Subsets of F_p with two prescribed moments, Part II: every exponent — the classification, the Cauchy-Liouville-Mirimanoff polynomials that decide the column n=3, and where this moment subset sum stops being PORC

Arithmetic geometry over finite fields · subsets with prescribed power sums

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Statements

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At a glance

What the paper contributes

$C_k(n,p)$ counts the n-subsets of $\mathbb{F}_p$ with $\Sigma t = \Sigma t^k = 0$. Part I settled $k = 3$; this note settles every $k \geq 3$.

- **The singularity criterion** (Proposition 2, Theorem 4): for a length-four stratum, singularity depends on k only through $\gcd(k-1, 6)$, by Mann's theorem, and the set of n with all length-four strata singular takes exactly three values.
- **…is the wrong criterion** (Proposition 5): what kills weight one is a genus-zero normalisation; at $k = 5$, $n = 6$ a singular stratum has genus 5. With the corrected criterion, $C_k(n,p)$ is Artin–Tate exactly for $n \in \{3,4,6\}$ ($k = 3$), $\{3,4\}$ ($k = 5$) and $n = 3$ otherwise (Theorem 8), and the surviving counts are written down (Theorem 9, Proposition 11).
- **The column n = 3** (Theorem 12, Propositions 15–16, Corollary 17): one term per root of Q_k whose cubic $Y^3 + uY - u$ splits; elementary exactly for $k \in \{3, 4, 5, 7\}$, because $\deg Q_k + 1 = \dim M_{2k}(SL_2(\mathbb{Z}))$.
- **Not PORC** (Proposition 13, Lemmas 14 and 21): some fibre has group S_3 — unconditionally, with forty printed certificates for $6 \leq k \leq 46$ — and a permutation-character identity shows it cannot cancel. Observation 20 separates the two failures of PORC by weight (Isham has the coexistence).
- The polynomials Q_k and the reduction to u are Helou's, Tzermias's and Nanninga's; the paper claims the arithmetic over $\mathbb{F}_p$.

🚀 **Start here: Theorem 8**: $C_k(n,p)$ is Artin–Tate exactly for $n \in 3,4,6$ ($k = 3$), $3,4$ ($k = 5$), 3 (every other k).

Main results:

- **Theorem 8**: $C_k(n,p)$ is Artin–Tate exactly for $n \in 3,4,6$ ($k = 3$), $3,4$ ($k = 5$), 3 (every other k)
- **Theorem 12**: the column $n = 3$ for every k, through Q_k; elementary exactly for $k \in 3, 4, 5, 7$
- **Corollary 17**: $C_k(3,p)$ is elementary iff $\dim M_{2k}(SL_2(Z)) = 1$, i.e. at the weights 6, 8, 10, 14

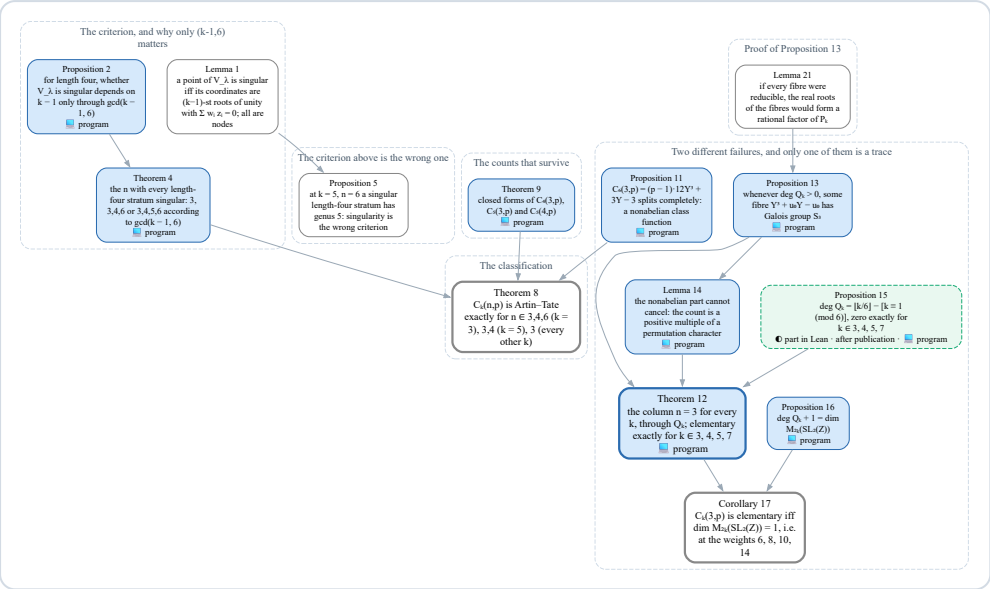
14 results | ✅ **0** entirely in Lean | ◐ **1** partly in Lean | 💻 **9** checked by a program | written proof only

(a result can carry several tags)

🗺️ Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each

statement's “Evidence & scope” block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



📐 Setting (copied from the paper, abstract, §1–§6)

(Abstract) Fix $k \geq 3$ and an odd prime p, and let $C_k(n,p)$ count the n-subsets $T \subseteq \mathbb{F}_p$ with $\sum_{t\in T} t = \sum_{t\in T} t^k = 0$. *(§1)* The Li–Wan partition sieve attaches to each partition $\lambda = (w_1, \dots, w_l) \vdash n$ the variety

$$V_\lambda \;=\; \Big\{ \textstyle\sum_i w_i x_i = 0, \;\; \sum_i w_i x_i^k = 0 \Big\} \;\subset\; \mathbf{P}^{\,l-1}.$$

(§2) Throughout, write $m = k-1$ and μ_m for the m-th roots of unity. *(§3)* For $\lambda = (w_1, \dots, w_l) \vdash n$ put

$$A_\lambda = \Big\{ x \in \mathbf{A}^{\,l} : \;\; \textstyle\sum_i w_i x_i = 0, \;\; \sum_i w_i x_i^k = 0 \Big\}, \qquad z_\lambda = \textstyle\prod_j j^{m_j}\, m_j!\,,$$

with m_j the number of parts of λ equal to j, so that A_λ is the affine cone over V_λ. *(§5)* Write $\chi_p(a) = \left(\frac{a}{p}\right)$. *(§6)* Under $e_1 = 0$ Newton's recursion reads $s_k = -e_2 s_{k-2} + e_3 s_{k-3}$ with $s_1 = 0$, $s_2 = -2e_2$, $s_3 = 3e_3$, so each s_k is a polynomial in e_2, e_3. […] every factor of s_k other than e_2 and e_3 is a polynomial Q_k in $u = e_2^3/e_3^2$.

$$f_u(Y) \;=\; Y^3 + uY - u, \qquad \text{so that} \qquad e_2 = e_3 = u, \quad \frac{e_2^3}{e_3^2} = u, \quad \text{disc } f$$

$$\mathfrak{D}_k \;=\; \mathrm{lc}(Q_k) \cdot \mathrm{disc}(Q_k) \cdot \mathrm{Res}\big(Q_k,\; u(4u+27)\big) \;\in\; \mathbb{Z} \smallsetminus \{0\}.$$

2 The criterion, and why only $\gcd(k-1,6)$ matters

Lemma 1. (the singular locus; the case $k=3$ is Segre's) p. 4

> *Let $\lambda=(w_1,\dots,w_l)$ with w_i positive and $\sum_i w_i=n$, and let V_λ be as in (1) over an algebraically closed field of characteristic 0 or $p>\max(n,k)$. A point $z\in V_\lambda$ is singular if and only if, after scaling, $z_i\in\mu_m$ for every i and $\sum_i w_i z_i=0$. All such singularities are ordinary double points.*

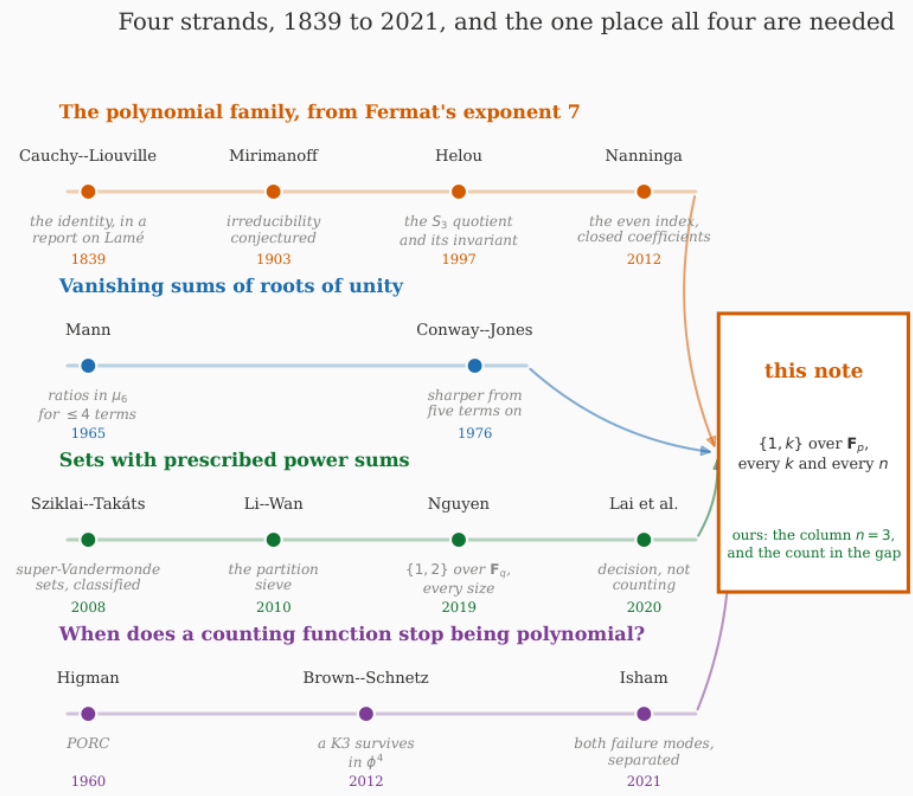
🆕 **Classical and new:** not ours, cited: for k = 3 Segre; Elsenhans–Jahnel [EJdisc, Fact 2.8]; for arbitrary degree Dimca–Sticlaru [DS, Thm. 2.4].

