Quantum state transfer through a lossy bus: fundamental error limits for every control pulse, with application to IBM’s meter-scale interconnects

Quantum state transfer · quantum interconnects · optimal control bounds

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Statements

Theorem 4.1 — Loss floor 4
Remark 4.2 — Ramped couplings 5
Lemma 5.1 7
Theorem 5.2 — Endpoint occupation 8
Lemma 6.1 — Frontier for partial transfer 10
Theorem 6.2 — Asymptotic floor 11
Theorem 6.3 — Attaining the constant 12
Lemma 7.1 — Coherence is an amplitude 14
Proposition 7.2 — Windows with a rate per element 15
Proposition 7.3 — Reciprocity 17
Remark 9.1 — Joint bound 18
Remark Lab — Check your link (the lab of this blueprint) 18

At a glance

What the paper contributes

A qubit hands its state to another qubit through a lossy bus (a resonator, a waveguide or a cable mode), with each coupler bounded by G. The paper gives error floors that hold for *every* control pulse, every duration and any number of bus modes, and applies them to IBM's meter-scale interconnect.

- **A floor at every duration** (Theorem 4.1, Isabelle): $P(T) \le e^{-\gamma T}/[1 + (\kappa - \gamma)/(G^2 T)]$.
- **The end qubits cannot be bypassed** (Theorem 5.2, Isabelle): the excitation must sit in the end qubits for an integrated time whose lower bound tends to $\pi/(2G)$ for a complete transfer.
- **The asymptotic constant** (Theorems 6.2 and 6.3): the perturbative optimum $2\sqrt{2\gamma(\kappa-\gamma)}/G$ becomes a lower bound over all controls, and an explicit pulse attains it with explicit error terms.
- **Dephasing and real devices** (Section 7): the coherence with the vacuum is an amplitude with rates $2/T_2$; a version with a rate for each element gives the floor of a real device; reciprocity makes the direction irrelevant.
- **IBM's links** (Section 8): floors 6.03–10.05 $\times 10^{-4}$ on $1 - F_B$, certified with ball arithmetic, against measured errors 11–16 times larger; which hardware change lowers the floor; three other platforms.

The lab **Check your link** computes the floor of your own link with the same code as the paper's tables.

Open in the laboratory, with the values of the paper: IBM Q240–Q247 · IBM Q241–Q246 · IBM Q242–Q245 · IBM Q243–Q244 · Heya et al. · Chang et al. · Burkhart et al. (1 − P) · unequal couplers (10.9 / 7.1 MHz) · g/2π = 22.5 MHz · towards 10^{-4} (design map). Each opens the three labs of the section with those values: the floor, the bound at the measured duration, the pulses, the error budget, the design map and the sliders.

🚀 **Start here: Theorem 4.1**: the loss floor: P(T) ≤ e^(−γT)/[1 + (κ−γ)/(G²T)] for every pulse and every duration.

Main results:

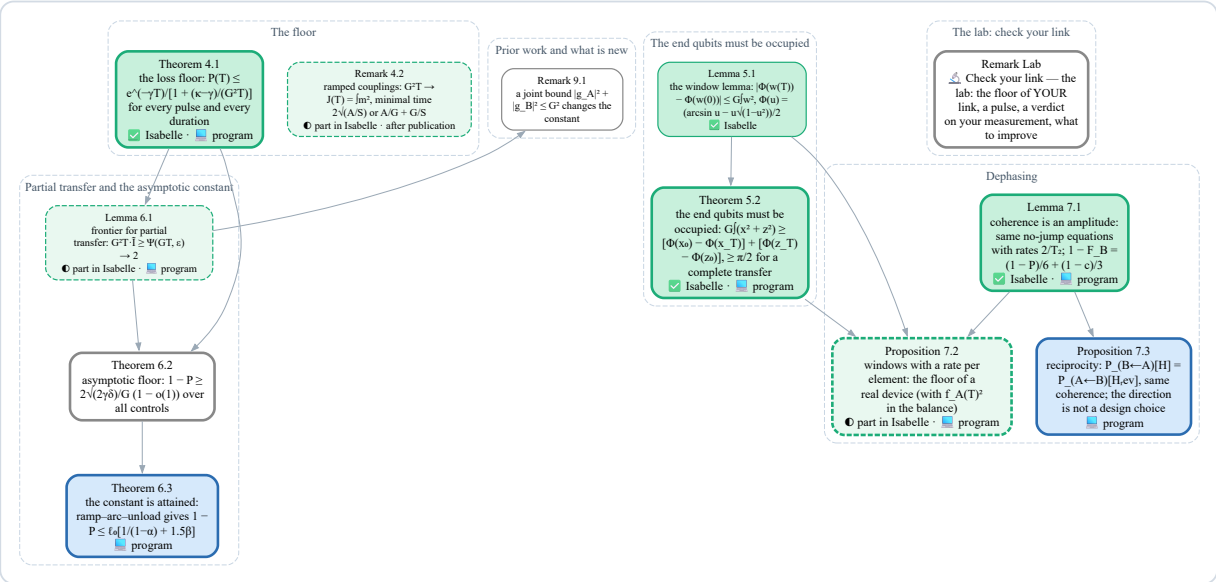
- **Theorem 4.1**: the loss floor: P(T) ≤ e^(−γT)/[1 + (κ−γ)/(G²T)] for every pulse and every duration
- **Theorem 5.2**: the end qubits must be occupied: G∫(x² + z²) ≥ [Φ(x₀) − Φ(x_T)] + [Φ(z_T) − Φ(z₀)], ≥ π/2 for a complete transfer
- **Theorem 6.2**: asymptotic floor: 1 − P ≥ 2√(2γδ)/G (1 − o(1)) over all controls
- **Theorem 6.3**: the constant is attained: ramp–arc–unload gives 1 − P ≤ ℓ₀[1/(1−α) + 1.5β]
- **Lemma 7.1**: coherence is an amplitude: same no-jump equations with rates 2/T₂; 1 − F_B = (1 − P)/6 + (1 − c)/3
- **Proposition 7.2**: windows with a rate per element: the floor of a real device (with f_A(T)² in the balance)
- **Proposition 7.3**: reciprocity: P_(B←A)[H] = P_(A←B)[H_rev], same coherence; the direction is not a design choice
- **Remark Lab**: 🔬 Check your link — the lab: the floor of YOUR link, a pulse, a verdict on your measurement, what to improve

