derivative - 4 * g
t * y t) (at t)"
    and dy: "\<forall>t\<in>{0..T}. (y has_real_derivative g t * w
t - 2 * a * y t) (at t)"
    and ini: "va 0 = 1" "u 0 = 0" "w 0 = 1" "y 0 = 0"
  shows "\<forall>t\<in>{0..T}. 0 < w t \<and> 0 \<le> y t \<and> 0
\<le> u t \<and> 2 * Om * u t < w t"
    and "\<forall>t\<in>{0..T}. (1 - w t) / 2 + 2 * u t < 1 / 2"
```

threshold_3G2 · Isabelle/MemoryAccessFloors/Unequal_Dephasing_Threshold.thy, line 20

```
theorem threshold_3G2:
  fixes q m G :: real
  assumes "0 < G" and "2 * G < (q + m) / 2" and "G < \<bar>q - m\
```

```
<bar> / 2"
  shows "sqrt (((q + m) / 2)\<^sup>2 - 4 * G\<^sup>2) < sqrt ((\
<bar>q - m\<bar> / 2)\<^sup>2 - G\<^sup>2) \<longleftrightarrow> q
* m < 3 * G\<^sup>2"
```

eventually_four_u_gt_w · Isabelle/MemoryAccessFloors/Unequal_Dephasing_Threshold.thy, line 301

```
theorem eventually_four_u_gt_w:
  fixes G a Om :: real
  assumes "0 < G" and "2 * G < a" and "kap G a < Om"
  shows "eventually (\<lambda>t. wf G a t < 4 * uf G a Om t)
at_top"
```

p_plus_2c_iff · Isabelle/MemoryAccessFloors/Unequal_Dephasing_Threshold.thy, line 376

```
lemma p_plus_2c_iff:
  fixes w u :: real
  shows "(1 - w) / 2 + 2 * u > 1 / 2 \<longleftrightarrow> w < 4 *
u"
```

resonant_below_five_twelfths · Isabelle/MemoryAccessFloors/Unequal_Dephasing_Threshold.thy, line 381

```
theorem resonant_below_five_twelfths:
  fixes G q m :: real
  assumes "0 < G" and "2 * G < (q + m) / 2" and "G < \<bar>q - m\
<bar> / 2" and "q * m < 3 * G\<^sup>2"
  shows "eventually (\<lambda>t. (1 - wf G ((q + m) / 2) t) / 2
                         + 2 * uf G ((q + m) / 2) (sqrt ((\<bar>q -
m\<bar> / 2)\<^sup>2 - G\<^sup>2)) t > 1 / 2) at_top"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Proposition 12, p. 15, doi:10.5281/zenodo.24250501

↳ Used in: Corollary 13 (p. 27)

Proof

Take $G = 1$. With $w = 1 - 2p$, y the imaginary part of ρ_{ab}, v_a the coherence left in the qubit and $u = c$ the one in the memory,

$$\dot w = -4gy, \quad \dot y = gw - \Phi y, \qquad \dot v_a = -\phi_q v_a - gu, \quad \dot u = gv_a - \phi_m u$$

from $w = 1, v_a = 1, y = u = 0$. Let $d = (\phi_q - \phi_m)/2$ (signed), $\delta = |d|$, $a = \Phi/2$, $\kappa = \sqrt{a^2 - 4}$, $\theta_\pm = (a \pm \kappa)/4$ (the roots of $4\theta^2 - \Phi\theta + 1$) and $\chi_\pm = -d \pm \Omega$ (the roots of $\chi^2 + 2d\chi + 1$); under the hypotheses of (a), $\kappa{>}0$ and $\Omega \geq 2$. With $z = \theta_- w - y$ and $h = v_a - \chi_+ u$ the two systems read

$$\dot y = \tfrac{g}{\theta_-} z + \left(\tfrac{g}{\theta_-} - 2a\right) y, \qquad \dot z = -\tfrac{g}{\theta_-} z + (1 - g)\, 2a\, y,$$
$$\dot u = gh + (g\chi_+ - \phi_m) u, \qquad \dot h = -(\phi_q + g\chi_+) h + (1 - g)(1 + \chi_+^2) u$$

whose off-diagonal coefficients are non-negative for $0 \le g \le 1$; so the cones $y, z \ge 0$ and $u, h \ge 0$ are invariant for bounded measurable controls, and $w>0$, $h>0$ at every finite time, while u may vanish during an initial wait. Use the regular storage function

$$H = \frac{4(v_a - \chi_- u)}{\theta_+ w - y}, \qquad H(0) = \frac{4}{\theta_+},$$

whose numerator and denominator are strictly positive, so that H is defined also when $u = 0$. Its logarithmic derivative is affine in g. At $g = 1$ numerator and denominator decay at $a - \Omega$ and $a - \kappa$, so $\dot H / H = \Omega - \kappa \le 0$ exactly when $\phi_q \phi_m \ge 3$ (that is $\delta^2 - 1 \le a^2 - 4$). At $g = 0$ the bound is $\delta(\delta \mp \Omega)/\Omega - \phi_m$ for $d \gtrless 0$; both equal $\delta^2/\Omega - a$, which is negative because $a^2\Omega^2 - \delta^4 \ge (\delta^2 + 3)(\delta^2 - 1) - \delta^4 = 2\delta^2 - 3>0$. Hence $\dot H \le 0$ almost everywhere for all allowed controls, and $H \le H(0)$ gives $v_a - \chi_- u \le w - y/\theta_+ \le w$. But $v_a - \chi_- u = h + 2\Omega u>2\Omega u$ at finite time, so $u<w/(2\Omega)$, including the case $\Omega = 2$. Restoring G gives (a), strict at finite time, and then $p + 2c - \frac{1}{2} = 2c - \frac{w}{2}<(\frac{1}{\Omega} - \frac{1}{2})w \le 0$. The resonant family tends to $r = 5/12$, so the infimum is not attained. For a product below 3 the same asymmetry hypothesis puts the resonant solutions in the hyperbolic regime, $\Omega>\kappa$, and the long-time comparison above proves (b).

Corollary 13. (a sufficient region below $2\sqrt{5}\,G$) p. 16

Unequal_Sufficient_Region.psuf_kappa, Unequal_Sufficient_Region.psuf_a, Unequal_Sufficient_Region.psuf_initial, Unequal_Sufficient_Region.exp_step, Unequal_Sufficient_Region.sinh_ratio_mono, Unequal_Sufficient_Region.resonant_ratio, Unequal_Sufficient_Region.coth_step, Unequal_Sufficient_Region.initial_interval Take the class of Proposition 12 and units $G = 1$, but with $1<\delta<\sqrt{5}$. Put $s = \operatorname{artanh}(\Omega/\delta)/\Omega$ and $L = \log(2/\Omega)/s$. If

$$\phi_q \phi_m \;\ge\; P_{\mathrm{suf}}(\delta) := \max\Big\{16 - 8\delta,\; 3 + 2\Omega L + L^2,\; \big(\tfrac{\delta^2}{\Omega} + L\big)^2 - \delta^2\Big\},$$

then every pulse has $p + 2c<\frac{1}{2}$ at every duration, and $r_{\mathrm{opt}} = 5/12$, not attained. For instance $P_{\mathrm{suf}}(2) = 3.6911216538\ldots$: at $|\phi_q - \phi_m| = 4G$ the threshold lies in $[3G^2, 3.6912\,G^2]$, where the envelope bound of Corollary 8 certifies only from $4G^2$.

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

psuf_kappa · Isabelle/MemoryAccessFloors/Unequal_Sufficient_Region.thy, line 18

```
theorem psuf_kappa:
  fixes L P delta Omega kappa :: real
  assumes d1: "1 < delta" and Om: "Omega = sqrt (delta^2 - 1)" and
ka: "kappa = sqrt (delta^2 + P - 4)"
    and L0: "0 < L" and h: "3 + 2 * Omega * L + L^2 \<le> P"
  shows "L \<le> kappa - Omega"
```

psuf_a · Isabelle/MemoryAccessFloors/Unequal_Sufficient_Region.thy, line 33

```
theorem psuf_a:
  fixes L P delta Omega a :: real
  assumes d1: "1 < delta" and Om: "Omega = sqrt (delta^2 - 1)" and
a_def: "a = sqrt (delta^2 + P)"
    and L0: "0 < L" and h: "(delta^2 / Omega + L)^2 - delta^2 \<le>
P"
  shows "L \<le> a - delta^2 / Omega"
```

psuf_initial · Isabelle/MemoryAccessFloors/Unequal_Sufficient_Region.thy, line 44

```
theorem psuf_initial:
  fixes P delta a :: real
  assumes d1: "1 < delta" and d4: "delta < 4" and a_def: "a = sqrt
(delta^2 + P)" and h: "16 - 8 * delta \<le> P"
  shows "4 \<le> a + delta"
```

exp_step · Isabelle/MemoryAccessFloors/Unequal_Sufficient_Region.thy, line 56

```
theorem exp_step:
  fixes Omega s tau L m :: real
  assumes O0: "0 < Omega" and O2: "Omega < 2" and s0: "0 < s" and
Ldef: "L = ln (2 / Omega) / s"
    and Lm: "L \<le> m" and st: "s \<le> tau"
  shows "(2 / Omega) * exp (- m * tau) \<le> 1"
```

sinh_ratio_mono · Isabelle/MemoryAccessFloors/Unequal_Sufficient_Region.thy, line 84

```
theorem sinh_ratio_mono:
  fixes u v :: real
  assumes "0 < u" "u \<le> v"
  shows "sinh u / u \<le> sinh v / v"
```

resonant_ratio · Isabelle/MemoryAccessFloors/Unequal_Sufficient_Region.thy, line 128

```
theorem resonant_ratio:
  fixes Omega kappa a tau :: real
  assumes Ok: "0 < Omega" "Omega \<le> kappa" and a0: "0 < a" and
t0: "0 < tau"
  shows "4 * (sinh (Omega * tau) / Omega) / (cosh (kappa * tau) +
(a / kappa) * sinh (kappa * tau))
           \<le> 4 / (a + Omega * cosh (Omega * tau) / sinh (Omega
* tau))"
```

coth_step · Isabelle/MemoryAccessFloors/Unequal_Sufficient_Region.thy, line 161

```
theorem coth_step:
  fixes Omega delta tau s :: real
  assumes O0: "0 < Omega" and d0: "0 < delta" and t0: "0 < tau" and
ts: "tau \<le> s"
    and th: "tanh (Omega * s) = Omega / delta"
  shows "delta \<le> Omega * cosh (Omega * tau) / sinh (Omega *
tau)"
```