↳ Used in: Proposition 5 (p. 8)

Proof

> V_λ is cut by a hyperplane H and the hypersurface $F=\sum_i w_i x_i^k$. A point is singular iff the two differentials are proportional, i.e. $kw_iz_i^{k-1}=\mu w_i$ for all i and some μ; since $w_i\neq 0$ this says $z_i^{k-1}=\mu/k$ is the same for all i, and rescaling makes it 1. The point must also lie on H, which is $\sum_i w_iz_i=0$.
>
> For the second statement, restrict to H the bilinear form $B(u,v)=\sum_i(w_i/z_i)u_iv_i$, which is the Hessian of F at z up to a nonzero scalar because $z_i^{k-2}=z_i^{-1}$. It is nondegenerate on the ambient space, and $B(z,u)=\sum_i w_iu_i$, so the B-orthogonal of H is $\langle z\rangle$. Since $\sum_i w_iz_i=0$ we have $z\in H$, whence $\operatorname{rad}(B|_H)=\langle z\rangle$; modulo the radial direction the Hessian is nondegenerate, which is an ordinary double point.

Figure 2



Where this note sits, and what it owes. Four strands of work that do not otherwise meet: the Cauchy–Liouville–Mirimanoff family, the arithmetic of vanishing sums of roots of unity, the counting of subsets with prescribed power sums, and the question of when a counting function stops being polynomial — in the order drawn. The argument below needs a piece from each — Helou and Tzermias for the family our polynomials turn out to be, Mann for the criterion, Li–Wan for the sieve, and the PORC/POFS vocabulary to say what the failure is — and the box on the right marks the part we claim, which is the smallest of the four.

Proposition 2. (the invariant is $\gcd(m, 6)$) p. 4

> *Let λ have length four. Whether V_λ is singular depends on m only through $\gcd(m, 6)$.*

NEW **Classical and new:** a step towards Theorem 4, resting on Mann's theorem [Mann, Thm. 1].

↳ Used in: Theorem 4 (p. 6)

Proof

> Decompose a vanishing relation $\sum_{i=1}^{4} w_i \zeta_i = 0$ into minimal vanishing subsums, a decomposition that never splits an individual term. A singleton cannot vanish, because

$|w_i\zeta_i| = w_i > 0$; and a block of three would leave a singleton. So the only possibilities are that the relation is itself minimal, or that it splits as $2+2$.

If it is minimal then, by Mann [24] , all ratios of participating roots have order dividing $\prod_{q\le 4 \text{ prime}} q = 6$; rotating one root to 1 puts the rest in $\mu_m \cap \mu_6 = \mu_{\gcd(m,6)}$, and whether such a relation exists is then a condition on $\gcd(m,6)$ alone. If instead it splits as $2+2$, each pair satisfies $w_i\zeta_i = -w_j\zeta_j$, so taking absolute values gives $w_i = w_j$ and then $\zeta_i = -\zeta_j$; this requires $-1 \in \mu_m$, that is $2 \mid m$, which is again a condition on $\gcd(m,6)$ alone.

Evidence & scope

- **Program:** the witness $1 + (-1) + \zeta_7 + (-\zeta_7) = 0$ in μ_{14}; vanishing pairs for $m \le 24$, weights ≤ 5; ten length-four partitions depend only on $\gcd(m,6)$ for $m \le 24$. `gates/P3d_mann_minimalidad.py`

Theorem 4. (the trichotomy) p. 5

For $m \ge 1$ let S_m be the set of $n \ge 3$ such that every $\lambda \vdash n$ of length four has V_λ singular. Then

$$S_m = \begin{cases} \{3\} & \text{if } m \text{ is odd,} \\ \{3,4,6\} & \text{if } 2 \mid m,\ 3 \nmid m, \\ \{3,4,5,6\} & \text{if } 6 \mid m. \end{cases}$$

Classical and new: ours — an elementary consequence of Mann [Mann] and the triangle inequality (the paper's card).

↳ Used in: Theorem 8 (p. 9)

Proof

There are no partitions of 3 of length four, so $3 \in S_m$ always. The remaining content is which of $(2,1,1,1)$, $(3,1,1,1)$, $(2,2,1,1)$ and $(1,1,1,1)$ admit a relation.

For $(1,1,1,1)$ the relation $1+1-1-1=0$ works as soon as $-1 \in \mu_m$, i.e. $2 \mid m$; and for m odd no relation exists, since by Proposition 2 we may take $m \in \{1,3\}$, and four unit-weight terms in μ_3 sum to zero only if each cube root of unity occurs equally often, which four terms cannot do. This is the condition $4 \in S_m$. The partition $(2,2,1,1)$ has the same relation when $2 \mid m$, and also $2 + 2\omega + \omega^2 + \omega^2 = 0$ when $3 \mid m$; that changes nothing, because whether $6 \in S_m$ is decided by $(3,1,1,1)$ below, which needs $2 \mid m$.

For $(2,1,1,1)$ there can be no proper vanishing subsum, because the weight 2 has no equal-weight partner; so the whole relation is minimal and, by Proposition 2, lives in μ_6 after normalising $\zeta_1 = 1$. Inspection gives $(\zeta_2, \zeta_3, \zeta_4)$ a permutation of $(-1, \omega, \omega^2)$ with ω a primitive cube root, and indeed $2 - 1 + \omega + \omega^2 = 2 - 1 - 1 = 0$. Equivalently, in the polynomial form of the criterion, Φ_6 divides the weight polynomial:

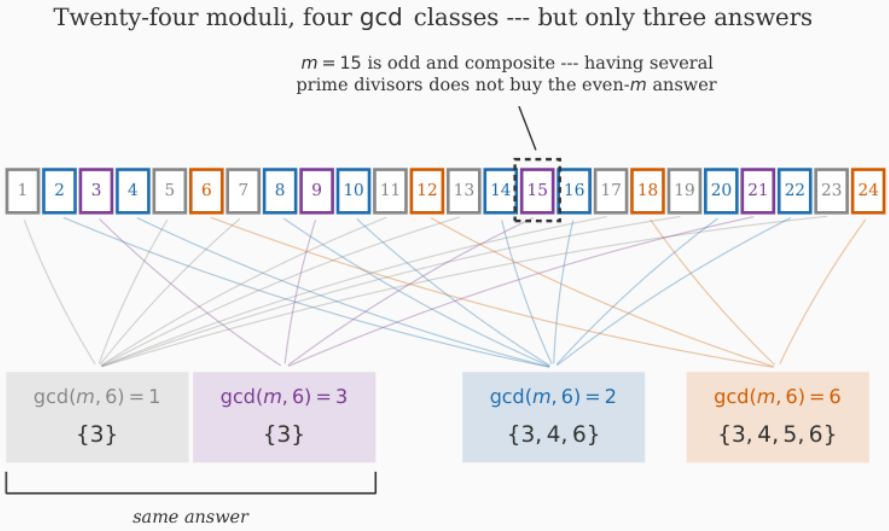
$$2 + X^2 + X^3 + X^4 = (X^2 - X + 1)(X^2 + 2X + 2).$$

So $(2,1,1,1)$ is singular exactly when $6 \mid m$, and then carries exactly six geometric singular points. Since $(2,1,1,1)$ is the only length-four partition of 5, this is precisely the condition $5 \in S_m$.

For $(3,1,1,1)$, equality in $|3\zeta_1| = |\zeta_2 + \zeta_3 + \zeta_4|$ forces $\zeta_2 = \zeta_3 = \zeta_4 = -\zeta_1$, which needs $-1 \in \mu_m$; so $(3,1,1,1)$ is singular exactly when $2 \mid m$, with exactly one singular point. Since $(2,2,1,1)$ is singular whenever $2 \mid m$, this gives $6 \in S_m \iff 2 \mid m$.

Finally, for $n \geq 7$ the partition $(n-3,1,1,1)$ has $n - 3 > 3$, so $|(n-3)\zeta_1| > |\zeta_2 + \zeta_3 + \zeta_4|$ and no relation can exist for any m; hence no $n \geq 7$ lies in S_m.