12 results | ✅ 0 entirely in Lean | ◐ 3 partly in Lean | ✅ 4 in Isabelle

💻 7 checked by a program | written proof only | (a result can carry several tags)

🗺️ **Map of the results (click a box to go to it)**

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



📐 Setting (copied from the paper, §3)

One excitation; emitter qubit A, receiver qubit B, bus C with $M \geq 1$ modes and an arbitrary Hermitian internal Hamiltonian $H_C(t)$ (detunings allowed). The qubits couple to the bus through vectors $g_A(t), g_B(t) \in \mathbb{C}^M$ with

$$\|g_A(t)\| \leq G, \qquad \|g_B(t)\| \leq G \qquad \text{for all } t, \tag{1}$$

each bounded separately (a "box"). The hypotheses are: H1, pure loss at zero temperature: jumps take the excitation to the vacuum, which does not repopulate B; qubits lose at rate γ, bus modes at rates collected in $K(t) \succeq \kappa \mathbb{1}$. H2, no direct A–B coupling; no drive on B. H3 (Section 7 only), Markovian dephasing: Lindblad operators $\sqrt{1/T_{1,j}}\,|0\rangle\langle j|$ and $\sqrt{2\gamma_{\varphi,j}}\,|j\rangle\langle j|$ with $\gamma_{\varphi,j} = 1/T_{2,j} - 1/(2T_{1,j})$. H4 (Section 8 only), one effective bus mode with $G = 2\pi \times 9$ MHz. G bounds the norm of the coupling vector, so a multimode bus with a per-mode bound of 9 MHz is a different constraint. Under H1–H2 the population in B is given exactly by the no-jump evolution of the unnormalised amplitudes $a(t)$, $c(t) \in \mathbb{C}^M$, $b(t)$:

$$\dot{a} = -i\, g_A^\dagger c - \tfrac{\gamma}{2} a, \quad \dot{b} = -i\, g_B^\dagger c - \tfrac{\gamma}{2} b, \quad \dot{c} = -iH_C c - i(g_A a + g_B b) - \tfrac{1}{2} K c, \tag{2}$$

with $a(0) = 1$, $b(0) = 0$, $c(0) = 0$, and $P(T) = |b(T)|^2$.

Rescaled frame. When $\kappa \geq \gamma$, put $\tilde{a} = e^{\gamma t/2} a$, $\tilde{c} = e^{\gamma t/2} c$, $\tilde{b} = e^{\gamma t/2} b$. The qubits become lossless and the bus keeps the loss $K - \gamma \succeq \delta \mathbb{1}$, $\delta := \kappa - \gamma \geq 0$. Write

$$x = |\tilde{a}|, \quad y = \|\tilde{c}\|, \quad z = |\tilde{b}|, \quad \tilde{n} = x^2 + y^2 + z^2, \quad \tilde{I} = \int_0^T y^2\, dt, \quad \tilde{P} = z(T)^2 = e^{\gamma T} P.$$

From (2) and (1), almost everywhere,

$$\dot{\tilde{n}} \leq -\delta\, y^2, \qquad |\dot{x}| \leq G\, y, \qquad |\dot{z}| \leq G\, y. \tag{3}$$

The lower bounds of Sections 4–6 use only (3) and $\tilde{n}(0) = 1$; this is why the number of modes, the phases and H_C never enter. The Isabelle proofs start from (3): the step from (2) to (3) is done on paper.

What this blueprint shows.

The numbered statements of the paper, with their proofs, dependencies and what checked them, under the sections that hold them (with the paper's numbers), and the figures and tables they use. The full text is in the PDF of the paper.