`initial_interval` · Isabelle/MemoryAccessFloors/Unequal_Sufficient_Region.thy, line 178

```
theorem initial_interval:
  fixes P delta Omega a kappa tau s :: real
  assumes d1: "1 < delta" and d4: "delta < 4"
    and Om: "Omega = sqrt (delta^2 - 1)" and a_def: "a = sqrt
(delta^2 + P)"
    and ka: "kappa = sqrt (delta^2 + P - 4)"
    and P3: "3 \<le> P" and P16: "16 - 8 * delta \<le> P"
    and t0: "0 < tau" and ts: "tau \<le> s" and th: "tanh (Omega *
s) = Omega / delta"
  shows "4 * (sinh (Omega * tau) / Omega) / (cosh (kappa * tau) +
(a / kappa) * sinh (kappa * tau)) \<le> 1"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Corollary 13, p. 16, doi:10.5281/zenodo.24250501

↳ Not used by another result of the paper.

Proof

Since $\phi_q\phi_m \geq 3$, the cones and the storage function H of the proof above are unchanged (they need $\kappa{>}0$, not $\Omega \geq 2$), and the vertex values give $\dot{H}/H \leq -m$ with $m = \min(\kappa - \Omega,\ a - \delta^2/\Omega)$; the last two terms of P_{suf} say $\kappa - \Omega \geq L$ and $a - \delta^2/\Omega \geq L$. Hence $v_a - \chi_- u \leq e^{-m\tau} w$, so $4u/w{<}(2/\Omega)e^{-m\tau} \leq (2/\Omega)e^{-Ls} = 1$ for $\tau \geq s$. For $\tau \leq s$, Corollary 8 bounds p and c by the resonant curves: the population is monotone ($\kappa{>}0$), and the coherence envelope follows c_{res} up to the peak of $e^{\phi_{\min}\tau}c_{\mathrm{res}}(\tau)$, which is at $\tanh(\Omega\tau) = \Omega/\delta$, that is at $\tau = s$, since $a - \phi_{\min} = \delta$. As $\cosh(x\tau)$ and $\sinh(x\tau)/x$ increase with x and $\kappa \geq \Omega$,

$$\frac{4c_{\mathrm{res}}}{1 - 2p_{\mathrm{res}}} = \frac{4\sinh(\Omega\tau)/\Omega}{\cosh\kappa\tau + (a/\kappa)\sinh\kappa\tau} \leq \frac{4}{a + \Omega\coth\Omega\tau} \leq \frac{4}{a + \delta}$$

the last step by $\phi_q\phi_m \geq 16 - 8\delta$, and strictly for $\tau{<}s$. So $p + 2c{<}\frac{1}{2}$ at every finite duration.

Proposition 14. ((a) , (b)) p. 17

MarBrem:corrused-byMarBrem:relaxanswered-byMarBrem:unifiedanswered-by MarBprop:slow2answered-byMarBprop:noiselossanswered-by Quasistatic_Average.qs_average, Quasistatic_Average.qs_average_uncorrelated,

Memory_Access_Quasistatic.qsA_ge, Memory_Access_Quasistatic.qsA_pop_coh, Memory_Access_Quasistatic.qs_pointwise (a) With quasi-static offsets as the only noise (no relaxation and no Markovian pure dephasing), the resonant pulse of duration $\pi/(2G)$ has

$$r \;\le\; \frac{\mathbb{E}\,\Delta^2}{12\,G^2} + \frac{\pi^2\,\mathbb{E}\,\bar{\xi}^{\,2}}{24\,G^2}, \qquad \text{that is} \qquad r \le \frac{8+\pi^2}{96}\,\frac{\sigma_q^2+\sigma_m^2}{G^2}$$

for offsets with $\mathbb{E}[\xi_q \xi_m] = 0$ (in particular centred uncorrelated offsets), whatever their distribution with finite second moments. (b) With relaxation rates γ_q, γ_m and independent centred Gaussian offsets, every pulse of every duration has

$$r \;\ge\; \frac{1}{2} - \frac{1}{6}\,\sup_{T\ge 0}\Big[B_p^{(0)}(T) + 2\,e^{-v_\perp T^2/2}\,B_c^{(0)}(T)\Big], \qquad v_\perp = \frac{\sigma_q^2\sigma_m^2}{\sigma_q^2+\sigma_m^2},$$

where $B_p^{(0)}, B_c^{(0)}$ are the envelopes of Corollary 8 with the relaxation rates alone ($\phi_q = \phi_m = 0$); set $v_\perp = 0$ when $\sigma_q = \sigma_m = 0$, and the bound then reduces to the relaxation-only envelope bound.

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

qs_average · Isabelle/MemoryAccessFloors/Quasistatic_Average.thy, line 176

```
theorem qs_average:
  fixes M :: "'a measure" and D xb :: "'a \<Rightarrow> real" and G
:: real
  assumes "prob_space M" and "D \<in> borel_measurable M" and "xb \
<in> borel_measurable M"
    and "integrable M (\<lambda>x. (D x)\<^sup>2)" and "integrable
M (\<lambda>x. (xb x)\<^sup>2)" and "0 < G"
  shows "qs_r M G D xb \<le> (\<integral>x. (D x)\<^sup>2 \
<partial>M) / (12 * G\<^sup>2) + pi\<^sup>2 * (\<integral>x. (xb
x)\<^sup>2 \<partial>M) / (24 * G\<^sup>2)"
```

qs_average_uncorrelated · Isabelle/MemoryAccessFloors/Quasistatic_Average.thy, line 205

```
corollary qs_average_uncorrelated:
  fixes M :: "'a measure" and xq xm :: "'a \<Rightarrow> real" and
G sq sm :: real
  assumes "prob_space M" and "xq \<in> borel_measurable M" and "xm
\<in> borel_measurable M"
    and "integrable M (\<lambda>x. (xq x)\<^sup>2)" and "integrable
M (\<lambda>x. (xm x)\<^sup>2)"
    and "(\<integral>x. xq x * xm x \<partial>M) = 0"
    and "(\<integral>x. (xq x)\<^sup>2 \<partial>M) = sq\<^sup>2"
and "(\<integral>x. (xm x)\<^sup>2 \<partial>M) = sm\<^sup>2" and
"0 < G"
  shows "qs_r M G (\<lambda>x. xq x - xm x) (\<lambda>x. (xq x + xm
x) / 2)
         \<le> (8 + pi\<^sup>2) * (sq\<^sup>2 + sm\<^sup>2) / (96 *
G\<^sup>2)"
```

qsA_ge · Isabelle/MemoryAccessFloors/Memory_Access_Quasistatic.thy,

line 88

```
lemma qsA_ge:
  fixes q :: real
  assumes "0 \<le> q"
  shows "1 - q / 2 \<le> qsA q"
```

qsA_pop_coh · Isabelle/MemoryAccessFloors/Memory_Access_Quasistatic.thy, line 111

```
lemma qsA_pop_coh:
  fixes q :: real
  assumes "0 \<le> q"
  shows "3 - (qsA q)\<^sup>2 - 2 * qsA q \<le> 2 * q"
```

qs_pointwise · Isabelle/MemoryAccessFloors/Memory_Access_Quasistatic.thy, line 125

```
theorem qs_pointwise:
  fixes q x :: real
  assumes "0 \<le> q"
  shows "3 - (qsA q)\<^sup>2 - 2 * qsA q * cos x \<le> 2 * q + x\
<^sup>2"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Proposition 14, p. 17, doi:10.5281/zenodo.24250501

↳ Not used by another result of the paper.

Proof

(a) For fixed offsets the amplitude transferred is $-iA\,e^{-i\bar{\xi}T_0}$ with $T_0=\pi/(2G)$, $A=\sin(\frac{\pi}{2}\sqrt{1+q})/\sqrt{1+q}$ and $q=\Delta^2/(4G^2)$. So $p=\mathbb{E}A^2$ and $c=|\mathbb{E}[Ae^{-i\bar{\xi}T_0}]|\geq\mathbb{E}[A\cos\bar{\xi}T_0]$, and $6r\leq\mathbb{E}\big[(3-A^2-2A)+2A(1-\cos\bar{\xi}T_0)\big]$. Now $A\leq 1$ and $A\geq 1-q/2$. Indeed, with $s=\sqrt{1+q}$ the claim is $\sin(\pi s/2)/s\geq(3-s^2)/2$: for $s\geq 2$ the left side is at least $-\frac{1}{2}$ and the right side at most $-\frac{1}{2}$; for $\sqrt{3}\leq s\leq 2$ the left side is non-negative and the right side is not positive; and for $s=1+x\leq\sqrt{3}$, $\sin(\pi s/2)=\cos(\pi x/2)\geq 1-\pi^2x^2/8\geq 1-\frac{3}{2}x^2-\frac{1}{2}x^3=s(3-s^2)/2$, because $\pi^2/8<\frac{3}{2}$. So $3-A^2-2A=(1-A)(3+A)\leq 2q$, and $2A(1-\cos x)\leq x^2$. (b) Write $\bar{\xi}=k\Delta+\xi_0$ with ξ_0 independent of Δ; its variance is $v_\perp$. The part ξ_0 is a common shift of the two levels: it commutes with every control and multiplies the coherence with the vacuum by $e^{-i\xi_0 T}$, whose average is $e^{-v_\perp T^2/2}$. For each value of Δ what remains is a pulse of the model with a detuning, so $p\leq B_p^{(0)}(T)$ and the modulus of its coherence is at most $B_c^{(0)}(T)$; average, and use the triangle inequality for c.