Figure 3



Proposition 2 and Theorem 4 together. What the picture shows and the statement does not is that the collapse is not injective: four gcd classes produce only three answers, because gcd $= 1$ and gcd $= 3$ agree. The circled $m = 15$ is the case that kills the natural guess — having several prime divisors is not what buys the even-m answer. The sets drawn here are the S_m of the trichotomy — which n have a *singular* length-four stratum — and not the live cells of Figure 1: the two coincide only at $k = 3$, and Section 3 is about why.

Evidence & scope

- **Program:** S_m from the definition by exhaustive enumeration over μ_m⁴, 1 ≤ m ≤ 18, 3 ≤ n ≤ 10; the factorisation by Φ₆ exactly. `gates/P3d_tricotomia_Sm.py`

3 The criterion above is the wrong one

Proposition 5. (the converse fails at $k = 5$, $n = 6$) p. 8

> *Let $k = 5$, so $m = 4$, and $\lambda = (3, 1, 1, 1)$. Then V_λ is a geometrically irreducible plane quintic with exactly one node, its normalisation has genus 5, and consequently $C_5(6, p)$ is not elementary — although $\lambda = (3, 1, 1, 1)$ and the other length-four partition $(2, 2, 1, 1)$ of 6 are both singular, so that $6 \in S_4$.*

NEW **Classical and new:** ours, and load-bearing (the paper's card); it does not affect Part I, which is the case k = 3.

↳ Not used by another result of the paper.

Proof

> The singular-point equation is $3z_1 + z_2 + z_3 + z_4 = 0$ with $z_i \in \mu_4$; equality in the triangle inequality forces $[z_1 : z_2 : z_3 : z_4] = [1 : -1 : -1 : -1]$, so there is exactly one singular point, a node by Lemma 1. The quintic is geometrically irreducible: two groups of components of degrees a and $5 - a$ would meet in $a(5 - a) \geq 4$ points counted with multiplicity, and with all singularities ordinary nodes those intersections cannot be concentrated in the single available node. Hence $g(\widetilde{V}_\lambda) = \frac{(5-1)(5-2)}{2} - 1 = 5$, and $[H^1]$ has dimension 10. Since $z_{(3,1,1,1)} = 3 \cdot 3! = 18$, the summand $\frac{1}{18}[H^1(\widetilde{V}_\lambda)]$ appears in (4) with the sign common to the whole length-four stratum, so it cannot cancel; and an expression polynomial in p with fixed-character coefficients is Artin–Tate and has only even weights.

4 The classification

Theorem 8. (all k, all n) p. 9

⚠ Erratum (2026-11-10): Editorial: in the bibliography, the first author of [GMPP] (Smooth symmetric systems over a finite field and applications) is N. Giménez, not «J. Giménez». No statement is affected.

⚠ Erratum (2026-11-10): Editorial: the legend drawn inside Figure 1 calls the paler nonabelian cells «nonabelian too, but the group is larger and not determined here»; the caption and the text determine the group, $S_3 \times C_2$, for $k = 8, 10, 11$, and only $k = 12$ is left undetermined. No statement is affected.

> *Say $C_k(n, p)$ is Artin–Tate if there are a number field $L/\mathbb{Q}$ finite Galois with group Γ, a number field E splitting Γ, and class functions $\alpha_0, \ldots, \alpha_d$ on Γ with values in E — equivalently, E-linear combinations of Artin characters — such that*
>
> $$C_k(n, p) \;=\; \sum_{j=0}^{d} p^{\,j}\, \alpha_j(\mathrm{Frob}_p) \qquad (4)$$
>
> *for all but finitely many p. Then*

$$\{\, n \geq 3 :\ C_k(n,p) \text{ is Artin--Tate} \,\} = \begin{cases} \{3,4,6\}, & k = 3, \\ \{3,4\}, & k = 5, \\ \{3\}, & k \neq 3, 5. \end{cases}$$

🆕 **Classical and new:** ours — proved, not extrapolated (the paper's card); the case k = 3 is Part I [P3c, Thm. 6 and Remark 7].

↳ Not used by another result of the paper.

Proof

The case $k = 3$ is [37 , Thm. 6 and Remark 7] : Theorem 6 there excludes elementary expressions for $n \notin \{3, 4, 6\}$, and Remark 7 notes that the same weight-one argument excludes coefficients given by arbitrary Artin-type class functions of finite image, which is the class (5); for $n = 3, 4, 6$ the closed forms there are elementary, hence Artin–Tate. For $n = 3$ there are no length-four strata and every V_λ with $\lambda \vdash 3$ is zero-dimensional, so the cones over them contribute only Artin–Tate classes and the count is an Artin-type class function in the sense of (5). It need not be a character: Proposition 11 produces the indicator of a conjugacy class, which is why (5) was stated for combinations rather than for traces. This is the column $n = 3$ of Figure 1, and Section 6 shows the class function need not be abelian. It remains to kill every $n \geq 4$ when $k \neq 3$, and every $n \geq 5$ when $k = 5$.

Two uniform witnesses do it, and Figure 4 is the bookkeeping.

Figure 1: The proof of Theorem 8 as an accounting. A node removes exactly one unit of genus, so past $k = 3$ the nodes are never enough. The two shaded columns are the two where a witness happens to be rational; $k = 5$ is shaded because its $n = 4$ witness degenerates to a conic, which is why $C_5(4, p)$ exists at all.

The witness at $n = 4$. Here $\lambda = (1, 1, 1, 1)$ and, eliminating x_4, the curve is $F_k(x, y, z) = x^k + y^k + z^k - (x + y + z)^k$. If k is even then m is odd, the curve is smooth by Theorem 4, and its genus is $\binom{k-1}{2} \geq 3$. If k is odd then, up to a nonzero scalar,

$$F_k = (x + y)(x + z)(y + z)\, R_{k-3}, \tag{8}$$

with R_{k-3} of degree $k - 3$. Every singular point of F_k corresponds to four unit roots summing to zero, hence lies on one of the three pairing lines; since all singularities are nodes, R_{k-3} can have no singularity — off the lines it would give an

unaccounted singular point of F_k, on a line it would give multiplicity at least three — and it cannot be reducible, by Bézout. So R_{k-3} is smooth of genus $\binom{k-4}{2}$. This is positive for odd $k \geq 7$ and zero exactly for $k = 3$ (no residual curve at all) and $k = 5$ (a conic). Hence $n = 4$ dies for every $k \neq 3, 5$, and survives for $k = 3, 5$.

The witness at $n \geq 5$. For $n \geq 7$ take $\lambda = (n-3, 1, 1, 1)$, smooth for every m by the last paragraph of Theorem 4, of genus $\binom{k-1}{2} > 0$ for $k \geq 4$. For $n = 6$ take $\lambda = (3, 1, 1, 1)$: smooth when k is even and carrying exactly one node when k is odd, so its normalisation has genus $\binom{k-1}{2}$ or $\binom{k-1}{2} - 1$, positive for every $k \geq 4$. For $n = 5$ take $\lambda = (2, 1, 1, 1)$: smooth unless $6 \mid m$, and when $6 \mid m$ it carries exactly six nodes, so with r geometric components $\sum_j g(\widetilde{C}_j) = \binom{k-1}{2} - 6 + r - 1 \geq 9$ because $6 \mid m$ forces $k \geq 7$.

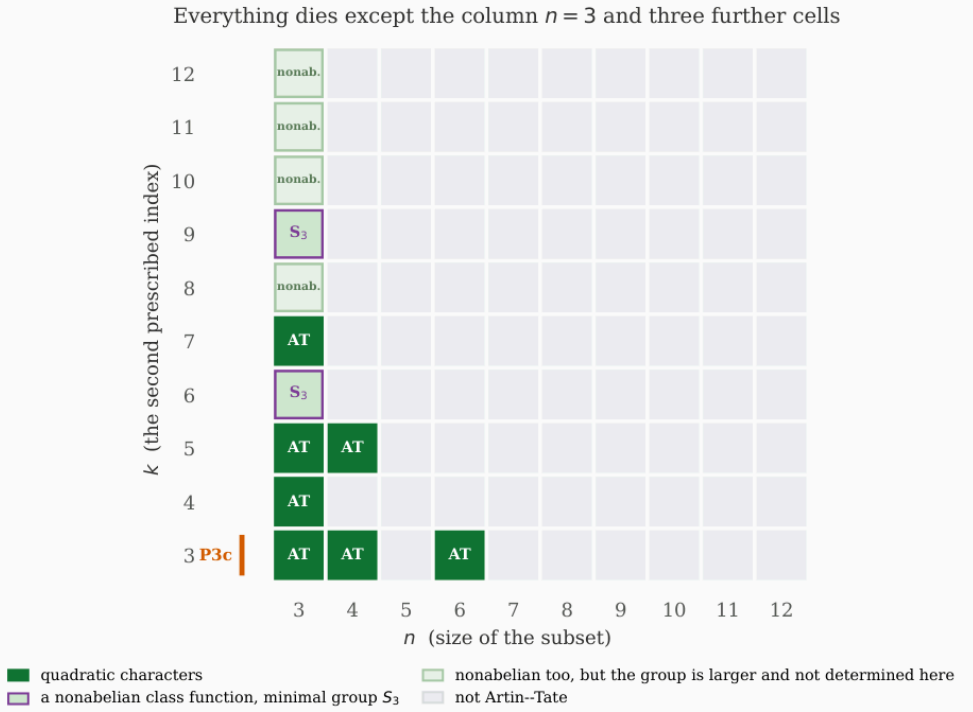
In each case a length-four stratum has normalisation of positive genus, so by (4) the class $\operatorname{gr}_1^W \mathcal{M}_{n,k}$ is a sum of effective classes with one common sign and is nonzero.

