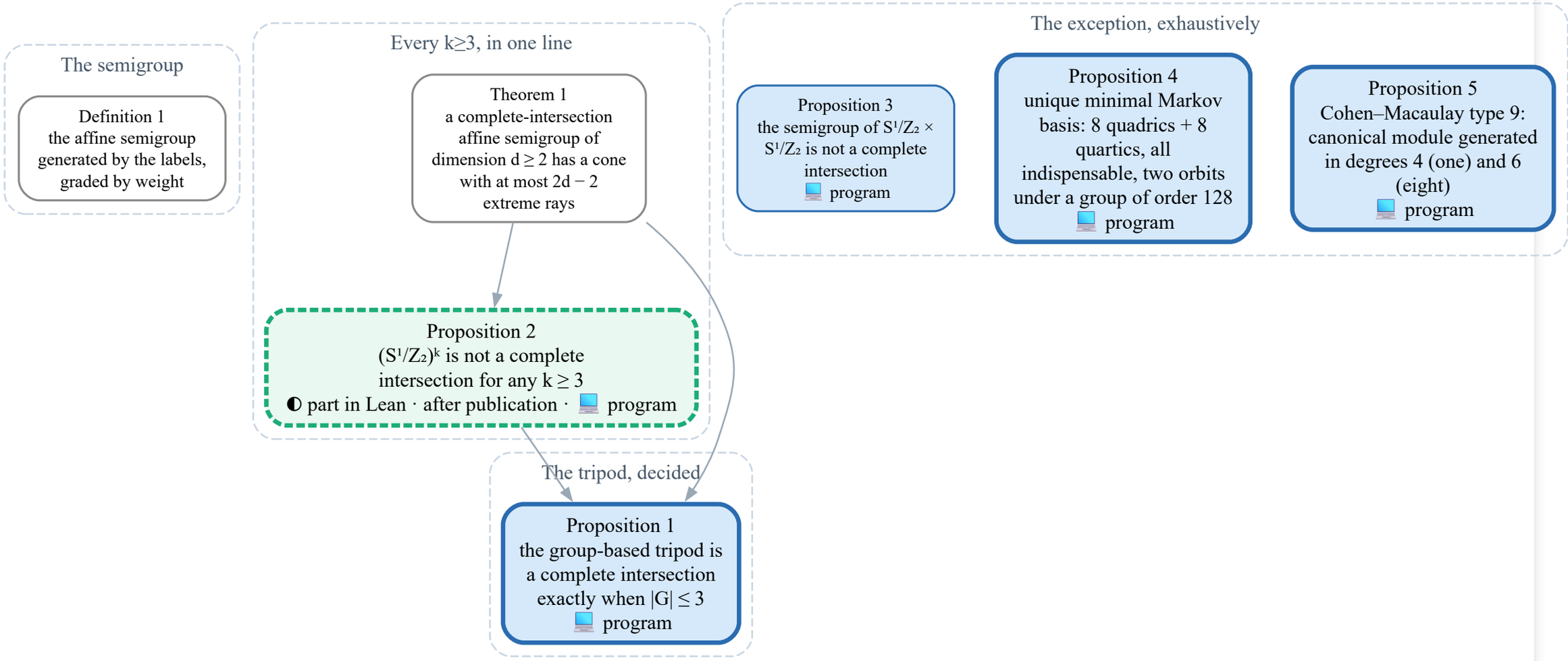
# Map of *An Affine Semigroup from Orbifold Boundary Conditions: cut, phylogenetic and hierarchical models in the unit-weight sector, and weighted configurations beyond them (Part IX-B)*

Combinatorial commutative algebra · affine semigroups and toric ideals

**7** results | ✅ **0** entirely in Lean | ◐ **1** partly in Lean | 💻 **5** checked by a program | written proof only | (a result can carry several tags)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



## What the paper contributes

The equivalence classes of orbifold boundary conditions are the elements of an **affine semigroup**: one generator per label of the alphabet of Part IX-A, graded by weight, embedded by its local data at the fixed points. The note identifies that semigroup and decides which of its cases are complete intersections.