4 The floor

Theorem 4.1. Isabelle **(Loss floor)** p. 7

Under H1–H2 with $\kappa \geq \gamma$, for every control satisfying (1) and every $T>0$,

$$\boxed{P(T) \leq \frac{e^{-\gamma T}}{1 + (\kappa - \gamma)/(G^2 T)}} \qquad (1)$$

Isabelle · ours **show**

`loss_floor` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Loss_Floor.thy`, line 73 — from the reduced inequalities (3), derivatives outside a countable set (piecewise-smooth controls)

```
theorem loss_floor:
  fixes n y z n' z' :: "real \<Rightarrow> real" and G T \<delta> :: real and S ::
"real set"
  assumes T: "0 < T" and G: "0 < G" and d0: "0 \<le> \<delta>" and S: "countable S"
    and nc: "continuous_on {0..T} n" and yc: "continuous_on {0..T} y" and zc:
"continuous_on {0..T} z"
    and y0: "\<forall>t\<in>{0..T}. 0 \<le> y t"
    and n0: "n 0 \<le> 1" and z0: "z 0 = 0" and zT: "(z T)\<^sup>2 \<le> n T"
    and der: "\<forall>t\<in>{0<..<T} - S. (n has_real_derivative n' t) (at t) \
<and> (z has_real_derivative z' t) (at t)"
    and bnd: "\<forall>t\<in>{0<..<T} - S. n' t \<le> - \<delta> * (y t)\<^sup>2 \
<and> \<bar>z' t\<bar> \<le> G * y t"
  shows "(z T)\<^sup>2 * (1 + \<delta> / (G\<^sup>2 * T)) \<le> 1"
```

Trust base: 🔒 kernel only. Isabelle: 1 result, no oracles

🆕 **Classical and new:** the paper found no earlier statement for finite T, lossy qubits, separately bounded couplings and any number of modes (§9); the norm balance itself is classical (Gorshkov et al. 2007), and Hughes, Goodwin and Horak 2024 is closest in method for a different task.

↳ Used in: Lemma 6.1 (p. 10), Theorem 6.2 (p. 11)

Proof

By (3), $\tilde{P} \leq \tilde{n}(T) \leq 1 - \delta \tilde{I}$. Since $z(0) = 0$ and $|\dot{z}| \leq G y$, $\sqrt{\tilde{P}} = z(T) \leq G \int_0^T y \leq G\sqrt{T \tilde{I}}$ by Cauchy–Schwarz, so $\tilde{I} \geq \tilde{P}/(G^2 T)$. Hence $\tilde{P}\,(1 + \delta/(G^2 T)) \leq 1$, and $P = e^{-\gamma T}\tilde{P}$.

💡 **In plain words (commentary, not part of the paper)**

A transfer has two ways to lose the excitation. If it goes slowly, the qubits decay: that is the factor $e^{-\gamma T}$. If it goes fast, the excitation must pass through the bus, and the bus loses more: with couplers of strength G, moving the excitation in time T forces the bus to hold it for an integrated time of at least $\tilde{P}/(G^2T)$ (Cauchy–Schwarz), and the bus loses at the excess rate $\kappa - \gamma$ while it holds it. No pulse escapes both. The bound holds for every shape, phase and duration and for any number of bus modes, because the proof only uses the three inequalities (3) of the Setting. For the loss ratio of an IBM link ($\kappa/\gamma = 4.2$) the best pulses found by optimal control lie within 4–12% of it (Figure 1). Machine-checked in Isabelle (`loss_floor`), for derivatives outside a countable set.

Evidence & scope

- **Program:** the best loss found by optimal control lies above the floor in all twelve cases of Figure 1; within 4–12
- **Program:** exact control of every IBM floor pipeline: with excess loss only in the bus the strengthened bound of Prop. 7.2 reduces to this theorem (9 cases, ratio 1.000000). `scripts/mision_suelo_cli.py`, `scripts/misiones/`

Remark 4.2. (Ramped couplings) p. 8

If in addition $\|\dot{g}_i(t)\| \le S$ (Euclidean norm of the coupling vector) and $g_i(0) = g_i(T) = 0$, then $\|g_i(t)\| \le m(t) := \min(St, S(T-t), G)$, and Cauchy–Schwarz with weight m gives (4) with G^2T replaced by $J(T) = \int_0^T m^2$ ($S^2T^3/12$ for $T \le 2G/S$, $G^2T - 4G^3/(3S)$ for $T \ge 2G/S$). With $A = \pi/\sqrt{2}$, reparametrising time with $s = \int_0^t m$ gives the minimal time $2\sqrt{A/S}$ when $S \le G^2/A$ (triangular envelope), $A/G + G/S$ otherwise (trapezoidal), attained without losses by $g_A = g_B = m$. (Proved on paper; not formalised.)