To conclude, fix an embedding $\iota \colon E \hookrightarrow \overline{\mathbb{Q}}_\ell$. Read through ι, the right-hand side of (5) is not the trace of a representation — its coefficients are fractional, as $\mathbf{1}_{\{1\}} = \frac{1}{6}\chi_{\mathrm{reg}}$ above already shows — but it is the trace of a well-defined *virtual* one, an element of the Grothendieck group

$$K_0^{\mathrm{ss}}\big(G_{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell\big) \otimes_{\mathbb{Z}} E$$

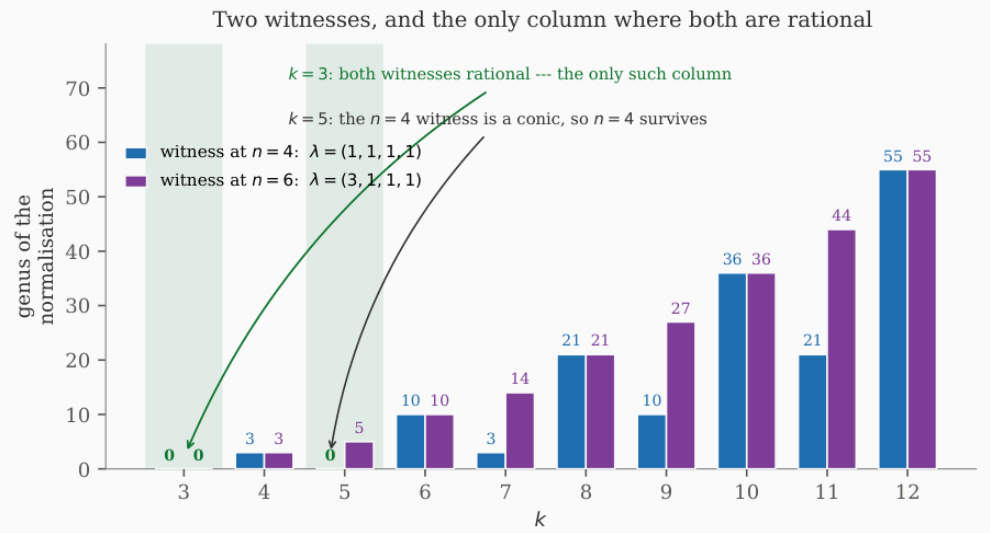
of semisimple $\overline{\mathbb{Q}}_\ell$-representations with coefficients extended to E. In that group the trace still determines the class, irreducible characters being E-linearly independent. Chebotarev gives agreement of traces at a set of primes of density one, so Brauer–Nesbitt forces agreement of the two classes there; and every weight occurring on the Artin–Tate side is even, while $\operatorname{gr}_1^W \mathcal{M}_{n,k} \neq 0$. The comparison is thus between virtual representations throughout, which is all the weight argument needs. The positive direction for $k = 5$, $n \in \{3, 4\}$ is Section 5.

Figure 1



The result of this note. A cell is live when $C_k(n, p)$ is Artin–Tate. The column $n = 3$ never dies, because a zero-dimensional variety always contributes an Artin character; but the class function is expressible in quadratic characters only for $k \in \{3, 4, 5, 7\}$ (Theorem 12), so "Artin–Tate" and "elementary" part company from $k = 6$ on. Among the nonabelian cells the two shades are not decoration: the darker ones, $k = 6$ and $k = 9$, are those where the group of the class function is exactly S_3, and the paler ones those where it is larger: $S_3 \times C_2$ for $k = 8, 10, 11$, and not determined here for $k = 12$. Which cells are which, and why, is settled after Theorem 12; Proposition 13 gives one S_3 *fibre* whenever $\deg Q_k > 0$, which is weaker than either and is all Lemma 14 needs. Everything else is dead, and the row $k = 3$ of the companion note is the only one with more than two live cells.

Figure 4



The proof of Theorem 8 as an accounting. A node removes exactly one unit of genus, so past $k = 3$ the nodes are never enough. The two shaded columns are the two where a witness happens to be rational; $k = 5$ is shaded because its $n = 4$ witness degenerates to a conic, which is why $C_5(4, p)$ exists at all.

5 The counts that survive

Theorem 9. p. 11

For $p>3$,

$$\boxed{C_4(3,p) = \frac{(p-1)\big(1+\chi_p(-3)\big)}{6}}, \qquad \boxed{C_5(3,p) = \frac{(p-1)\big(4}{6}}$$

and

$$\boxed{C_5(4,p) = \frac{p-1}{24}\Big(4p - 17 - 6\chi_p(-2) - 3\chi_p(-1)\Big)}.$$

At $p = 3$ the three counts are 0, 1 and 0; the two formulas for $k = 5$ also hold at $p = 5$, where they give 2 and 1.

Classical and new: ours — identities, not fits; the first recovers the size-three super-Vandermonde sets of Sziklai and Takáts [ST, Thm. 1.12].

↳ Used in: Theorem 8 (p. 9)

Proof

> Write e_j for the elementary symmetric functions of the elements of T and s_j for the power sums. Under $e_1 = 0$, Newton's identities give $s_4 = 2e_2^2$ and $s_5 = -5e_2e_3$.
>
> For three elements and $s_4 = 0$ one gets $e_2 = 0$, and the distinct triples with $e_1 = e_2 = 0$ are exactly the sets $a\{1, \omega, \omega^2\}$ with ω a primitive cube root, which exist iff $\chi_p(-3) = 1$; there are $(p-1)/3$ of them in that case and 0 otherwise, which is the first formula.
>
> For three elements and $s_5 = 0$, away from characteristic 5, either $e_2 = 0$ or $e_3 = 0$. The triples with $e_3 = 0$ and $e_1 = 0$ are $\{0, a, -a\}$, of which there are $(p-1)/2$; the two families are disjoint, giving the second formula.
>
> For four elements with $e_1 = 0$ the identity $s_5 = -5e_2e_3$ still holds. Let A be the family with $e_2 = 0$ and B the family with $e_3 = 0$. Vanishing of e_3 together with $e_1 = 0$ makes the root polynomial even, so $B = \{\{\pm a, \pm b\} : a, b \neq 0,\ b \neq \pm a\}$ and $|B| = (p-1)(p-3)/8$. The condition $e_2 = 0$ is the quadratic moment condition $s_2 = 0$, whose distinct-coordinate sieve gives $|A| = (p-1)\big(p - 5 - 6\chi_p(-2)\big)/24$. The intersection consists of the sets $\{a, -a, ia, -ia\}$ with $i^2 = -1$, so $|A \cap B| = (p-1)\big(1 + \chi_p(-1)\big)/8$. Inclusion and exclusion give the third formula.
>
> At $p = 5$ the argument for the two counts with $k = 5$ does not apply, since $s_5 = -5e_2e_3$ vanishes identically; but $x^5 = x$ on $\mathbb{F}_5$, so $s_5 = s_1$ and the only condition is $s_1 = 0$. Translation by t shifts the sum of an n-subset by nt, and $n = 3, 4$ are invertible mod 5, so the $\binom{5}{n}$ subsets are spread evenly over the five sums: 2 for $n = 3$ and 1 for $n = 4$, which are the values of the two formulas.

Evidence & scope

- **Program:** against exhaustive enumeration over F_p: 16, 15 and 12 primes; |B| and |A ∩ B| separately. `gates/P3d_tricotomia_Sm.py`
- **Program:** the k = 4 triples are the sets a1, ω, ω², 28 primes 5 ≤ p ≤ 113. `gates/literatura_super_vandermonde.py`

6 Two different failures, and only one of them is a trace

Proposition 11. p. 13

> *For $p \nmid 66$,*
>
> $$\boxed{C_6(3,p) = (p-1)\cdot \mathbf{1}\{\, 2Y^3+3Y-3 \text{ splits completely over } \mathbb{F}$$
>
> *The cubic $2Y^3+3Y-3$ is irreducible over $\mathbb{Q}$ with discriminant $-1188 = -36\cdot 33$, so its Galois group is S_3 and complete splitting is not determined by any congruence on p.*

Classical and new: ours (the paper's card).