Evidence & scope

- **Literature:** Companion paper: this statement is used in paper B (Robust qubit–memory transfer against quasi-static frequency noise costs time and exposure),

Remark 10 (silent link, not printed in the paper).
https://karlesmarin.github.io/blueprints/robust-offset/

- **Literature:** Companion paper: this statement has its open question answered in paper B (Robust qubit–memory transfer against quasi-static frequency noise costs time and exposure), Remark 9 (silent link, not printed in the paper).
https://karlesmarin.github.io/blueprints/robust-offset/
- **Literature:** Companion paper: this statement has its open question answered in paper B (Robust qubit–memory transfer against quasi-static frequency noise costs time and exposure), Remark 11 (silent link, not printed in the paper).
https://karlesmarin.github.io/blueprints/robust-offset/
- **Literature:** Companion paper: this statement has its open question answered in paper B (Robust qubit–memory transfer against quasi-static frequency noise costs time and exposure), Proposition 5 (silent link, not printed in the paper).
https://karlesmarin.github.io/blueprints/robust-offset/
- **Literature:** Companion paper: this statement has its open question answered in paper B (Robust qubit–memory transfer against quasi-static frequency noise costs time and exposure), Proposition 8 (silent link, not printed in the paper).
https://karlesmarin.github.io/blueprints/robust-offset/

Proposition 15. (the price of one stopping time) p. 19

Equal_Rate_Threshold.gap_lower_bound, Equal_Rate_Threshold.gap_at_midpoint, Equal_Rate_Threshold.gap_min_value, Equal_Rate_Threshold.wph1_deriv, Equal_Rate_Threshold.wph1_stationary, Equal_Rate_Threshold.wph1_negative Let $\varepsilon = \max(\gamma_q, \gamma_m, \phi_q, \phi_m)/G$.

1. *Without dephasing (*$\Phi = 0$*),* $c_{\mathrm{res}} = \sqrt{p_{\mathrm{res}}}$*, so* $r_{\mathrm{sep}} = r_{\mathrm{opt}} = r_{\mathrm{res}}$*: the resonant pulse stopped at its first maximum is exactly optimal for* r *and for* F_e*.*
2. *In general the population maximum does not move with the dephasing to first order, the coherence maximum comes earlier by* $\Phi/(2G^2)$*, and*

$$r_{\mathrm{res}} - r_{\mathrm{sep}} = \frac{\Phi^2}{48\,G^2} + O(\varepsilon^3).$$

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

gap_lower_bound · Isabelle/MemoryAccessFloors/Equal_Rate_Threshold.thy, line 315

```
lemma gap_lower_bound:
  fixes G a b t :: real
  shows "G\<^sup>2 * (a - b)\<^sup>2 / 12 \<le> (G\<^sup>2 / 6) *
((t - a)\<^sup>2 + (t - b)\<^sup>2)"
```

gap_at_midpoint · Isabelle/MemoryAccessFloors/Equal_Rate_Threshold.thy, line 326

```
lemma gap_at_midpoint:
  fixes G a b :: real
  shows "(G\<^sup>2 / 6) * (((a + b) / 2 - a)\<^sup>2 + ((a + b) /
2 - b)\<^sup>2) = G\<^sup>2 * (a - b)\<^sup>2 / 12"
```

gap_min_value · Isabelle/MemoryAccessFloors/Equal_Rate_Threshold.thy, line 333

```
lemma gap_min_value:
  fixes G a b Phi :: real
  assumes "G \<noteq> 0" and "a - b = Phi / (2 * G\<^sup>2)"
  shows "G\<^sup>2 * (a - b)\<^sup>2 / 12 = Phi\<^sup>2 / (48 * G\
<^sup>2)"
```

wph1_deriv · Isabelle/MemoryAccessFloors/Equal_Rate_Threshold.thy, line 364

```
lemma wph1_deriv:
  fixes G Phi om t :: real
  assumes "0 < om" and "om\<^sup>2 = 4 * G\<^sup>2 - Phi\<^sup>2 /
4"
  shows "((wph1 Phi om) has_real_derivative - Phi * wph1 Phi om t -
4 * G\<^sup>2 * wph Phi om t) (at t)"
```

wph1_stationary · Isabelle/MemoryAccessFloors/Equal_Rate_Threshold.thy, line 386

```
theorem wph1_stationary:
  fixes Phi om :: real
  assumes "0 < om"
  shows "wph1 Phi om (pi / om) = 0"
```

wph1_negative · Isabelle/MemoryAccessFloors/Equal_Rate_Threshold.thy, line 392

```
theorem wph1_negative:
  fixes Phi om t :: real
  assumes "0 < om" and "0 < t" and "t < pi / om"
  shows "wph1 Phi om t < 0"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Proposition 15, p. 19, doi:10.5281/zenodo.24250501

↳ Used in: Proposition 16 (p. 34)

Proof

(a) With $\phi_q = \phi_m = 0$ the populations follow from the amplitudes (a, b) of the one-excitation state, and the coherence with the vacuum obeys the same equations with the same rates $\gamma_q/2, \gamma_m/2$; hence $p_{\mathrm{res}} = |b|^2$ and $c_{\mathrm{res}} = |b|$ peak at the same time. There $p_{\mathrm{res}} = P_\phi$ and $c_{\mathrm{res}} = c_*$ (Corollary 9 with $\Phi = 0$), so the two sides of Corollary 9 coincide.

(b) Let $t_0 = \pi/(2G)$. The coherence has the closed form of Corollary 9; it peaks where $\tan \omega_c t = \omega_c/\bar{\eta}$, $\bar{\eta} = (\eta_q + \eta_m)/2 = \bar{\gamma}/2 + \Phi/2$, that is $t_c = t_0 - \bar{\eta}/G^2 + O(\varepsilon^2/G)$, with $c''_{\mathrm{res}}(t_c) = -G^2 + O(\varepsilon G^2)$. The population peak t_p is a non-degenerate maximum ($p''_{\mathrm{res}} = -2G^2 + O(\varepsilon G^2)$), so by the implicit function theorem it is a smooth function of the four rates and $t_p = t_0 + \sum_j a_j \gamma_j + b\,\Phi + O(\varepsilon^2/G)$. With $\Phi = 0$, $t_p = t_c$ by (a), so the a_j are those of t_c: $t_0 - \bar{\gamma}/(2G^2)$. With $\gamma_q = \gamma_m = 0$ the population is $\frac{1}{2}(1 - w)$ with $\ddot{w} + \Phi\dot{w} + 4G^2 w = 0$, $w(0) = 1$, $\dot{w}(0) = 0$; then $\dot{w} = -e^{-\Phi t/2}(\omega_p + \Phi^2/4\omega_p)\sin \omega_p t$, the peak is at $t_p = \pi/\omega_p = t_0 + O(\Phi^2/G^3)$ and $b = 0$. Hence $t_p - t_c = \Phi/(2G^2) + O(\varepsilon^2/G)$.

The branch t_p is the global maximum, $P_\phi = p_{\mathrm{res}}(t_p)$. With $h = (\gamma_q + \gamma_m)/2$ and $\Delta = (\gamma_q - \gamma_m)/2$, the Laplace transform of p_{res} is $2G^2/Q(s + h)$ with $Q(z) = z^3 + \Phi z^2 + (4G^2 - \Delta^2)z - \Phi\Delta^2$; for small rates its roots are one real $z_0 \in [0, h]$ and a conjugate pair, so $p_{\mathrm{res}}(t) = Ae^{-\ell t}[1 - e^{-dt}(\cos \omega t + (d/\omega)\sin \omega t)]$ with $\ell = h - z_0 \geq 0$, $d = (\Phi + 3z_0)/2 \geq 0$, $\omega \to 2G$. With $P = 2\pi/\omega$ and $M = \max_{[0,P]} p_{\mathrm{res}}$, $p_{\mathrm{res}}(s + nP) = e^{-(\ell+d)nP} p_{\mathrm{res}}(s) + (1 - e^{-dnP})Ae^{-\ell(s+nP)}$ for $0 \leq s < P$, $n \geq 1$, and both terms on the right are at most M (the baseline $Ae^{-\ell t}$ is at most M past the first zero of the oscillating factor), with weights summing to at most one; so no later period exceeds the first, whose maximum is t_p by uniform convergence to $\sin^2 Gt$ and local concavity. The coherence $c \leq (G/\omega_c)e^{-\bar{\eta}t}$ has its global maximum at t_c.

The joint minimum lies near t_0. For $t \geq 2t_0$ the coherence deficit is at least $c_* - c(t) \geq C\varepsilon$, with $C > 0$ independent of the direction of the (small, non-negative) rate vector, so $r_{\mathrm{res}}(t) - r_{\mathrm{sep}} \geq C\varepsilon/3$ there; on $[0, t_0/2] \cup [3t_0/2, 2t_0]$ the ideal joint excess is positive and stays so for small rates; a candidate near t_0 has excess $O(\varepsilon^2)$; and on the first window the ideal curvature, $(G^2/3)[\sin Gt - \cos 2Gt] \geq G^2\sqrt{2}/6$, keeps the excess strictly convex. So the unique local minimum near t_0 is the global one.

Expansion. In $\tau = Gt$, Taylor's theorem centred at each maximum gives

$$r_{\mathrm{res}}(t) - r_{\mathrm{sep}} = \frac{(\tau - \tau_p)^2 + (\tau - \tau_c)^2}{6} + O\big(\varepsilon[(\tau - \tau_p)^2 + (\tau - \tau_c)^2] \text{-}$$

the minimiser is $t_r = (t_p + t_c)/2 + O(\varepsilon^2/G)$, the remainder there is $O(\varepsilon^3)$, and $r_{\mathrm{res}} - r_{\mathrm{sep}} = G^2(t_p - t_c)^2/12 + O(\varepsilon^3) = \Phi^2/(48G^2) + O(\varepsilon^3)$.

Proposition 16. (when the resonant pulse is exactly optimal) p. 20

Memory_Floors_Pieces.f_res_min, Memory_Floors_Pieces.f_env_large, Memory_Floors_Pieces.f_env_small Let $\nu = \min(\eta_q, \eta_m)$ and $\Phi = \phi_q + \phi_m$. Fix $G{>}0$ and a nonnegative rate direction $(\hat\gamma_q, \hat\gamma_m, \hat\varphi_q, \hat\varphi_m)$, normalised to have maximum component one, and set $(\gamma_q, \gamma_m, \phi_q, \phi_m) = \varepsilon G(\hat\gamma_q, \hat\gamma_m, \hat\varphi_q, \hat\varphi_m)$. All statements below concern $\varepsilon \downarrow 0$ along this fixed direction; the required upper bound on ε may depend on the direction.

1. If $4\nu{>}\Phi$, then for all small enough ε, $r_{\mathrm{opt}} = r_{\mathrm{res}}$: the resonant pulse, stopped at its best duration, is exactly optimal for the average fidelity.
2. In general $r_{\mathrm{res}} - r_{\mathrm{time}} = \max(0, \Phi - 4\nu)^2/(48G^2) + O(\varepsilon^3)$: the undecided width between the bound of Corollary 10 and the resonant pulse is of second order. Its quadratic term vanishes when $4\nu \geq \Phi$; exact equality is asserted in part (a) under the strict condition $4\nu{>}\Phi$.

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

`f_res_min` · `Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy`, line 315