Isabelle · ours · part **show**

Only a part is in Isabelle: the minimal time 2√(A/S) or A/G + G/S, after publication; the replacement G²T → J(T) in the loss floor is on paper

`ramped_transfer_time` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Ramp_Time.thy`, line 670 — the minimal time: with the derivatives of the end amplitudes bounded by the trapezoidal envelope times the root of the bus occupation, T ≥ 2√(A/S) or A/G + G/S, A = π/√2 (countable exceptional set)

```
theorem ramped_transfer_time:
  fixes x z x' z' :: "real \<Rightarrow> real" and S G T :: real and E :: "real set"
  assumes S: "0 < S" and G: "0 < G" and T: "0 \<le> T" and E: "countable E"
    and cx: "continuous_on {0..T} x" and cz: "continuous_on {0..T} z"
    and D: "\<forall>t\<in>{0..T}. 0 \<le> x t \<and> 0 \<le> z t \<and> (x t)^2 + (z
t)^2 \<le> 1"
    and x0: "x 0 = 1" and zT: "z T = 1"
    and der: "\<forall>t\<in>{0<..<T} - E. (x has_real_derivative x' t) (at t) \<and>
(z has_real_derivative z' t) (at t)"
    and bnd: "\<forall>t\<in>{0<..<T} - E. \<bar>x' t\<bar> \<le> ramp_env S G T t *
sqrt (1 - (x t)^2 - (z t)^2)
     \<and> \<bar>z' t\<bar> \<le> ramp_env S G T t * sqrt (1 - (x t)^2 - (z t)^2)"
  shows "ramp_Tmin (pi / sqrt 2) S G \<le> T"
```

`ramp_min_time_conv` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Ramp_Time.thy`, line 616 — the converse: that time suffices for the envelope integral to reach A

```
theorem ramp_min_time_conv:
  fixes A S G T :: real
  assumes A: "0 < A" and S: "0 < S" and G: "0 < G" and T: "0 < T"
    and hT: "ramp_Tmin A S G \<le> T"
  shows "A \<le> ramp_I S G T"
```

`ramp_env_integral` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Ramp_Time.thy`, line 427 — the

integral of the triangular or trapezoidal envelope

```
theorem ramp_env_integral:
  fixes S G T :: real
  assumes S: "0 < S" and G: "0 < G" and T: "0 < T"
  shows "((\<lambda>t. ramp_env S G T t) has_integral ramp_I S G T) {0..T}"
```

`ramp_envelope` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Extensions.thy`, line 159 — couplings that vanish at 0 and T, are Lipschitz with constant S and bounded by G stay under the envelope