↳ Used in: Theorem 8 (p. 9)

Proof

> Newton with $e_1 = 0$ gives $s_6 = -2e_2^3 + 3e_3^2$, and a solution with distinct entries has $e_2 e_3 \neq 0$. So $s_6 = 0$ says exactly
>
> $$\frac{e_2^3}{e_3^2} = \frac{3}{2},$$
>
> a condition on the scaling invariant alone. Scaling the three roots by t sends $e_j \mapsto t^j e_j$, so $t = e_2/e_3$ makes e_2 and e_3 equal, and their common value is $e_2^3/e_3^2 = \frac{3}{2}$: the normalised polynomial is $Y^3 + \frac{3}{2}Y - \frac{3}{2}$, one half of $2Y^3+3Y-3$. When that splits, its nonzero scalings give exactly $p-1$ distinct subsets. The discriminant is computed directly; it is not a square, so the Galois group is S_3 and not C_3, and Chebotarev distinguishes the identity class from a 3-cycle inside any fixed cyclotomic congruence condition.

Evidence & scope

- **Program:** against exhaustive enumeration in the 17 primes $5 \le p \le 71$ with $p \nmid 66$; density of split primes 0.160 below 400. `gates/P3d_tricotomia_Sm.py`

Theorem 12. (the column $n = 3$) p. 14

> *For every prime $p \nmid 6k\mathfrak{D}_k$,*

$$\frac{6\,C_k(3,p)}{p-1} = \big[e_2 \mid s_k\big]\big(1+\chi_p(-3)\big) + 3\big[e_3 \mid s_k\big] \\ +6\,\#\big\{u_0 \in \mathbb{F}_p :\ Q_k(u_0)=0,\ f_{u_0} \text{ splits compl}$$

with f_u as in (7). Consequently $C_k(3,p)$ is elementary — expressible in quadratic characters of fixed integers — if and only if Q_k is constant, that is, if and only if s_k is a monomial in e_2 and e_3. That happens exactly for

$$\boxed{k \in \{3,4,5,7\}}.$$

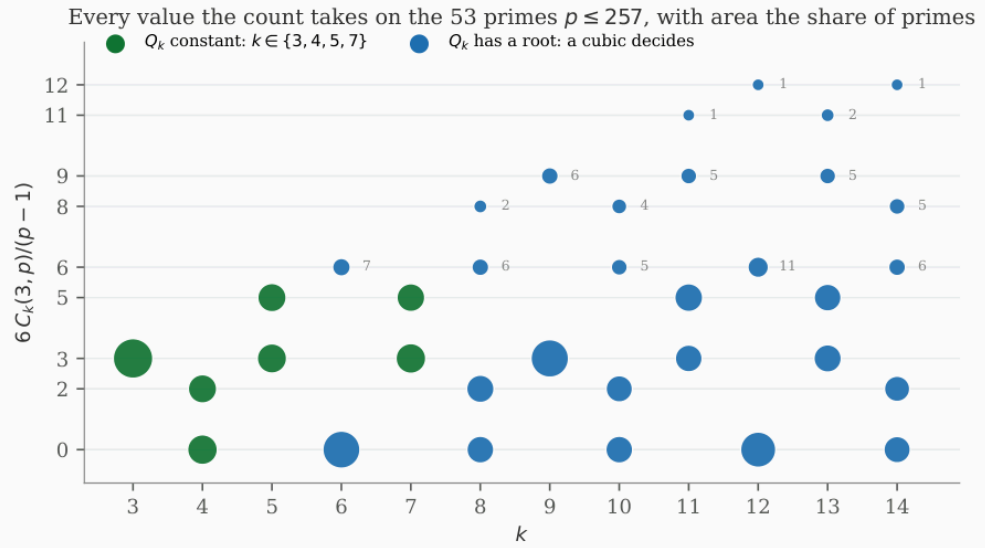
🆕 **Classical and new:** ours — the count, not the polynomials: Q_k and the reduction to u are in Helou, Tzermias and Nanninga [Helou; Tzermias; NanA].

↳ Used in: Corollary 17 (p. 22)

Proof

The factor e_2 contributes the triples with $e_1 = e_2 = 0$, which are the sets $a\{1, \omega, \omega^2\}$ and number $(p-1)(1+\chi_p(-3))/6$; the factor e_3 contributes $\{0, a, -a\}$, of which there are $(p-1)/2$; and the two families meet only in the zero triple. Each root u_0 of Q_k is an invariant of the scaling action, and the subsets realising it form a single orbit of $\mathbb{F}_p^\times$: if T has invariant $u_0 \neq 0$ and $tT = T$ then $t^2 e_2 = e_2$ and $t^3 e_3 = e_3$, and $e_2 e_3 \neq 0$ forces $t^2 = t^3 = 1$, hence $t = 1$. The action is therefore free and the orbit has exactly $p-1$ elements, so u_0 contributes $p-1$ subsets when f_{u_0} splits into three distinct factors over $\mathbb{F}_p$ and none otherwise. This proves the displayed identity. For the criterion, one direction is immediate: if Q_k is constant the identity has only the two character terms. The other is not, and it is where the work is: if $\deg Q_k > 0$ then Proposition 13 and Lemma 14 show the fibre contribution is not PORC, so *a fortiori* it cannot be written in quadratic characters. For the list $\{3,4,5,7\}$, the four monomials are $s_3 = 3e_3$, $s_4 = 2e_2^2$, $s_5 = -5e_2e_3$ and $s_7 = 7e_2^2e_3$; that no other k joins them is not a computation but Proposition 15 below, which gives $\deg Q_k$ in closed form for every k and shows it vanishes exactly there.

Figure 5



Theorem 12 measured rather than stated: every value $6C_k(3,p)/(p-1)$ takes on the 53 primes $p \le 257$ (omitting $p \mid k$), with the area of each disc the share of primes giving it and the count printed for the rare ones. The green columns are the four elementary k; everywhere else the values climb in steps of 6, one step per root of Q_k whose cubic splits, on top of the 0 or 2 contributed by e_2 and the 3 contributed by e_3.

Evidence & scope

- **Program:** the identity against exhaustive enumeration for 3 ≤ k ≤ 14 and primes 5 ≤ p ≤ 257 with p ∤ 6kD_k (620 pairs), with two failing controls. `gates/P3d_columna_n3.py`

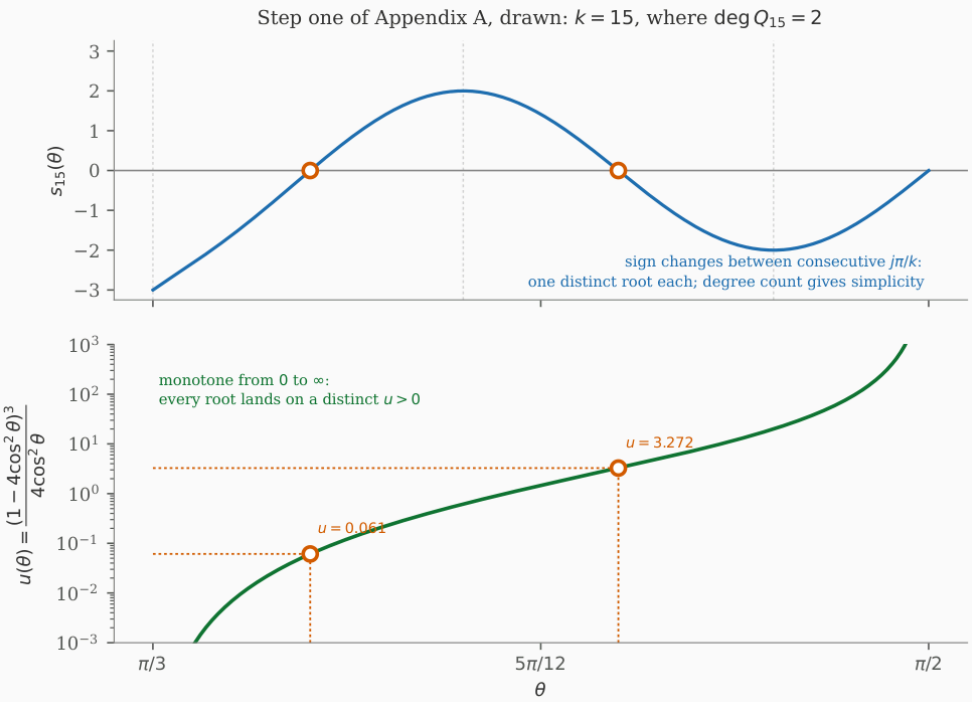
Proposition 13. (an S_3-fibre exists) p. 14

> *If* $\deg Q_k > 0$ *then* Q_k *has a root* u_0 *for which* $f_{u_0}(Y) = Y^3 + u_0 Y - u_0$ *has Galois group* S_3 *over* $\mathbb{Q}(u_0)$.

Classical and new: unconditional, for every k (the paper's card): the inequality A < 2^d for k ≥ 38 and forty printed certificates for 6 ≤ k ≤ 46.

↳ Used in: Theorem 12 (p. 15), Lemma 14 (p. 18)

Figure 8



Step one of this appendix, for $k = 15$, an example where Q_k has two roots — with one root nothing would show that they are *simple*, which is half the statement. The smallest such k is 12, by Proposition 15; 15 is drawn because its two roots are further apart on the u-axis and the guides do not collide. Above, s_k on the interval, changing sign once between consecutive $j\pi/k$; each change is a root, and they can be counted. Below, the invariant $u(\theta)$ of (10), which climbs monotonically from 0 to ∞, so each root above lands on a positive value of u and on a different one. The two halves of "real, positive and simple" are the two properties of this curve. The lower axis is logarithmic and clipped to the band where the roots fall; the curve leaves it at both ends, which is what "from 0 to ∞" means.