```
theorem f_res_min:
  fixes D t :: real
  shows "D\<^sup>2 / 2 \<le> f_res D t" and "f_res D (D / 2) = D\
<^sup>2 / 2"
```

`f_env_large` · `Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy`, line 331

```
theorem f_env_large:
  fixes D x t :: real
  assumes "0 < D" and "D / 2 \<le> x" and "0 \<le> t"
  shows "D\<^sup>2 / 2 \<le> f_env D x t" and "f_env D x (D / 2) =
D\<^sup>2 / 2"
```

`f_env_small` · `Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy`, line 355

```
theorem f_env_small:
  fixes D x t :: real
  assumes "0 < D" and "0 \<le> x" and "x < D / 2" and "0 \<le> t"
  shows "2 * x * (D - x) \<le> f_env D x t"
    and "f_env D x (D - x) = 2 * x * (D - x)"
    and "D\<^sup>2 / 2 - 2 * x * (D - x) = (D - 2 * x)\<^sup>2 / 2"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Proposition 16, p. 20, doi:10.5281/zenodo.24250501

↳ Not used by another result of the paper.

Proof

Write $\tau = T - t_c$ and $\Delta = t_p - t_c$. The weighted curve $e^{\nu s} c_{\mathrm{res}}(s)$ peaks at $s_* = t_c + \nu/G^2 + O(\varepsilon^2/G)$, so $B_c = c_{\mathrm{res}}$ up to s_*; likewise $B_p = p_{\mathrm{res}}$ up to a time $\geq t_p$. To second order, $P_\phi + 2c_* - (p + 2c)$ is $G^2[(\tau - \Delta)^2 + \tau^2]$ for the resonant pulse, and for the envelopes the same up to $\tau = x := \nu/G^2$ and $G^2[(\tau - \Delta)^2 + 2x\tau - x^2]$ beyond. The first is minimal at $\tau = \Delta/2$, with value $G^2\Delta^2/2$; the second, on $\tau \geq x$, at $\tau = \max(x, \Delta - x)$. If $x<\Delta/2$, the envelope minimum is $G^2 \cdot 2x(\Delta - x)$, and the difference with $G^2\Delta^2/2$ is $G^2(\Delta - 2x)^2/2 = (\Phi - 4\nu)^2/(8G^2)$; dividing by 6 gives (b).

(a) If $x>\Delta/2$, i.e. $4\nu>\Phi$, the minimum of the envelope expression is the resonant one, attained at $\tau = \Delta/2<x$. To make this exact, let $\mu = \min(\gamma_q, \gamma_m)$ and let s_p and s_c be the first maxima of $e^{\mu T} p_{\mathrm{res}}(T)$ and $e^{\nu T} c_{\mathrm{res}}(T)$ respectively. The first weighted curve is the resonant population with the nonnegative shifted relaxation rates $\gamma_j - \mu$, so the global-maximum argument of Proposition 15 applies to it. The second is the resonant coherence with rates $\eta_j - \nu \geq 0$, whose first maximum is global by its closed form. Each weighted curve increases up to its first maximum. Hence each envelope agrees with its resonant curve up to that maximum and is a nonincreasing exponential thereafter; the first derivatives agree at the joins. The expansions give $s_c - t_r = (4\nu - \Phi)/(4G^2) + O(\varepsilon^2/G)$ and $s_p - t_r = (2\mu + \Phi)/(4G^2) + O(\varepsilon^2/G)$. Along a fixed direction with $4\nu>\Phi$, the exact resonant maximiser t_r therefore lies strictly before both joins for sufficiently small ε; the case $\Phi = 0$ also follows directly from Proposition 15(a). In a fixed neighbourhood of t_0, $B_p + 2B_c$ is strictly concave until the later join: the negative curvature of any remaining resonant branch dominates the $O(\varepsilon^2 G^2)$ curvature of an exponential branch. After both joins its derivative is $-\mu B_p - 2\nu B_c \leq 0$. Before either join it equals the resonant objective, whose global maximum is t_r. Thus its exact global supremum is attained at t_r, where both envelopes coincide with the resonant curves, and Corollary 10 gives $r_{\mathrm{opt}} = r_{\mathrm{res}}$.

Proposition 17. (the memory cycle with a reset) p. 21

Memory_Floors_Pieces.chan_comp, Memory_Floors_Pieces.valid_chan_comp, Memory_Floors_Pieces.cycle_channel, Memory_Floors_Pieces.cycle_floor Markovian rates; write, wait a time t in the memory with no coupling, read back into the qubit, and assume that the transmon is reset to its ground state between the write and the read (what it still holds is lost, as in the channel $\mathcal{E}_{p,c}$). Then the write–wait–read channel is $\mathcal{E}_{p,c}$ with $p = p_w p_r e^{-\gamma_m t}$ and $c = c_w c_r e^{-\eta_m t}$, the read obeys the same bounds as the write, and

$$r_{\mathrm{cycle}} \geq \frac{1 - P_\phi^2 e^{-\gamma_m t}}{6} + \frac{1 - c_*^2 e^{-\eta_m t}}{3}.$$

> *Here $P_\phi = \sup_{s\ge 0} p_{\mathrm{res}}(s)$ and $c_* = \sup_{s\ge 0} c_{\mathrm{res}}(s) = \sqrt{P_*(2\eta_q, 2\eta_m; G)}$, with the continuous and hyperbolic forms of Theorem 1 used at critical damping and overdamping. The known accumulated phase is corrected at the output.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

`chan_comp` · `Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy`, line 36

```
theorem chan_comp: "chan p2 c2 (chan p1 c1 r) = chan (p1 * p2) (c1
* c2) r"
```

`valid_chan_comp` · `Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy`, line 45

```
theorem valid_chan_comp:
  assumes "valid_chan p1 c1" and "valid_chan p2 c2"
  shows "valid_chan (p1 * p2) (c1 * c2)"
```

`cycle_channel` · `Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy`, line 58

```
theorem cycle_channel:
  "chan pr cr (chan e1 e2 (chan pw cw r)) = chan (pw * e1 * pr) (cw
* e2 * cr) r"
```

`cycle_floor` · `Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy`, line 72

```
theorem cycle_floor:
  fixes p1 p2 c1 c2 P C e1 e2 :: real
  assumes "0 \<le> p1" and "p1 \<le> P" and "0 \<le> p2" and "p2 \
<le> P"
    and "0 \<le> c1" and "c1 \<le> C" and "0 \<le> c2" and "c2 \
<le> C"
    and "0 \<le> e1" and "e1 \<le> 1" and "0 \<le> e2" and "e2 \
<le> 1"
  shows "(1 - P\<^sup>2 * e1) / 6 + (1 - C\<^sup>2 * e2) / 3 \<le>
rfloor (p1 * p2 * e1) (c1 * c2 * e2)"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Proposition 17, p. 21, doi:10.5281/zenodo.24250501

↳ Used in: Proposition 18 (p. 38)

Proof

> $\mathcal{E}_{p_2,c_2} \circ \mathcal{E}_{p_1,c_1} = \mathcal{E}_{p_1p_2,c_1c_2}$ by composing the matrices of the channel above (6); the wait is $\mathcal{E}_{e^{-\gamma_m t},e^{-\eta_m t}}$; the read is the write with the two elements exchanged, and P_*, P_ϕ, c_* are symmetric under that exchange (the resonant transfer is moreover reciprocal). Bound each factor.

Proposition 18. (the cycle without reset, to first order) p. 22

Memory_Cycle_No_Reset.change_of_variables_ineq, Memory_Cycle_No_Reset.F_derivative_pi_half, Memory_Cycle_No_Reset.every_minimiser_below_pi_half, Cycle_Utility_Threshold.rho_concave, Cycle_Utility_Threshold.rho_zero_lt, Cycle_Utility_Threshold.rho_ge_rho_one, Cycle_Utility_Threshold.F_t_difference, Cycle_Utility_Threshold.F_t_ge, Cycle_Utility_Threshold.rho_slope_gt Fix a direction of non-negative Markovian rates and scale it by ε. Fix, independently of ε, a write–wait–read control with no reset ($|g| \le G$, any phase and detuning; a wait of time t with no coupling) whose zero-rate propagator returns the excitation to the qubit up to a known phase, corrected at the output. Put

$$\rho(P) = \tfrac{1}{3}\left[\gamma_q P + \phi_q P(2-P) + \gamma_m(1-P) + \phi_m(1-P^2)\right],$$
$$F_t(\theta) = \frac{2}{G}\int_0^\theta \rho(\cos^2 u)\,du + t\,\rho(\cos^2\theta).$$

Then $r_{\mathrm{cycle}} = r_1 + O(\varepsilon^2)$ for each control, with

$$r_1 \;\ge\; r_1^*(t) := \min_{0\le\theta\le\pi/2} F_t(\theta),$$

attained by a resonant write stopped at a minimiser θ_*, the wait, and the inverse resonant read (the same pulse with the coupling phase inverted, the phase of the wait compensated). If $\gamma_m + \phi_m > 0$, then $\theta_* < \pi/2$ for every $t > 0$: the optimal cycle never completes the swap, and $r_1^*(t)$ lies strictly below $F_t(\pi/2)$, the first-order part of the floor of Proposition 17. Accessing the memory pays at this order exactly for $t > t_c$, where, with $A(\theta) = \frac{2}{G}\int_0^\theta \rho(\cos^2 u)\,du$ and $d(\theta) = \rho(1) - \rho(\cos^2\theta)$,

$$t_c = \begin{cases} +\infty, & \gamma_m + \phi_m \ge \gamma_q + \phi_q, \\ \min\left\{ \dfrac{A(\theta)}{d(\theta)} \;:\; 0 < \theta \le \dfrac{\pi}{2},\; d(\theta) > 0 \right\}, & \gamma_m + \phi_m < \gamma_q + \phi_q : \end{cases}$$

in the first case $d \le 0$ and not accessing ($\theta = 0$) is optimal for every t; in the second t_c is finite, not accessing and a nonzero parking angle tie at t_c, and since $F_t'(0) = \frac{2}{G}\rho(1) > 0$ the onset is a jump to a finite angle. If $0 < \gamma_m + \phi_m < \gamma_q + \phi_q$, then every minimiser tends to $\pi/2$ as $t \to \infty$, and

$$\frac{\pi}{2} - \theta_* \sim \delta_* = \frac{\gamma_m + \phi_m}{Gt\,(\gamma_q + 2\phi_q - \gamma_m)}, \qquad F_t(\tfrac{\pi}{2}) - r_1^*(t) = \frac{(\gamma_m +}{3G^2(\gamma_q + 2}$$

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

change_of_variables_ineq · Isabelle/MemoryAccessFloors/Memory_Cycle_No_Reset.thy, line 36