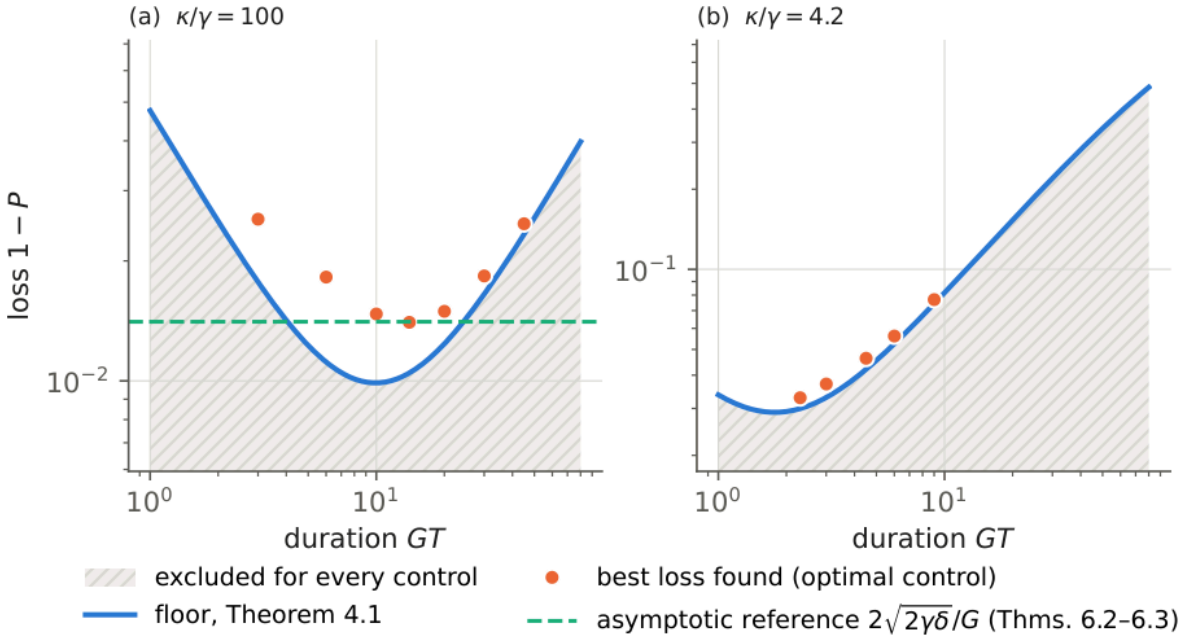
```
lemma ramp_envelope:
  fixes g :: "real \<Rightarrow> 'a::real_normed_vector" and S G T :: real
  assumes lip: "\<forall>s\<in>{0..T}. \<forall>t\<in>{0..T}. norm (g s - g t) \<le>
S * \<bar>s - t\<bar>"
    and g0: "g 0 = 0" and gT: "g T = 0" and gG: "\<forall>t\<in>{0..T}. norm (g t) \
<le> G"
    and t: "t \<in> {0..T}"
  shows "norm (g t) \<le> ramp_env S G T t"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Isabelle: 4 results, no oracles

🆕 **Classical and new:** proved on paper in the paper (which says it is not formalised); the minimal-time part was formalised in Isabelle after publication. The replacement G²T → J(T) in the loss floor is on paper.

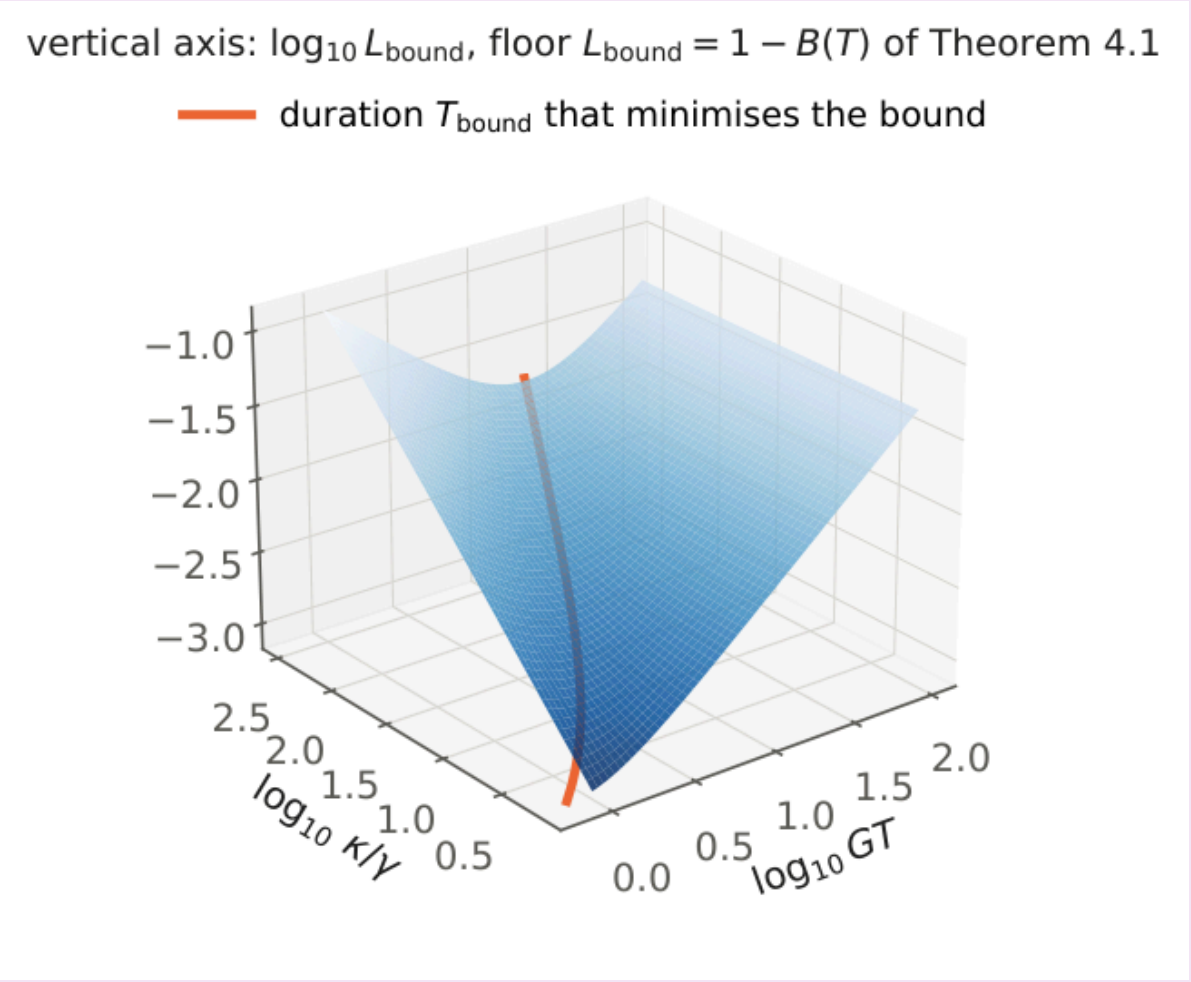
↳ Not used by another result of the paper.

🖼️ **Figure 1**



The floor of Theorem 4.1 against the duration, with $G = 1$. Hatched: losses that no control reaches. Orange: best losses found by optimal control (one mode, 40 piecewise-constant steps, 4 starts), not proven optima. (a) $\gamma = 5 \times 10^{-4}$, $\kappa = 0.05$: the best loss found is 44–63% above the floor at short durations and approaches it at long ones; near T_{asym} it reaches the asymptotic reference $2\sqrt{2\gamma\delta}/G$ of Theorems 6.2–6.3 (dashed), a factor $\sqrt{2}$ above the floor's minimum; the point at $GT = 14$ lies 0.3% below it, which the asymptotic statement allows. (b) $\gamma = 0.00826$, $\kappa = 0.0347$ ($\kappa/\gamma = 4.2$, the ratio of IBM's device): the floor is within 4–12% of the best loss found.

Figure 2



The floor $L_{\text{bound}} = 1 - B(T)$ of Theorem 4.1 as a function of the duration GT and of the loss ratio κ/γ, at $\gamma/G = 5 \times 10^{-4}$ (logarithmic axes). Short transfers pay the bus loss, long ones the qubit loss; the orange curve marks the exact duration T_{bound} that minimises the bound (Section 4).

5 The end qubits must be occupied

Lemma 5.1. Isabelle p. 9

> *Let w be absolutely continuous and $y \geq 0$ continuous on $[0, T]$, with $w \geq 0$, $w^2 + y^2 \leq 1$, and $|\dot{w}| \leq G\, y$ outside a null set. Then $|\Phi(w(T)) - \Phi(w(0))| \leq G \int_0^T w^2\, dt$.*

Isabelle · ours show

`endpoint_window_ac` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Endpoints.thy`, line 395 — absolutely continuous w, exceptions in a null set (every measurable bounded control)