Evidence & scope

- **Program:** for $6 \le k \le 46$ with $d > 0$: roots of Q_k real, positive and simple (Sturm), P_k irreducible modulo the printed prime (Table 3). `gates/P3d_s3_certificados.py`
- **Program:** the raw leading coefficient 2 or k and A | k, by a separate route for $3 \le k \le 400$. `gates/P3d_lc_Qk.py`

Lemma 14. (the nonabelian part cannot cancel) p. 15

> *Let $L/\mathbb{Q}$ be a finite Galois extension containing the roots of Q_k and of every f_{u_0} for u_0 a root of Q_k, and let $G = \mathrm{Gal}(L/\mathbb{Q})$. Fix a G-orbit $\mathcal{O}$ of roots of Q_k, let H be the stabiliser of one of them, and let $K = \ker\big(H \to S_3\big)$ for the action of H on the three roots of that fibre's cubic. For p unramified in L and $p \nmid 6k\mathfrak{D}_k$, let $N_{\mathcal{O}}(p)$ be the number of roots of Q_k in $\mathcal{O}$ that lie in $\mathbb{F}_p$ and whose cubic splits completely. Then*

$$\boxed{|H/K|\; N_{\mathcal{O}}(p) \;=\; \chi_{E[G/K]}(\mathrm{Frob}_p)}, \qquad \text{so} \qquad 6\,N_{\mathcal{O}}(p) = \frac{6}{|H/}$$

with E a splitting field for G as in (5), the factor $6/|H/K|$ being a positive rational. If moreover $H/K \simeq S_3$ — which Proposition 13 provides for some orbit, and where the factor is 1 — then $E[G/K]$ has an irreducible constituent of dimension greater than one. Since the orbits enter Theorem 12 with coefficient $+1$ each, that constituent cannot cancel, and $C_k(3,p)$ is not PORC.

🆕 **Classical and new:** ours (the paper's card).

↳ Used in: Theorem 12 (p. 15)

Proof

A coset gK is determined by the pair consisting of the root $g(u_0) \in \mathcal{O}$ and the labelling of the three roots of $f_{g(u_0)}$ induced by g; the fibre of this correspondence over a root is H/K, of size 6 exactly when $H/K \simeq S_3$. Now Frob_p fixes gK exactly when it fixes $g(u_0)$ and acts trivially on the three roots of its cubic, that is, exactly when that root lies in $\mathbb{F}_p$ and its cubic splits completely. Each such root therefore contributes $|H/K|$ fixed cosets and no other coset is fixed, which is the displayed identity, the character of a permutation representation being the number of fixed points.

For the second claim, H/K acts on the subset $\{hK\} \subseteq G/K$ by left translation, and that action is regular, so the image of G in $\mathrm{Sym}(G/K)$ contains a subgroup isomorphic to $H/K \simeq S_3$ and is not abelian. A representation all of whose irreducible constituents are one-dimensional has abelian image; hence $E[G/K]$ has a constituent of dimension at least two. The coefficient field must be E and not $\mathbb{Q}$ here: over E every irreducible of an abelian group is one-dimensional, whereas over $\mathbb{Q}$ an abelian group can have irreducibles of larger dimension — already $\mathbb{Q}[C_3] = \mathbb{Q} \times \mathbb{Q}(\zeta_3)$ has one of dimension two — so over $\mathbb{Q}$ the dimension alone would not exclude an abelian quotient. A PORC function is, by (5), one whose coefficients factor through an abelian quotient, and no such coefficient contains that constituent.

Evidence & scope

- **Program:** the identity 6N = χ_E[G/K](Frob_p) at k = 6 in 75 primes below 400, both sides computed independently, with a cyclic decoy.

gates/P3d_caracter_regular_S3.py

Proposition 15. (the degree of Q_k) p. 16

> *For every $k \geq 3$,*
>
> $$\boxed{\deg Q_k = \left\lfloor \frac{k}{6} \right\rfloor - \left[\, k \equiv 1 \pmod 6 \,\right]}.$$
>
> *In particular $\deg Q_k = 0$ exactly for $k \in \{3, 4, 5, 7\}$, and since the degree is nondecreasing along each residue class modulo 6, no further k can return to zero.*

Lean · ours, in part **show**

added to the Lean repository after publication (21-oct-2026): the count # {(a, b) : 2a + 3b = k} = ⌊k/6⌋ + 1 − [k ≡ 1 mod 6] for every k

- `P3d.card_degreePairs`
- `P3d.card_degreePairs_eq_one_iff`

✅ Checked when this page was built: DegreeCount.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — card_degreePairs, card_degreePairs_eq_one_iff: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean DegreeCount.lean` (SHA-256 of the compiled file: `55e85b8ae935ef0d`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 2 results, standard axioms only

Proof: in this note · **Covers:** not formalized · **Cite:** Lemma 1, p. 4, doi:10.5281/zenodo.22886543

🆕 **Classical and new:** for odd k the number is Helou's r_k [Helou]; the paper claims neither, only a self-contained proof for all k.

↳ Used in: Theorem 12 (p. 15)

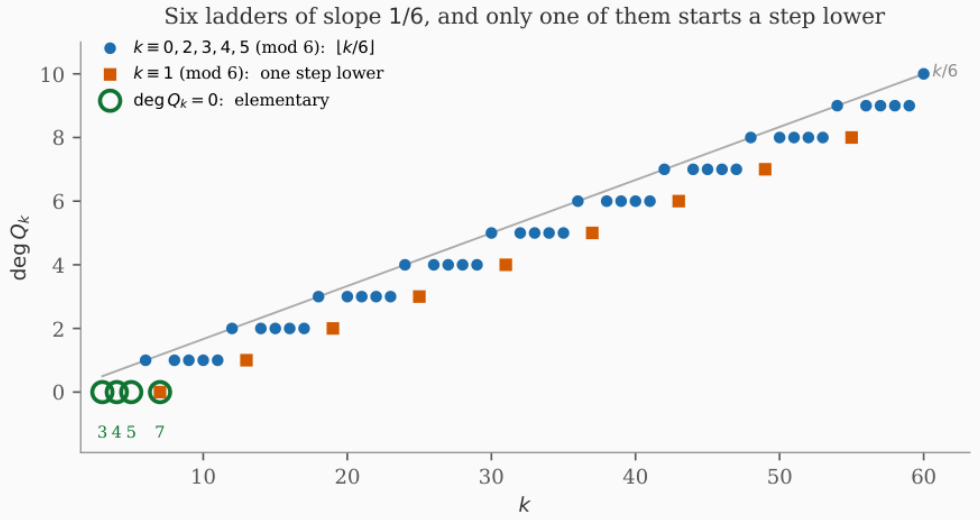
Proof

Newton–Waring at $e_1 = 0$ is (9) below, and no coefficient there vanishes, so the monomials of s_k are exactly the pairs $(a, b) \geq 0$ with $2a + 3b = k$. Those pairs form a chain: b runs over the integers with $b \equiv k \pmod 2$ and $0 \leq b \leq k/3$, and a is determined. Dividing out the common monomial leaves consecutive powers of $u = e_2^3/e_3^2$, whence

$$\deg Q_k = \#\{(a, b) \geq 0 : \; 2a + 3b = k\} - 1.$$

> Counting the admissible b gives $\lfloor k/6 \rfloor + 1$ in every residue class except $k \equiv 1 \pmod 6$, where the parity constraint costs one, and subtracting 1 is the formula.

Figure 6



Proposition 15. The grey line is $k/6$. What the picture adds to the formula is that the exceptional class is a single ladder rather than a scatter of exceptions, and that the four circled points are the only ones that will ever touch zero.

Evidence & scope

- **Program:** both sides independently for 3 ≤ k ≤ 90, and every proper factor of s_k homogeneous in u. `gates/P3d_grado_Qk.py`

Proposition 16. (the degree is a dimension of modular forms) p. 18

> *Substitute $e_2 = -3E_4$ and $e_3 = 2E_6$, with E_4, E_6 the normalised Eisenstein series. Then $\mathcal{F}_k = s_k(-3E_4, 2E_6)$ is a modular form of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$, and since $M_*(\mathrm{SL}_2(\mathbb{Z})) = \mathbf{C}[E_4, E_6]$ a basis of M_{2k} is indexed by the solutions of $2a + 3b = k$ — the same set that indexes the terms of s_k. Hence*
>
> $$\boxed{\dim M_{2k} = \deg Q_k + 1},$$
>
> *and $\dim M_{2k} = 1$ exactly for $k \in \{3, 4, 5, 7\}$.*

Proof: in this note · **Covers:** not formalized · **Cite:** Proposition 16, p. 18, doi:10.5281/zenodo.22886543

> **Classical and new:** the dictionary M_* = C[E₄, E₆] is standard [Milne, Prop. 4.16]; what is used is the coincidence of index sets.