```
theorem change_of_variables_ineq:
  fixes f theta theta' :: "real \<Rightarrow> real" and a b G ::
real
  assumes ab: "a \<le> b" and G: "0 < G" and fc: "continuous_on
UNIV f"
    and f0: "\<forall>s\<in>{a..b}. 0 \<le> f (theta s)"
    and der: "\<forall>s\<in>{a..b}. (theta has_vector_derivative
theta' s) (at s within {a..b})"
    and bd: "\<forall>s\<in>{a..b}. \<bar>theta' s\<bar> \<le> G"
  shows "(1 / G) * \<bar>integral {min (theta a) (theta b)..max
(theta a) (theta b)} f\<bar>
           \<le> integral {a..b} (\<lambda>s. f (theta s))"
```

F_derivative_pi_half · Isabelle/MemoryAccessFloors/Memory_Cycle_No_Reset.thy, line 191

```
theorem F_derivative_pi_half:
  "(F has_real_derivative (2 / (3 * G)) * (gm + fm)) (at (pi / 2))"
```

every_minimiser_below_pi_half · Isabelle/MemoryAccessFloors/Memory_Cycle_No_Reset.thy, line 261

```
corollary every_minimiser_below_pi_half:
  assumes pos: "0 < gm + fm" and th0: "th0 \<in> {0..pi / 2}" and
min: "\<forall>th\<in>{0..pi / 2}. F th0 \<le> F th"
  shows "th0 < pi / 2" and "F th0 < F (pi / 2)"
```

rho_concave · Isabelle/MemoryAccessFloors/Cycle_Utility_Threshold.thy, line 17

```
lemma rho_concave:
  assumes "0 \<le> fq" "0 \<le> fm"
  shows "concave_on {0..1} (rho gq fq gm fm)"
```

rho_zero_lt · Isabelle/MemoryAccessFloors/Cycle_Utility_Threshold.thy, line 47

```
lemma rho_zero_lt:
  assumes "0 \<le> fq" "0 \<le> fm" "gm + fm < gq + fq" "0 < P" "P
\<le> 1"
  shows "rho gq fq gm fm 0 < rho gq fq gm fm P"
```

rho_ge_rho_one · Isabelle/MemoryAccessFloors/Cycle_Utility_Threshold.thy, line 34

```
lemma rho_ge_rho_one:
  assumes "0 \<le> fq" "0 \<le> fm" "gq + fq \<le> gm + fm" "0 \
<le> P" "P \<le> 1"
  shows "rho gq fq gm fm 1 \<le> rho gq fq gm fm P"
```

F_t_difference · Isabelle/MemoryAccessFloors/Cycle_Utility_Threshold.thy, line 68

```
lemma F_t_difference:
  assumes "A 0 = 0"
  shows "F_t A r t \<theta> - F_t A r t 0 = A \<theta> - t * (r 1 -
r ((cos \<theta>)\<^sup>2))"
```

F_t_ge · Isabelle/MemoryAccessFloors/Cycle_Utility_Threshold.thy, line 73

```
lemma F_t_ge:
  assumes "A 0 = 0" "0 \<le> A \<theta>" "0 \<le> t" "r 1 - r ((cos
\<theta>)\<^sup>2) \<le> 0"
  shows "F_t A r t 0 \<le> F_t A r t \<theta>"
```

rho_slope_gt · Isabelle/MemoryAccessFloors/Cycle_Utility_Threshold.thy, line 91

```
lemma rho_slope_gt:
  fixes gq fq gm fm :: real
  assumes "0 \<le> fq" "0 \<le> fm" "gm + fm < gq + fq"
  shows "fq + fm < gq + 2 * fq - gm"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Proposition 18, p. 22, doi:10.5281/zenodo.24250501

↳ Not used by another result of the paper.

Proof

Along the lossless trajectory, put $P(s)=\cos^2\theta(s)$ for the population of the qubit. To first order the error is the sum of the contributions of single jumps; a relaxation or dephasing jump at time s leaves an error that depends on the path only through $P(s)$, because the lossless evolution is unitary in the one-excitation block and, the cycle being closed, the ideal output transported back to time s is the lossless state at s, so the overlap of the jumped state with it is fixed by $P(s)$ (without the closure a full write followed by no read leaves an error of order one at zero rates). This gives $r_1=\int\rho(P(s))\,ds$ over the whole cycle (the weight $P(2-P)$ of qubit dephasing collects the population and the coherence terms of r). With $\dot P=2\,\mathrm{Im}(g\bar a b)$ one has $|\dot P|\le 2|g|\sqrt{P(1-P)}$, that is $|\dot\theta|\le|g|\le G$. During the write θ runs from 0 to θ_w, so, as $\rho\ge 0$, $\int\rho\,ds\ge\frac{1}{G}\int\rho(\cos^2\theta)|\dot\theta|\,ds\ge\frac{1}{G}\int_0^{\theta_w}\rho(\cos^2 u)\,du$; the read gives the same bound, and the wait costs exactly $t\rho(\cos^2\theta_w)$. Hence $r_1\ge F_t(\theta_w)\ge r_1^*(t)$, with equality for $\dot\theta=G$, the resonant pulse. Since $\frac{d}{d\theta}\rho(\cos^2\theta)=-\rho'(\cos^2\theta)\sin 2\theta$ vanishes at $\pi/2$, $F_t'(\pi/2)=\frac{2}{G}\rho(0)=\frac{2}{3G}(\gamma_m+\phi_m)>0$, so no minimiser is at $\pi/2$. Next, $F_t(\theta)-F_t(0)=A(\theta)-t\,d(\theta)$, which gives t_c. When $\rho(0)<\rho(1)$, the set where $\rho<\rho(1)$ is an interval containing $P=0$ (ρ is concave, below), so $\{d>0\}$ is an interval with included right end $\pi/2$, where $d(\pi/2)=\rho(1)-\rho(0)>0$ and A/d is finite; at its excluded left end A/d diverges (either $d\to 0$ with $A>0$, or the end is $\theta=0$, where $A\simeq 2\rho(1)\theta/G$ and $d=O(\theta^2)$), so the minimum is attained away from zero. The density is concave in P ($\rho''=-\frac{2}{3}(\phi_q+\phi_m)\le 0$), so $\rho\ge\min\{\rho(0),\rho(1)\}$ on $[0,1]$: if $\rho(0)\ge\rho(1)$, that is $\gamma_m+\phi_m\ge\gamma_q+\phi_q$, then $d\le 0$ and $\theta=0$ is optimal; if $\rho(0)<\rho(1)$, then $P=0$ is the unique minimum of ρ, and since $F_t/t\to\rho(\cos^2\theta)$

uniformly every minimiser tends to $\pi/2$. Near $\pi/2$, with $\delta = \pi/2 - \theta$, $F_t = F_t(\pi/2) - \frac{2}{G}\rho(0)\delta + t\rho'(0)\delta^2 + O(\delta^3 + t\delta^4)$ and $\rho'(0) = (\gamma_q + 2\phi_q - \gamma_m)/3 > \frac{1}{3}(\phi_q + \phi_m) \geq 0$; minimising gives the stated δ_* and gain.

A Equal relaxation rates: an elementary proof

Proposition 19. Isabelle p. 33

> *Memory_Access_Dephasing.P1, Memory_Access_Dephasing.P3, Memory_Access_Extras.memory_dephasing_equal_rates_free Let* $\gamma_q = \gamma_m = \gamma$ *and* $\Phi < 4G$. *Put* $\omega(\beta) = 2G - \frac{\Phi}{2}\sin 2\beta$, $R(\beta) = -\int_0^\beta \Phi \sin^2/\omega$ *and* $\tau(\beta) = \int_0^\beta 1/\omega$. *For every control and every* T,
>
> $$p_b(T) \leq \max_{0 \leq \beta \leq \pi} \tfrac{1}{2}\, e^{-\gamma\tau(\beta)}\left(1 - e^{R(\beta)}\cos\beta\right) = P_\phi,$$
>
> *and* $p_b(T) \leq p_{\mathrm{res}}(T)$ *for every* $T \leq \tau(\pi/2)$.