```
lemma endpoint_window_ac:
  fixes w y w' :: "real \<Rightarrow> real" and G T :: real and S :: "real set"
  assumes T: "0 < T" and G: "0 < G" and S: "negligible S"
    and wac: "absolutely_continuous_on {0..T} w" and yc: "continuous_on {0..T} y"
    and nrm: "\<forall>t\<in>{0..T}. 0 \<le> w t \<and> 0 \<le> y t \<and> (w t)\
<^sup>2 + (y t)\<^sup>2 \<le> 1"
    and der: "\<forall>t\<in>{0<..<T} - S. (w has_real_derivative w' t) (at t) \
<and> \<bar>w' t\<bar> \<le> G * y t"
```

```
  shows "\<bar>Phi (w T) - Phi (w 0)\<bar> \<le> G * integral {0..T} (\<lambda>t. (w
t)\<^sup>2)"
```

endpoint_window · Isabelle/AFP/Quantum_Speed_Limits/Relay_Endpoints.thy, line 228 — earlier version, exceptions in a countable set

```
lemma endpoint_window:
  fixes w y w' :: "real \<Rightarrow> real" and G T :: real and S :: "real set"
  assumes T: "0 < T" and G: "0 < G" and S: "countable S"
    and wc: "continuous_on {0..T} w" and yc: "continuous_on {0..T} y"
    and nrm: "\<forall>t\<in>{0..T}. 0 \<le> w t \<and> 0 \<le> y t \<and> (w t)\
<^sup>2 + (y t)\<^sup>2 \<le> 1"
    and der: "\<forall>t\<in>{0<..<T} - S. (w has_real_derivative w' t) (at t) \
<and> \<bar>w' t\<bar> \<le> G * y t"
  shows "\<bar>Phi (w T) - Phi (w 0)\<bar> \<le> G * integral {0..T} (\<lambda>t. (w
t)\<^sup>2)"
```

Trust base: 🔒 kernel only. Isabelle: 2 results, no oracles

↳ Used in: Theorem 5.2 (p. 8), Proposition 7.2 (p. 15)

Proof

Where $w<1$, $|\frac{d}{dt}\Phi(w)| = w^2|\dot{w}|/\sqrt{1-w^2} \le G\,w^2\,y/\sqrt{1-w^2} \le G\,w^2$, because $y \le \sqrt{1-w^2}$. At $w = 1$ the derivative of Φ does not exist, so apply the estimate to $\Phi(\lambda w)$ with $0<\lambda<1$: Φ is Lipschitz on $[0,\lambda]$, hence $\Phi(\lambda w)$ is absolutely continuous, the fundamental theorem of calculus for absolutely continuous functions applies, and almost everywhere

$$\left|\tfrac{d}{dt}\Phi(\lambda w)\right| = \frac{\lambda^3 w^2|\dot{w}|}{\sqrt{1-\lambda^2 w^2}} \le G\,w^2\,\frac{\lambda^3\sqrt{1-w^2}}{\sqrt{1-\lambda^2 w^2}} \le G\,w^2.$$

Let $\lambda \to 1$; Φ is continuous at the end points.

Theorem 5.2. ✅ Isabelle **(Endpoint occupation)** 📕 p. 9

Let x, z be absolutely continuous and y continuous on $[0,T]$, with $x, y, z \ge 0$, $x^2+y^2+z^2 \le 1$ and $|\dot{x}|, |\dot{z}| \le G\,y$ outside a null set. Then

$$G\int_0^T (x^2+z^2)\,dt \;\ge\; [\Phi(x(0)) - \Phi(x(T))] + [\Phi(z(T)) - \Phi(z(0))].$$

In particular, for a complete transfer ($x: 1 \to 0$, $z: 0 \to 1$) and every T,

$$\boxed{G\int_0^T (x^2+z^2)\,dt \ge \frac{\pi}{2}.}$$

Isabelle · ours **show**

endpoint_occupation_ac · Isabelle/AFP/Quantum_Speed_Limits/Relay_Endpoints.thy, line 535 — the complete-transfer form (x: 1 → 0, z: 0 → 1), absolutely continuous, null exceptional set; the general inequality is the window lemma applied twice

```
theorem endpoint_occupation_ac:
  fixes x z y x' z' :: "real \<Rightarrow> real" and G T :: real and S :: "real set"
```