↳ Used in: Corollary 17 (p. 22)

Evidence & scope

- **Program:** dim M_2k = deg Q_k + 1 for 3 ≤ k ≤ 60; Waring's formula against the recursions. `gates/P3d_familia_Qk.py`

Corollary 17. (the criterion is a dimension) p. 18

For every $k \geq 3$,

$$\boxed{C_k(3,p) \text{ is elementary} \iff \dim M_{2k}(\mathrm{SL}_2(\mathbb{Z})) = 1.}$$

Proof: in this note · **Covers:** not formalized · **Cite:** Corollary 17, p. 18, doi:10.5281/zenodo.22886543

NEW **Classical and new:** ours — but only the joining of two halves (the paper's card).

↳ Not used by another result of the paper.

Proof

By Theorem 12, $C_k(3,p)$ is elementary if and only if $\deg Q_k = 0$, and by Proposition 16 that is $\dim M_{2k} = 1$. The one-dimensional weights are $2k \in \{4,6,8,10,14\}$, and $2k = 4$ is excluded by $k \geq 3$; the remaining four are the list of Theorem 12.

A Proof of Proposition 13

Lemma 21. (the real roots form a factor) p. 25

Suppose f_u is reducible over $\mathbb{Q}(u)$ for every root u of Q_k, and let $y(u) \in (0,1)$ be the real root of f_u. Then the set $\{y(u) : Q_k(u) = 0\}$ is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the primitive integral multiple h of $\prod_u (Y - y(u))$ lies in $\mathbb{Z}[Y]$, has degree d, divides P_k, and has leading coefficient $B \mid A$.

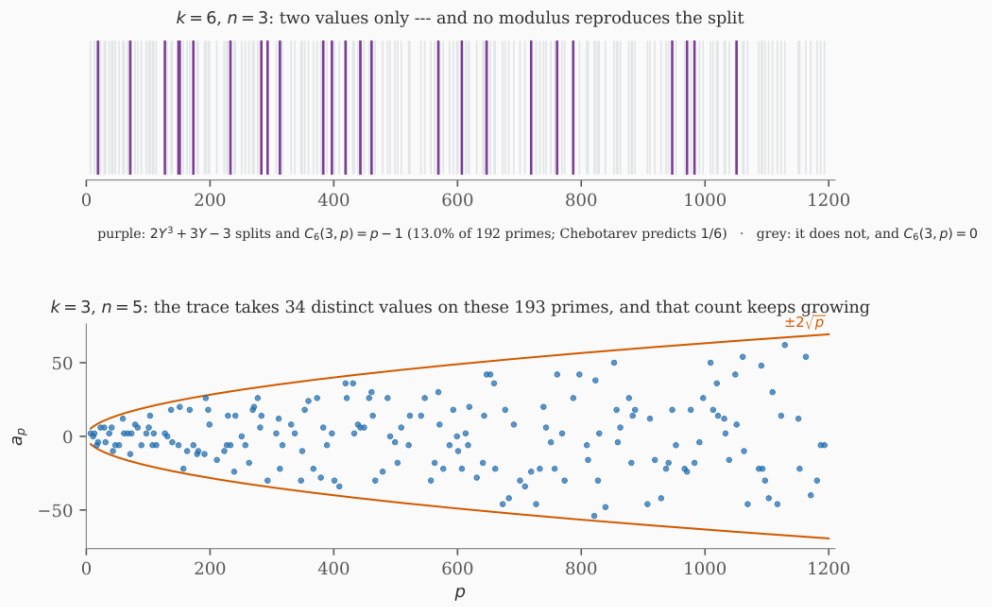
Proof: in this note · **Covers:** not formalized · **Cite:** Lemma 21, p. 25, doi:10.5281/zenodo.22886543

↳ Used in: Proposition 13 (p. 17)

Proof

> Every root of Q_k is real by step one, so every conjugate of u is real and $\mathbb{Q}(u)$ is totally real. A reducible cubic f_u has a root $r \in \mathbb{Q}(u)$. For each embedding σ of $\mathbb{Q}(u)$, $\sigma(r)$ is a real root of $f_{\sigma(u)}$, whose discriminant is negative, so it is the only one: $\sigma(r) = y(\sigma(u))$. Hence $r = y(u)$, and the conjugates of $y(u)$ are the values of y at the conjugates of u. The $y(u)$ are distinct, since $u = y^3/(1-y)$ recovers u, so the product has degree d and divides P_k in $\mathbb{Q}[Y]$; by Gauss's lemma its primitive integral multiple divides P_k in $\mathbb{Z}[Y]$, whence $B \mid A$.

Figure 7



Observation 20, measured. Both panels show a failure of polynomiality on residue classes, and they are different failures: above, a two-valued count whose partition of the primes no modulus reproduces; below, a growing set of observed Frobenius traces. The "not POFS" conclusion comes from weight one and not from this numerical growth: one quantity is bounded by a theorem, while the observed support of the other keeps growing across these windows. A reader who wants a single picture of why "not PORC" is too coarse a verdict for this classification should look at these two panels together.

Tables

Every table of the paper, in its order: click one to unfold it, with where the paper cites it, how its numbers were checked, and its data as CSV. The typeset table is one click away, on its page of the PDF.

Table 1. Theorem 4 in three rows. The tinted rows are the two where something beyond $n = 3$ is singular; the bottom one is where $n = 5$ enters, and the certificate is the divisibility by Φ_6 that puts it there. Note that this table answers the *singularity* question, which Section 3 shows is not the same as the Artin–Tate question.

$m = k - 1$	$\gcd(m, 6)$	S_m	certificate
odd	1 or 3	$\{3\}$	—
$2 \mid m, 3 \nmid m$	2	$\{3, 4, 6\}$	$1 + 1 - 1 - 1 = 0$ in μ_2
$6 \mid m$	6	$\{3, 4, 5, 6\}$	$2 + X^2 + X^3 + X^4 = \Phi_6 \cdot (X^2 + 2X + 2)$

Colour: grey = nothing.

p. 6 in the PDF ⬇ CSV

cited in §2 The criterion, and why only gcd(k-1,6) matters

Table 2. Every count this note and its companion put in closed form, with the reason the length-four obstruction failed to appear in each. The last row is tinted differently because it is closed but not in quadratic characters — it is the subject of Section 6.

count	closed form	why no obstruction	checked
$C_3(n, p)$, $n = 3, 4, 6$	see [P3c]	rational, or none	11/33 primes
$C_4(3, p)$	$\frac{p-1}{6}\big(1 + \chi_p(-3)\big)$	no length-four stratum	16 primes
$C_5(3, p)$	$\frac{p-1}{6}\big(4 + \chi_p(-3)\big)$	no length-four stratum	15 primes
$C_5(4, p)$	$\frac{p-1}{24}\big(4p - 17 - 6\chi_p(-2) - 3\chi_p(-1)\big)$	the witness is a conic	12 primes
$C_7(3, p)$	$\frac{p-1}{6}\big(4 + \chi_p(-3)\big)$	s_7 is a monomial	53 primes
$C_6(3, p)$	$(p-1) \cdot \mathbf{1}\{2Y^3 + 3Y - 3 \text{ splits}\}$	*a nonabelian class function*	17 primes

Colour: red = no / fails.

p. 13 in the PDF ⬇ CSV cited in §5 The counts that survive

Table 3. The forty certificates behind Proposition 13. For each k with $\deg Q_k > 0$ in the range, $d = \deg Q_k$, $A = \mathrm{lc}(Q_k)$, and p is a

prime modulo which P_k — of degree $3d$ — remains irreducible, which proves it irreducible over $\mathbb{Q}$. Each row is checkable in one line of any computer algebra system and requires nothing of ours to be believed. Shade of A is the margin $2^d/A$ by which the inequality holds — dark where it barely does, pale where it holds by orders of magnitude — and the tinted rows are the seven where it does not hold at all. The two codes together are the shape of the argument: the margin grows exponentially in k while $A \le k$ grows at most linearly, so the tinted rows can only occur early, and they do. The table is generated by `gates/P3d_s3_certificados.py`, not typed.

k	d	A	p	k	d	A	p	k	d	A	p
6†	1	2	7	21†	3	21	97	35†	5	35	113
8	1	1	11	22	3	2	281	36	6	2	89
9†	1	3	2	23	3	1	23	37	5	1	31
10†	1	2	11	24	4	2	71	38	6	2	5
11	1	1	2	25	3	5	59	39	6	39	11
12	2	2	5	26	4	2	37	40	6	2	7
13	1	1	3	27	4	9	113	41	6	1	179
14	2	2	5	28	4	2	13	42	7	2	31
15†	2	15	11	29	4	1	71	43	6	1	5
16	2	1	29	30	5	2	11	44	7	2	191
17	2	1	11	31	4	1	23	45	7	45	29
18	3	2	61	32	5	1	101	46	7	2	17
19	2	1	13	33†	5	33	41				
20	3	2	29	34	5	2	29				

Colour: blue shade = the number's size within its column.

p. 28 in the PDF · ⬇ CSV · cited in §7 Where this sits · cited in §8 What is left open · cited in §A Proof of Proposition 13

References

1

P. Belkale and P. Brosnan, *Matroids, motives, and a conjecture of Kontsevich*, Duke Math. J. **116** (2003), no. 1, 147–188. 🔎 find it

2

C. Birkenhake and H. Lange, *Complex Abelian Varieties*, 2nd ed., Grundlehren der mathematischen Wissenschaften **302**, Springer, Berlin, 2004. 🔎 find it

3

F. Brown and O. Schnetz, *A K3 in* ϕ^4, Duke Math. J. **161** (2012), no. 10; arXiv:1006.4064.

4

A. Dimca and G. Sticlaru, *Waring rank of symmetric tensors, and singularities of some projective hypersurfaces*, arXiv:1902.01351, Thm. 2.4.