Isabelle · ours **show**

P1 · Isabelle/MemoryAccessFloors/Memory_Access_Dephasing.thy, line 526

```
theorem P1:
  assumes "t \<in> {0..T}"
  shows "(X t)\<^sup>2 + (y t)\<^sup>2 \<le> 1"
```

P3 · Isabelle/MemoryAccessFloors/Memory_Access_Dephasing.thy, line 1067

```
theorem P3:
  assumes t: "t \<in> {0..T}" and th: "t \<le> t_half"
  shows "dp_Xr G \<Phi> t \<le> X t"
```

memory_dephasing_equal_rates_free · Isabelle/MemoryAccessFloors/Memory_Access_Extras.thy, line 830

```
theorem memory_dephasing_equal_rates_free:
  fixes G \<Phi> \<gamma> T t :: real and X y w :: "real \
<Rightarrow> real"
  assumes G: "0 < G" and Ph0: "0 \<le> \<Phi>" and Ph4: "\<Phi> <
4 * G" and g: "0 \<le> \<gamma>" and T: "0 \<le> T"
    and dX: "\<And>t. t \<in> {0..T} \<Longrightarrow> (X
has_real_derivative - 2 * w t * y t) (at t within {0..T})"
    and dy: "\<And>t. t \<in> {0..T} \<Longrightarrow> (y
has_real_derivative - \<Phi> * y t + 2 * w t * X t) (at t within
{0..T})"
    and wb: "\<And>t. t \<in> {0..T} \<Longrightarrow> \<bar>w t\
<bar> \<le> G"
    and X0: "X 0 = 1" and y0: "y 0 = 0" and t: "t \<in> {0..T}"
```

```
  shows "\<exists>s. 0 \<le> s \<and> exp (- \<gamma> * t) * (1 -
X t) / 2 \<le> exp (- \<gamma> * s) * (1 - dp_Xr G \<Phi> s) / 2"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Proposition 19, p. 33, doi:10.5281/zenodo.24250501

↳ Not used by another result of the paper.

Proof

Let $X = e^{\gamma t}(p_a - p_b)$, $y = 2e^{\gamma t}|\rho_{ab}|$ and $\rho^2 = X^2 + y^2$; note $e^{\gamma t}(p_a + p_b) = 1$. From (2), $\frac{d}{dt}\rho^2 = -2\Phi y^2$ whatever the control, and $|\dot{X}| \le 2Gy$. In polar form $X = \rho\cos\beta$, $y = \rho\sin\beta$:

$$\frac{d}{dt}\ln\rho = -\Phi\sin^2\beta, \qquad \dot{\beta} = 2w - \tfrac{\Phi}{2}\sin 2\beta, \qquad |w| \le G,$$

which is the structure of Theorem 1, and the exact form of the first-order inequality of [34 , Eq. (25)] (there the term in $\sin 2\beta$ is dropped); the resonant pulse is $w \equiv G$, and the end points are handled as there. The angle need not be continued across the half-line $y = 0$, $X \le 0$: if the trajectory touches it at a time s, then for $t \ge s$, $X(t)^2 \le \rho(t)^2 \le \rho(s)^2 = X(s)^2$ gives $X(t) \ge X(s)$ and so $p_b(t) = \frac{1}{2}e^{-\gamma t}(1 - X(t)) \le p_b(s)$; it is enough to prove the bound up to the first such time, and there by continuity. (i) $W = \ln\rho - R(\beta)$ is non-increasing, since $\dot{W} = -\Phi\sin^2\beta\cdot 2(G - w)/\omega$; and by comparison of scalar equations $\beta(t) \le \beta_{\mathrm{res}}(t)$, so $T \ge \tau(\beta(T))$. (ii) If $X(T) \le 0$ then $2p_b = e^{-\gamma T}(1 + \rho|\cos\beta|) \le e^{-\gamma\tau(\beta)}(1 + e^{R(\beta)}|\cos\beta|) = 2p_{\mathrm{res}}(\tau(\beta))$. (iii) If $X(T){>}0$ and $T \ge \tau(\pi/2)$ then $2p_b{<}e^{-\gamma T} \le e^{-\gamma\tau(\pi/2)} = 2p_{\mathrm{res}}(\tau(\pi/2))$. (iv) If $X(T){>}0$ and $T{<}\tau(\pi/2)$ then $\frac{d}{dt}\ln X = -2w\tan\beta \ge -2G\tan\beta_{\mathrm{res}}(t) = \frac{d}{dt}\ln X_{\mathrm{res}}$, so $X(T) \ge X_{\mathrm{res}}(T)$ and $p_b(T) \le p_{\mathrm{res}}(T)$.

Figure 5

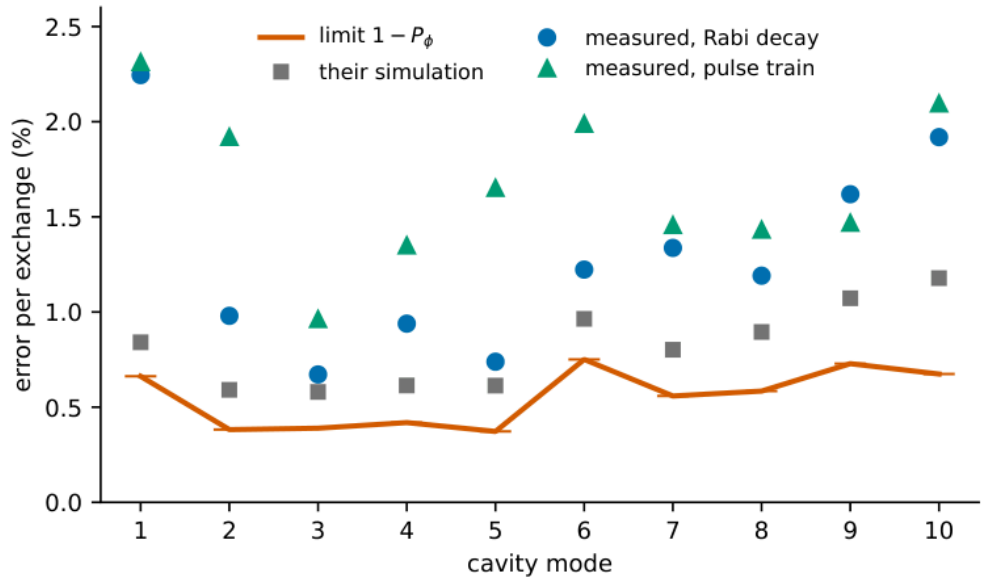
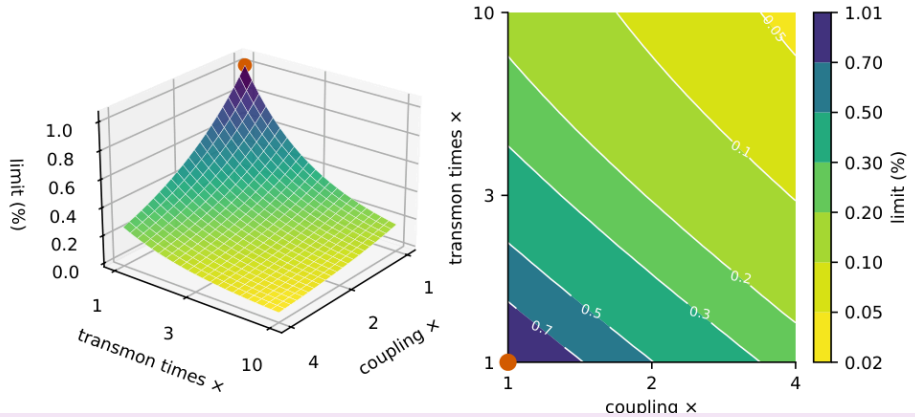


Table 3 as a picture: no measured value and no simulated value falls below the limit.

Figure 6



The limit r_{res} for mode 1 of [47] as the coupling and the transmon coherence times (T_1 and T_2 together) are multiplied by a factor; the phonon mode is kept as it is. Left: the surface. Right: the same as level curves, in per cent. The dot is the present device.

Tables

Every table of the paper, in its order: click one to unfold it, with where the paper cites it, how its numbers were checked, and its data as CSV. The typeset table is one click away, on its page of the PDF.

Table 1. The phonon-memory processor of [MRQC]: $G = 2\pi \times 0.296$ MHz, transmon $T_1 = 30\,\mu$s and echo $T_2 = 23\,\mu$s, three phonon modes. The difference $r_{\mathrm{res}} - r_{\mathrm{sep}}$ is $4.2 \cdot 10^{-6}$, $4.2 \cdot 10^{-6}$ and $4.1 \cdot 10^{-6}$ in absolute infidelity. Measured: $1.71\,\%$ per exchange, average of the three modes, by randomised benchmarking.

mode	$1-P_*$	$1-P_\phi$	$1-c_*$	r_{sep} (%)	r_{res} (%)
1 ($T_1 = 196$, $T_2 = 368\,\mu$s)	0.0161	0.0216	0.0192	1.0001	1.0005
2 ($T_1 = 137$, $T_2 = 250\,\mu$s)	0.0170	0.0226	0.0197	1.0335	1.0339
3 ($T_1 = 64$, $T_2 = 127\,\mu$s)	0.0204	0.0259	0.0213	1.1429	1.1433

Colour: blue shade = the number's size within its column.

p. 23 in the PDF ⬇ CSV cited in §1 The question cited in §5 The write channel cited in §7 Which parameter is worth improving cited in §9 What backs each statement

Table 2. Model benchmark and applicability. Each device is set against the limit for the quantity closest to what it reports; only the first row is a direct comparison in the same metric. HBAR gives lifetimes only, the Rutgers coupling is inferred from the area and duration of a pulse with ramps (not the instantaneous maximum), and the randomised benchmarking of the cascaded memory is a protocol-level error, not the population of an isolated transfer. In average infidelity r_{res} the limits are: 1.00–$1.14\,\%$ [MRQC]; 0.59–$0.78\,\%$ with echo times and 0.62–$0.80\,\%$ with Ramsey times [HBAR]; $0.16\,\%$ (echo) and $0.41\,\%$ (Ramsey) [Cascaded]; $17.6\,\%$ and $18.2\,\%$ for oscillators A and B with the Ramsey time of the transmon and the echo time of the oscillator, $17.8\,\%$ and $18.4\,\%$ with Ramsey times for both [Caltech] (there the resonant pulse is not proved optimal, and the certified lower bounds r_{sep} are 17.48, 18.12, 17.70 and $18.27\,\%$, about 0.1 point lower); Table 3 for [Rutgers]. For [HBAR], the six modes of their sample B.2 with the times read from their Fig. 4(e); that the exchange lasts $\pi/(2g_0)$ is read from the period of their Fig. 10(b), $1.655 \pm 0.02\,\mu$s against $\pi/g_0 = 1.678\,\mu$s.

experiment	quantity measured	measured	limit for that quantity
ETH phonon processor [MRQC], three modes; resonant exchange	average infidelity of one exchange, inferred from two randomised benchmarkings	1.71 % (mean of the modes)	1.00–1.14 %
ETH long-lived HBAR [HBAR], six modes; resonant exchange	none: the paper reports lifetimes	not reported	0.59–0.78 % (average infidelity, echo)
Rutgers multimode cavity [Rutgers], ten modes; sideband $\lvert f,0\rangle \leftrightarrow \lvert g,1\rangle$	population lost per π pulse	0.67–2.2 % (Rabi decay), 0.96–2.3 % (pulse train)	0.37–0.75 %
Cascaded memory [Cascaded], buffer ↔	failure to recover the initial state, per full exchange (raw	0.358 %	population not transferred: 0.32 %

mode S2; parametric beam splitter	beam-splitter RB); the part that survives post-selection is 0.037 %		(relaxation only), 0.39 % (echo dephasing)
Caltech mechanical memory [Caltech], oscillator B; resonant exchange	population of one phonon after preparation	0.506 ± 0.033	at most 0.721 (any dephasing), 0.614 (Markovian, Ramsey)

Colour: red = no / fails.

p. 25 in the PDF | ⬇ CSV | cited in §6 Five devices