```
  assumes T: "0 < T" and G: "0 < G" and S: "negligible S"
    and xac: "absolutely_continuous_on {0..T} x" and zac: "absolutely_continuous_on
{0..T} z" and yc: "continuous_on {0..T} y"
    and nrm: "\<forall>t\<in>{0..T}. 0 \<le> x t \<and> 0 \<le> z t \<and> 0 \<le> y
t \<and> (x t)\<^sup>2 + (y t)\<^sup>2 + (z t)\<^sup>2 \<le> 1"
    and bd: "x 0 = 1" "x T = 0" "z 0 = 0" "z T = 1"
    and der: "\<forall>t\<in>{0<..<T} - S. (x has_real_derivative x' t) (at t) \
<and> (z has_real_derivative z' t) (at t)
      \<and> \<bar>x' t\<bar> \<le> G * y t \<and> \<bar>z' t\<bar> \<le> G * y t"
  shows "pi / 2 \<le> G * integral {0..T} (\<lambda>t. (x t)\<^sup>2 + (z t)\
<^sup>2)"
```

`endpoint_occupation` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Endpoints.thy`, line 349 — earlier version, exceptions in a countable set

```
theorem endpoint_occupation:
  fixes x z y x' z' :: "real \<Rightarrow> real" and G T :: real and S :: "real set"
  assumes T: "0 < T" and G: "0 < G" and S: "countable S"
    and xc: "continuous_on {0..T} x" and zc: "continuous_on {0..T} z" and yc:
"continuous_on {0..T} y"
    and nrm: "\<forall>t\<in>{0..T}. 0 \<le> x t \<and> 0 \<le> z t \<and> 0 \<le> y
t \<and> (x t)\<^sup>2 + (y t)\<^sup>2 + (z t)\<^sup>2 \<le> 1"
    and bd: "x 0 = 1" "x T = 0" "z 0 = 0" "z T = 1"
    and der: "\<forall>t\<in>{0<..<T} - S. (x has_real_derivative x' t) (at t) \
<and> (z has_real_derivative z' t) (at t)
      \<and> \<bar>x' t\<bar> \<le> G * y t \<and> \<bar>z' t\<bar> \<le> G * y t"
  shows "pi / 2 \<le> G * integral {0..T} (\<lambda>t. (x t)\<^sup>2 + (z t)\
<^sup>2)"
```

Trust base: 🔒 kernel only. Isabelle: 2 results, no oracles

🆕 **Classical and new:** the paper found no earlier statement (§9).

↳ Used in: Proposition 7.2 (p. 15)

Proof

Apply Lemma 5.1 to (x, y) and to (z, y); $x^2 + y^2 \le 1$ and $z^2 + y^2 \le 1$ follow from the hypothesis.

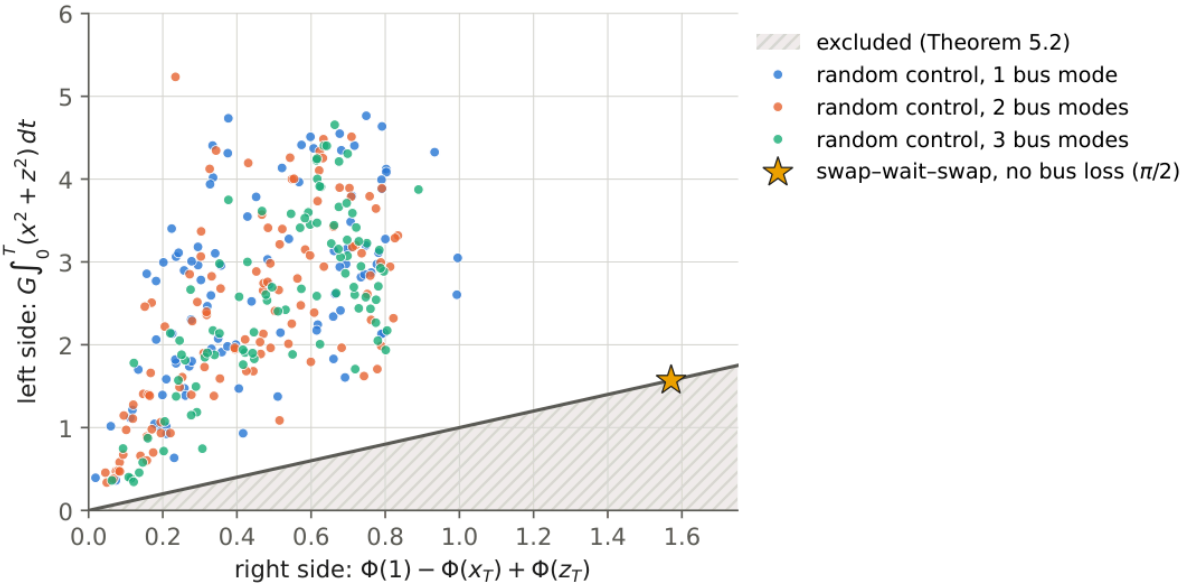
💡 **In plain words (commentary, not part of the paper)**

Protocols such as STIRAP keep the bus nearly empty. This theorem is the other half of the picture: the excitation then has to sit in the end qubits. The function $\Phi(u) = \frac{1}{2}(\arcsin u - u\sqrt{1-u^2})$ cannot change faster than G times the occupation u^2, because the flow out of a qubit is at most G times the bus amplitude, which is at most $\sqrt{1-u^2}$. Emptying A moves Φ by $\pi/4$ and filling B by $\pi/4$ more, so a complete transfer keeps the excitation in the end qubits for an integrated time of at least $\pi/(2G)$, however slowly or cleverly it is done. Swap–wait–swap without bus loss attains it. Machine-checked in Isabelle: the window lemma (`endpoint_window_ac`, every measurable bounded control) and the complete-transfer form (`endpoint_occupation_ac`); the general inequality is the lemma applied twice.

🛡️ **Evidence & scope**

- **Program:** 300 random controls (1–3 modes, complex phases, losses): left side over right side at least 2.11. `scripts/falsar_extremos.py`, `scripts/verificar_v201.py`
- **Program:** $\Phi(1) = \pi/4$ and the swap value $\pi/2$ in exact arithmetic (Sage). `scripts/certificado_sage.sage`

Figure 3



Theorem 5.2 on 300 random controls (piecewise-constant complex couplings with $\|g\| \le G$, random bus Hamiltonian and losses, 1–3 bus modes; qubits lossless). Each point is one control: horizontal, the right side $\Phi(1) - \Phi(x_T) + \Phi(z_T)$; vertical, the occupation $G\int_0^T (x^2+z^2)$. The theorem excludes the hatched region. The star is swap–wait–swap without bus loss, a complete transfer that lies on the boundary at $\pi/2$.

6 Partial transfer and the asymptotic constant

Lemma 6.1. (Frontier for partial transfer) p. 9

Under (3) with $\tilde{n}(0) = 1$, $x(0) = 1$, $y(0) = z(0) = 0$, let $\tilde{P} = 1 - \varepsilon$ with $\varepsilon \le 1/2$. There is $y_c \in [0,1]$ with

$$G^2 T\,\tilde{I} \ge \left(\sqrt{2(1-\varepsilon-y_c^2)_+} - \sqrt{\varepsilon}\right)_+^2 \quad \text{and} \quad G\,\tilde{I} \ge \tfrac{\sqrt{2}}{6}\,y_c^3.$$

Consequently $G^2 T\,\tilde{I} \ge \Psi(GT, \varepsilon)$, where

$$\Psi(\tau,\varepsilon) := \min_{y\in[0,1]} \max\left(1-\varepsilon,\ \left(\sqrt{2(1-\varepsilon-y^2)_+} - \sqrt{\varepsilon}\right)_+^2, \tau\tfrac{\sqrt{2}}{6}y^3\right),$$

and $\Psi(\tau,\varepsilon) \to 2$ as $\tau \to \infty$ and $\varepsilon \to 0$.

Isabelle · ours · part **show**

Only a part is in Isabelle: the two inequalities from the reduced inequalities; Ψ and its limit 2 are on paper

`partial_frontier` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Loss_Floor.thy`, line 229 — the two inequalities of the lemma from the reduced inequalities (countable exceptional set); Ψ and its limit on paper