5

M. du Sautoy and M. Vaughan-Lee, *Non-PORC behaviour of a class of descendant p-groups*, J. Algebra **361** (2012), 287–312; arXiv:1106.5530.

6

A.-S. Elsenhans and J. Jahnel, *The discriminant of a cubic surface*, arXiv:1006.0721 (2010), Fact 2.8 and Examples 2.9.

7

S. R. Ghorpade and G. Lachaud, *Étale cohomology, Lefschetz theorems and number of points of singular varieties over finite fields*, Moscow Math. J. **2** (2002), 589–631; corrected version arXiv:0808.2169, Prop. 3.2.

8

C. Goldstein, *Gabriel Lamé et la théorie des nombres: "une passion malheureuse"?*, Bulletin de la Sabix **44** (2009), 131–139. The letter to Dirichlet is in the NachlaSS Dirichlet, Staatsbibliothek zu Berlin. 🔎 find it

9

E. W. Howe and H. J. Zhu, *On the existence of absolutely simple abelian varieties of a given dimension over an arbitrary field*, J. Number Theory **92** (2002), no. 1, 139–163. 🔎 find it

10

G. Higman, *Enumerating p-groups. II: Problems whose solution is PORC*, Proc. London Math. Soc. (3) **10** (1960), 566–582. 🔎 find it

11

X. Cao, W.-S. Chou and J. Gu, *On the number of solutions of certain diagonal equations over finite fields*, Finite Fields Appl. **42** (2016), 225–252. 🔎 find it

12

F. N. Castro, I. Rubio, P. Guan and R. Figueroa, *On systems of linear and diagonal equation of degree $p^i + 1$ over finite fields of characteristic p*, Finite Fields Appl. **14** (2008), 648–657. 🔎 find it

13

A. Cauchy and J. Liouville, *Rapport sur un mémoire de M. Lamé relatif au dernier théorème de Fermat*, C. R. Acad. Sci. Paris **9** (1839), 359–363. 🔎 find it

14

J. Giménez, G. Matera, M. Pérez and M. Privitelli, *Smooth symmetric systems over a finite field and applications*, J. Algebra (2024); arXiv:2312.09477.

15

J. F. Gottig, M. Pérez and M. Privitelli, *An approach to the moments subset sum problem through systems of diagonal equations over finite fields*, arXiv:2401.06964 (2024).

16

C. Helou, *Cauchy–Mirimanoff polynomials*, C. R. Math. Rep. Acad. Sci. Canada **19** (1997), no. 2, 51–57. 🔎 find it

17

K. Isham, *Problems relating to arcs in projective space and subrings in $\mathbb{Z}^n$*, Ph.D. dissertation, University of California, Irvine, 2021; https://escholarship.org/uc/item/0ms4x869. See §2.3.3, Prop. 2.3.7, Ex. 2.3.8 and §2.4.2.

18

T. Lai, A. Marino, A. Robinson and D. Wan, *Moment subset sums over finite fields*, Finite Fields Appl. **62** (2020), 101607; arXiv:1910.05894.

19

E. Kani and M. Rosen, *Idempotent relations and factors of Jacobians*, Math. Ann. **284** (1989), no. 2, 307–327. 🔎 find it

20

H. Karapetyan and R. Hambardzumyan, *Irreducibility and locus of complex roots of polynomials related to Fermat's Last Theorem*, arXiv:2510.00020 (2025–2026).

21

J. Li and D. Wan, *A new sieve for distinct coordinate counting*, Sci. China Math. **53** (2010), 2351–2362. 🔎 find it

22

A. Blokhuis, G. Marino, F. Mazzocca and O. Polverino, *On almost small and almost large super-Vandermonde sets in* $\mathrm{GF}(q)$, Des. Codes Cryptogr. **84** (2017), 197–201. 🔎 find it

23

J. H. Conway and A. J. Jones, *Trigonometric diophantine equations (On vanishing sums of roots of unity)*, Acta Arith. **30** (1976), 229–240, Thm. 5. 🔎 find it

24

H. B. Mann, *On linear relations between roots of unity*, Mathematika **12** (1965), 107–117, Thm. 1; DOI:10.1112/S0025579300005210. A modern statement with a full proof is Schwartz–Solymosi–de Zeeuw [26 , Thm. 3.1] .

25

P. Ribenboim, *Fermat's Last Theorem for Amateurs*, Springer, New York, 1999, pp. 226–230. 🔎 find it

26

R. Schwartz, J. Solymosi and F. de Zeeuw, *Rational distances with rational angles*, Mathematika **58** (2012), 409–418; arXiv:1008.3671, Thm. 3.1.

27

M. Moisio and K. Ranto, *Elliptic curves and explicit enumeration of irreducible polynomials with two coefficients prescribed*, Finite Fields Appl. **14** (2008), 798–815. 🔎 find it

28

J. S. Milne, *Modular Functions and Modular Forms*, course notes, v1.31 (2017), Prop. 4.16. 🔎 find it

29

S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22** (1916), no. 9, 159–184. 🔎 find it

30

L. J. Mordell, *On Mr Ramanujan's empirical expansions of modular functions*, Proc. Cambridge Philos. Soc. **19** (1917), 117–124. 🔎 find it

31

P. Deligne, *La conjecture de Weil. I*, Publ. Math. IHES **43** (1974), 273–307. find it

32 D. Mirimanoff, *Sur l'équation* $(x+1)^l - x^l - 1 = 0$, Nouv. Ann. Math. **3** (1903), 385–397. find it

33 P. M. Nanninga, *Cauchy–Mirimanoff and related polynomials*, J. Aust. Math. Soc. **92** (2012), no. 2, 269–280. find it

34 P. M. Nanninga, *Cauchy–Mirimanoff and related polynomials*, Ph.D. thesis, Australian National University, 2013. find it

35 Y. Zhang, D. Zheng, X. Wang and M. Yuan, *Some new results on the second moment k-subset sum problem over finite fields*, Finite Fields Appl. **117** (2027), 102877; DOI:10.1016/j.ffa.2026.102877. find it

36 J. Nguyen, *Higher moments subset sum problem over finite fields*, Ph.D. dissertation, University of California, Irvine, 2019; https://escholarship.org/uc/item/2cr0w697.

37 C. Marín, *Subsets of* $\mathbb{F}_p$ *with two prescribed moments, Part I: the pair* $\{1,3\}$ *— exact counts of totally split squarefree polynomials with a non-consecutive pair of prescribed coefficients, and exactly when those counts are elementary*, 2026; doi:10.5281/zenodo.22886540.

38 C. Marín, *The order generated by a column of the* $\mathrm{SU}(n)_k$ *S-matrix is locally a numerical semigroup ring, Part II*, 2026; doi:10.5281/zenodo.22864266.

39 J. Tate, *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144. find it

40 W. C. Waterhouse, *Abelian varieties over finite fields*, Ann. Sci. École Norm. Sup. (4) **2** (1969), 521–560. find it

41 P. Tzermias, *On Cauchy–Liouville–Mirimanoff polynomials*, Canad. Math. Bull. **50** (2007), no. 2, 313–320. find it

42

P. Tzermias, *On Cauchy–Liouville–Mirimanoff polynomials II*, Funct. Approx. Comment. Math. **46** (2012), no. 1, 15–27; DOI:10.7169/facm/2012.46.1.2. 🔎 find it

43

I. Todhunter, *Theory of Equations*, Macmillan and Co., London, 1861. 🔎 find it

44

T. Muir, *Cauchy's theorem regarding the divisibility of* $(x+y)^n+(-x)^n+(-y)^n$, Messenger of Math. **8**(2) (1878), 119–120. 🔎 find it

45

P. Sziklai and M. Takáts, *Vandermonde sets and super-Vandermonde sets*, Finite Fields Appl. **14** (2008), 1056–1067; DOI:10.1016/j.ffa.2008.06.004. Theorem 1.12. 🔎 find it

46

J. R. Stembridge, *Counting points on varieties over finite fields related to a conjecture of Kontsevich*, Ann. Comb. **2** (1998), no. 4, 365–385. 🔎 find it

47

M. Stanojkovski and C. Voll, *Hessian matrices, automorphisms of p-groups, and torsion points of elliptic curves*, Math. Ann. **381** (2021), 593–629; arXiv:1912.09860, §1.5.2.