Table 3. Rutgers device [Rutgers]: error of one exchange in population, in %. $G = \pi/(2\tau_\pi)$ with τ_π from their Fig. 2(e) (it includes the ramps); transmon $T_1(f \to e) = 28.85\ \mu$s and Ramsey T_2 of the g–f superposition $36.58\ \mu$s, measured without the pump; cavity times from their Table S1. The last three columns are read from the markers of their Fig. 2(f).

mode	τ_π (ns)	limit $1 - P_\phi$	simulation [Rutgers]	Rabi decay	pulse train
1	328.1	0.662	0.841	2.245	2.313
2	189.2	0.382	0.591	0.980	1.920
3	193.0	0.390	0.580	0.672	0.963
4	207.2	0.419	0.614	0.939	1.350
5	184.2	0.373	0.613	0.739	1.652
6	369.7	0.751	0.964	1.223	1.989
7	272.1	0.559	0.802	1.337	1.457
8	287.2	0.585	0.895	1.191	1.433
9	359.2	0.729	1.072	1.619	1.469
10	332.9	0.674	1.178	1.919	2.096

Colour: blue shade = the number's size within its column.

p. 26 in the PDF | ⬇ CSV | cited in §6 Five devices | cited in §9 What backs each statement

Table without a number, in the text of §7 Which parameter is worth improving

change	r_{res} (%)	relative
phonon times $\times 10$	0.9346	0.93
transmon T_2 raised to $2T_1$	0.5393	0.54
coupling $\times 2$	0.5040	0.50
transmon times $\times 10$	0.1685	0.17

transmon $\times 10$ and coupling $\times 2$	0.0843	0.08

Colour: blue shade = the number's size within its column.

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Table without a number, in the text of §C Both couplings at once: model-dependent numerical comparisons

scheme	error (%)
two exchanges in series, coupling g	3.3
both sidebands of one tone ($k = gJ_1(1.841) = 0.582\,g$ each), two modes only	2.6
the same with the neighbouring modes at $+1$ and -2 free spectral ranges	8.6
two-harmonic modulation that cancels the third sideband, with neighbours	3.0

Colour: blue shade = the number's size within its column.

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Table without a number, in the text of §D Formula sheet

quantity	formula
maximum population, relaxation only	$P_* = \frac{4G^2}{4G^2+\gamma_q\gamma_m}\exp\!\Big(-\frac{\bar\gamma}{\Omega}\arctan\frac{2\Omega}{\bar\gamma}\Big)$ with $\Omega^2 = G^2-(\gamma_q-\gamma_m)^2/16>0$ and $\bar\gamma=(\gamma_q+\gamma_m)/2$; in general $P_* = \frac{4G^2}{4G^2+\gamma_q\gamma_m}e^{-\bar\gamma t_*}$ with t_* as in Theorem 1 (over damping: $P_* = \frac{4G^2}{4G^2+\gamma_q\gamma_m}\big(\frac{\bar\gamma-2\kappa}{\bar\gamma+2\kappa}\big)^{\bar\gamma/(2\kappa)}$, $\kappa^2=-\Omega^2$)
first order	$P_* \approx \exp(-\pi(\gamma_q+\gamma_m)/(4G))$, and P_* is never below it
maximum coherence	$c_* = \frac{G}{\sqrt{G^2+\eta_q\eta_m}}\exp\!\Big(-\frac{\eta_q+\eta_m}{2\omega_c}\arctan\frac{2\omega_c}{\eta_q+\eta_m}\Big)$ with $\eta = 1/T_2$ and $\omega_c^2 = G^2-(\eta_q-\eta_m)^2/4>0$; at critical and over damping use the continuous or hyperbolic forms of Theorem 1 with rates $2\eta_q, 2\eta_m$ and take the square root of the population maximum
maximum population with dephasing	$P_\phi = \sup_t (e^{Mt})_{21}$, M in 5
fixed duration T	$p \le B_p(T) = e^{-\mu T}\max_{s\le T} e^{\mu s}p_{\mathrm{res}}(s)$, $c \le B_c(T) = e^{-\nu T}\max_{s\le T} e^{\nu s}c_{\mathrm{res}}(s)$
write channel	$r = 1-F = \frac{1-p}{6}+\frac{1-c}{3}$, $1-F_e = \frac{3}{2}\,r$
limits	$r_{\mathrm{sep}} \le \frac{1}{2}-\frac{1}{6}\sup_T\big[B_p+2B_c\big] \le r_{\mathrm{opt}} \le r_{\mathrm{res}}$ with $r_{\mathrm{sep}} = \frac{1-P_\phi}{6}+\frac{1-c_*}{3}$; $r_{\mathrm{res}}-r_{\mathrm{sep}} = \frac{\Phi^2}{48G^2}+O(\varepsilon^3)$, $=0$ if $\Phi=0$ (Prop. 15)
worst case	$1-F_{\mathrm{wc}} \ge 1-P_\phi$, with equality if $3P_\phi - 2c_{\mathrm{res}}(t_p) \le 1$
quasi-static offsets as the only	$r \le \frac{\mathbb{E}\,\Delta^2}{12G^2}+\frac{\pi^2\,\mathbb{E}\,\bar\xi^{\,2}}{24G^2}$; one noisy element: $r \le \frac{8+\pi^2}{96}\,\frac{\sigma^2}{G^2}$

noise (no relaxation, no Markovian pure dephasing; Prop. 14(a)), square pulse	
quasi-static Gaussian noise, every pulse	$r \geq \frac{1}{2} - \frac{1}{6}\sup_T\left[B_p^{(0)} + 2e^{-v_\perp T^2/2}B_c^{(0)}\right]$, $v_\perp = \frac{\sigma_q^2\sigma_m^2}{\sigma_q^2+\sigma_m^2}$ (independent centred Gaussian offsets, Prop. 14(b); $v_\perp = 0$ if both variances vanish)
resonant pulse exactly optimal	along a fixed rate direction, $r_{\mathrm{opt}} = r_{\mathrm{res}}$ for sufficiently small rates if $4\min(\eta_q,\eta_m) > \phi_q + \phi_m$; else $r_{\mathrm{res}} - r_{\mathrm{time}} = \frac{(\Phi-4\nu)^2}{48G^2} + O(\varepsilon^3)$ (Prop. 16)
very unequal dephasing, no relaxation	$\lvert\phi_q - \phi_m\rvert \geq 2\sqrt{5}G$: $r_{\mathrm{opt}} = 5/12$ iff $\phi_q\phi_m \geq 3G^2$ (fixed phase, zero detuning, $0 \leq g \leq G$; Prop. 12); for $1 < \delta < \sqrt{5}$ ($G = 1$), $r_{\mathrm{opt}} = 5/12$ if $\phi_q\phi_m \geq P_{\mathrm{suf}}(\delta)$, $P_{\mathrm{suf}}(2) = 3.6911\ldots$ (Cor. 13)
memory cycle with reset	$r_{\mathrm{cycle}} \geq \frac{1-{P_\phi}^2 e^{-\gamma_m t}}{6} + \frac{1-{c_*}^2 e^{-\eta_m t}}{3}$ (Prop. 17)
memory cycle without reset, first order	$r_1 \geq \min_\theta\left[\frac{2}{G}\int_0^\theta \rho(\cos^2 u)\,du + t\rho(\cos^2\theta)\right]$, $\rho(P) = \frac{1}{3}[\gamma_q P + \phi_q P(2-P) + \gamma_m(1-P) + \phi_m(1-P^2)]$; closing cycles; if $\gamma_m + \phi_m \geq \gamma_q + \phi_q$ no access is optimal for every wait; otherwise access pays for $t > t_c = \min_{d>0} A(\theta)/d(\theta)$; if $0 < \gamma_m + \phi_m < \gamma_q + \phi_q$, parking short of the swap by $\pi/2 - \theta_* \sim \delta_* = \frac{\gamma_m+\phi_m}{Gt(\gamma_q+2\phi_q-\gamma_m)}$ as $t \to \infty$ (Prop. 18)
stopping rule	stop the resonant pulse when $\lvert a\rvert/\lvert b\rvert = \gamma_m/(2G)$

⬇ CSV

References

1 I. Arrazola, P. Bertet, Y. Chu and P. Rabl, Engineering protected cavity-QED interactions through pulsed dynamical decoupling, npj Quantum Inf. 11, 197 (2025); we used arXiv:2505.02929v2.

2 D. Basilewitsch, J. Fischer, D. M. Reich, D. Sugny and C. P. Koch, Fundamental bounds on qubit reset, Phys. Rev. Research 3, 013110 (2021); we used arXiv:2001.09107v2.

3 F. Beaudoin, A. Blais and W. A. Coish, Hamiltonian engineering for robust quantum state transfer and qubit readout in cavity QED, New J. Phys. 19, 023041 (2017); we used arXiv:1602.05090v3.

4

A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin and R. J. Schoelkopf, Cavity quantum electrodynamics for superconducting electrical circuits: an architecture for quantum computation, Phys. Rev. A 69, 062320 (2004). 🔎 find it

5

U. Boscain, T. Chambrion and J.-P. Gauthier, On the K+P problem for a three-level quantum system: optimality implies resonance, J. Dyn. Control Syst. 8, 547–572 (2002); we used arXiv:math/0204233.

6

B. Bonnard, M. Chyba and D. Sugny, Time-minimal control of dissipative two-level quantum systems: the generic case, IEEE Trans. Automat. Control 54, 2598–2610 (2009); we used arXiv:0809.4138v1.

7

B. Bonnard and D. Sugny, Time-minimal control of dissipative two-level quantum systems: the integrable case, SIAM J. Control Optim. 48, 1289–1308 (2009); we used arXiv:0809.4130v1.

8

A. B. Bozkurt, O. Golami, Y. Yu, H. Tian and M. Mirhosseini, A mechanical quantum memory for microwave photons, arXiv:2412.08006v1 (2024).

9

H. K. Cummins, G. Llewellyn and J. A. Jones, Tackling systematic errors in quantum logic gates with composite rotations, Phys. Rev. A 67, 042308 (2003); we used arXiv:quant-ph/0208092v1.

10