- **The dictionary** (unit-weight sector), as an equality of point configurations: the cut configurations of $P_3$, $C_4$ and the wheel $W_4$ (Sturmfels–Sullivant); group-based models on claw trees; a mixed-order variant not located in the literature; binary hierarchical models of cross-polytope boundaries.
- **Thirteen complete intersections** with explicit gluing trees (unitary, orthogonal and symplectic alphabets); seven are weighted, outside every framework of the dictionary.
- **The tripod, decided** (Proposition 1): the group-based model on the tripod is a complete intersection exactly when $|G| \leq 3$, for every finite abelian group.
- **Every $k \geq 3$** (Proposition 2): $(S^1/\mathbb{Z}_2)^k$ is not a complete intersection, by the extreme-ray bound of Fischer–Morris–Shapiro (Theorem 1, cited), and not a cut ideal, by a count of edges.
- **The exceptional case** $S^1/\mathbb{Z}_2 \times S^1/\mathbb{Z}_2$: not a complete intersection by an exhaustive sweep (Proposition 3); a unique minimal Markov basis of 8 + 8 indispensable binomials in two orbits under an automorphism group of order 128 (Proposition 4); Cohen–Macaulay type 9 (Proposition 5).
- **Inherited**, and said so in the Ledger: the gluing criterion and the extreme-ray bound, the cut-ring invariants of $W_4$ (normality, degree 64, codimension 7), the group-based codimensions.

🚀 **Start here: Proposition 1**: the group-based tripod is a complete intersection exactly when $|G| \leq 3$.

**Main results:**

- **Proposition 1**: the group-based tripod is a complete intersection exactly when $|G| \leq 3$
- **Proposition 2**: $(S^1/Z_2)^k$ is not a complete intersection for any $k \geq 3$
- **Proposition 4**: unique minimal Markov basis: 8 quadrics + 8 quartics, all indispensable, two orbits under a group of order 128
- **Proposition 5**: Cohen–Macaulay type 9: canonical module generated in degrees 4 (one) and 6 (eight)

## 🧭 Fields: where the paper lives and what it connects

**Tools taken from other branches**

- **Affine semigroups (complete intersections)** → the gluing criterion and the extreme-ray bound of Fischer–Morris–Shapiro (Cor. 3.4), Delorme in rank one, the Hilbert

series of a gluing (Assi–García-Sánchez–Ojeda) — Proposition 1, Proposition 2, Proposition 3
- **Cut ideals and algebraic statistics** → cut ideals of graphs (Sturmfels–Sullivant 2008, Table 1 for the wheel $W_4$), Engström's $K_4$-minor theorem, group-based phylogenetic models (Sturmfels–Sullivant 2005) and their local description (Casanellas–Fernández-Sánchez–Michałek) — Proposition 1, Proposition 3, Proposition 4
- **Markov bases and lattice polytopes** → fibre connectivity and indispensable binomials; Ehrhart theory and Stanley's non-negativity of h*; Danilov–Stanley canonical module of a normal semigroup ring — Proposition 4, Proposition 5
- **Exact computation** → exhaustive sweeps over partitions (Hermite normal form, graded reachability), integer Farkas certificates for extreme rays, fibre enumeration, automorphism group of the configuration; every script with a decoy — Proposition 1, Proposition 3, Proposition 4, Proposition 5

**What the results serve**

- **Orbifold boundary conditions in gauge–Higgs unification** ← the relations among the classes whose generators Part IX-A determines: equivalence relations of the physics sources read as gluing elements — Proposition 3, Proposition 4
- **Weighted (non-abelian) flow configurations** ← gluing trees of the weighted alphabets as first data on the question the note ends on: which weighted flow configurations arise from a group