D. Daems, A. Ruschhaupt, D. Sugny and S. Guérin, Robust quantum control by a single-shot shaped pulse, Phys. Rev. Lett. 111, 050404 (2013); we used arXiv:1304.4016, Eqs. (7)–(10).

11

H.-Y. Huang, M. Broughton, J. Cotler, S. Chen, J. Li, M. Mohseni, H. Neven, R. Babbush, R. Kueng, J. Preskill and J. R. McClean, Quantum advantage in learning from experiments, Science 376, 1182–1186 (2022); we used arXiv:2112.00778v1.

12

R. Garcia-Belles et al., Loss mechanisms in high-coherence multimode mechanical resonators coupled to superconducting circuits, PRX Quantum 7, 033013 (2026), doi:10.1103/vszq-z1sh; we used arXiv:2602.22117v1.

13

A. V. Gorshkov, A. André, M. Fleischhauer, A. S. Sørensen and M. D. Lukin, Universal approach to optimal photon storage in atomic media, Phys. Rev. Lett. 98, 123601 (2007); arXiv:quant-ph/0604037.

14

A. V. Gorshkov, A. André, M. D. Lukin and A. S. Sørensen, Photon storage in Λ-type optically dense atomic media. I. Cavity model, Phys. Rev. A 76, 033804 (2007); arXiv:quant-ph/0612082.

15

P. Groszkowski, M. Koppenhöfer, H.-K. Lau and A. A. Clerk, Reservoir-engineered spin squeezing: macroscopic even-odd effects and hybrid-systems implementations, Phys. Rev. X 12, 011015 (2022); we used arXiv:2104.10363v2.

16

X. Huang, X. Zhang and D. Zhao, Beyond spectroscopic strong coupling: operational benchmarks for cavity-to-magnon quantum transfer, arXiv:2610.09921v1 (2026).

17

C. Marín, Robust qubit–memory transfer against quasi-static frequency noise costs time and exposure: exact bounds from Dubins curves, companion paper, Zenodo (2026), doi:10.5281/zenodo.24250503.

18

J. Huang et al., Fast sideband control of a weakly coupled multimode bosonic memory, arXiv:2503.10623v1 (2025).

19

N. Khaneja, R. Brockett and S. J. Glaser, Time optimal control in spin systems, Phys. Rev. A 63, 032308 (2001). 🔎 find it

20

A. Auffèves, D. Gerace, J.-M. Gérard, M. França Santos, L. C. Andreani and J.-P. Poizat, Controlling the dynamics of a coupled atom-cavity system by pure dephasing, Phys. Rev. B 81, 245419 (2010); we used arXiv:1002.3753.

21

B. M. Escher, G. Bensky, J. Clausen and G. Kurizki, Optimized control of quantum state transfer from noisy to quiet qubits, J. Phys. B: At. Mol. Opt. Phys. 44, 154015 (2011); we used the preprint arXiv:1010.5211v1, whose title (“Dynamical control of quantum state transfer within hybrid open systems”) and author list (with L. Davidovich) differ from the published version.

22

A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou and D. N. Christodoulides, Observation of $\mathcal{PT}$-symmetry breaking in complex optical potentials, Phys. Rev. Lett. 103, 093902 (2009). 🔎 find it

23

B. A. Tay and Y. S. H’ng, Entanglement generation across exceptional points in a two-qubit open quantum system: the role of initial states, Phys. Rev. E

112, 024133 (2025); we used arXiv:2508.16101v1, Eqs. (12)–(13) and (B2)–(B4).

24 J. Inoue, Upper limit of pure dephasing time of a paired-qubit with anisotropic Heisenberg coupling, J. Phys. Soc. Jpn. 93, 073801 (2024), doi:10.7566/JPSJ.93.073801; Eqs. (5) and (8).

25 F. Kayatz, J. Mielke and G. Burkard, Resonator-mediated quantum gate between distant charge qubits, Quantum Sci. Technol. 9, 045048 (2024); we used arXiv:2406.06442v2, Sec. IV and Fig. 3(b).

26 A. Kubica, A. Haim, Y. Vaknin, H. Levine, F. Brandão and A. Retzker, Erasure qubits: overcoming the T_1 limit in superconducting circuits, Phys. Rev. X 13, 041022 (2023); we used arXiv:2208.05461v1.

27 M. Lapert, Y. Zhang, S. J. Glaser and D. Sugny, Towards the time-optimal control of dissipative spin-1/2 particles in nuclear magnetic resonance, J. Phys. B: At. Mol. Opt. Phys. 44, 154014 (2011); we used the authors' version hal-00642391. 🔎 find it

28 M. Lapert, Y. Zhang, M. Braun, S. J. Glaser and D. Sugny, Singular extremals for the time-optimal control of dissipative spin 1/2 particles, Phys. Rev. Lett. 104, 083001 (2010); we used arXiv:1003.0288v1.

29 M. Lapert, E. Assémat, S. J. Glaser and D. Sugny, Understanding the global structure of two-level quantum systems with relaxation: vector fields organized through the magic plane and the steady-state ellipsoid, Phys. Rev. A 88, 033407 (2013) (abstract only). 🔎 find it

30 E. B. Lee and L. Markus, Foundations of Optimal Control Theory, Wiley, 1967. 🔎 find it

31 H. Levine, A. Haim, J. S. C. Hung, N. Alidoust et al., Demonstrating a long-coherence dual-rail erasure qubit using tunable transmons, Phys. Rev. X 14, 011051 (2024); we used arXiv:2307.08737v3.

32 Z. Li et al., A cascaded random access quantum memory, arXiv:2503.13953v2 (2026).

33 L. V. Lokutsievskiy, A. N. Pechen and M. I. Zelikin, Time-optimal state transfer for an open qubit, J. Phys. A: Math. Theor. 57, 275302 (2024); we used arXiv:2401.10099v1.

34

X.-J. Lu, X. Chen, A. Ruschhaupt, D. Alonso, S. Guérin and J. G. Muga, Fast and robust population transfer in two-level quantum systems with dephasing noise and/or systematic frequency errors, Phys. Rev. A 88, 033406 (2013); we used arXiv:1305.6127v1.

35

S. Luo, Y. Fang, Y. Wang and S. Chesi, Principle of least action for quasi-adiabatic state transfers with dissipation, arXiv:2402.12807v2 (2024).

36

J. Minguzzi, J. C. Owens, O. Painter, M. H. Matheny et al., Sub-MHz SWAP spectroscopy of two-level systems in superconducting qubits, arXiv:2610.11125v1 (2026).

37

V. Mukherjee, A. Carlini, A. Mari, T. Caneva, S. Montangero, T. Calarco, R. Fazio and V. Giovannetti, Speeding up and slowing down the relaxation of a qubit by optimal control, Phys. Rev. A 88, 062326 (2013); we used arXiv:1307.7964v1.

38

M. A. Nielsen, A simple formula for the average gate fidelity of a quantum dynamical operation, Phys. Lett. A 303, 249 (2002). 🔎 find it

39

M. Papič, A. Auer and I. de Vega, Characterizing physical error contributions of quantum gates, arXiv:2305.08916v3.

40

I. Pietikäinen, O. Černotík, A. Eickbusch, A. Maiti, J. W. O. Garmon, R. Filip and S. M. Girvin, Strategies and trade-offs for controllability and memory time of ultra-high-quality microwave cavities in circuit quantum electrodynamics, PRX Quantum 5, 040307 (2024); we used arXiv:2403.02278v2.

41

L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience, 1962. 🔎 find it

42

D. Stefanatos, N. Khaneja and S. J. Glaser, Optimal control of coupled spins in the presence of longitudinal and transverse relaxation, Phys. Rev. A 69, 022319 (2004). 🔎 find it

43

D. Sugny, C. Kontz and H. R. Jauslin, Time-optimal control of a two-level dissipative quantum system, Phys. Rev. A 76, 023419 (2007); we used arXiv:0708.3794v1.

44

L. Van Damme, Q. Ansel, S. J. Glaser and D. Sugny, Robust optimal control of two-level quantum systems, Phys. Rev. A 95, 063403 (2017); we used arXiv:1704.07653.

45

Y.-D. Wang, R. Zhang, X.-B. Yan and S. Chesi, Optimization of STIRAP-based state transfer under dissipation, New J. Phys. 19, 093016 (2017); we used arXiv:1603.01701v1 (authors there: Wang, Yan and Chesi).

46

R. Wang, J. Ding, C. Wang, Y. Zhang et al., Dressed-state relaxation in coupled qubits as a source of two-qubit gate errors, arXiv:2601.11316v1 (2026).

47

Y. Yang, I. Kladarić et al., Mechanical resonator-based quantum computing, arXiv:2601.07825v1 (2026).