```
theorem partial_frontier:
  fixes x z p x' z' p' :: "real \<Rightarrow> real" and G T \<epsilon> :: real and S
:: "real set"
  assumes T: "0 < T" and G: "0 < G" and S: "countable S"
    and cx: "continuous_on {0..T} x" and cz: "continuous_on {0..T} z" and cp:
"continuous_on {0..T} p"
    and D: "\<forall>t\<in>{0..T}. 0 \<le> x t \<and> 0 \<le> z t \<and> 0 \<le> p t
\<and> (x t)\<^sup>2 + (z t)\<^sup>2 + p t \<le> 1"
    and x0: "x 0 = 1" and z0: "z 0 = 0" and p0: "p 0 = 0"
    and e0: "0 \<le> \<epsilon>" and e1: "\<epsilon> \<le> 1 / 2" and zT: "1 - \
<epsilon> \<le> (z T)\<^sup>2"
    and nT: "\<forall>t\<in>{0..T}. 1 - \<epsilon> \<le> (x t)\<^sup>2 + (z t)\
<^sup>2 + p t"
    and der: "\<forall>t\<in>{0<..<T} - S. (x has_real_derivative x' t) (at t) \<and>
(z has_real_derivative z' t) (at t) \<and> (p has_real_derivative p' t) (at t)"
    and bnd: "\<forall>t\<in>{0<..<T} - S. \<bar>x' t\<bar> \<le> G * sqrt (p t) \
<and> \<bar>z' t\<bar> \<le> G * sqrt (p t) \<and> p' t \<le> 2 * sqrt 2 * G * sqrt
(p t)"
  shows "\<exists>yc\<in>{0..1}. (max 0 (sqrt (2 * (1 - \<epsilon> - yc\<^sup>2)) -
sqrt \<epsilon>))\<^sup>2 \<le> G\<^sup>2 * T * integral {0..T} p
                    \<and> sqrt 2 / 6 * yc ^ 3 \<le> G * integral {0..T} p"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Isabelle: 1 result, no oracles

↳ Used in: Theorem 6.2 (p. 11), 9.1 (p. 18)

Proof

$x(T)^2 \le \tilde{n}(T) - z(T)^2 \le \varepsilon$, so $x(T) \le \sqrt{\varepsilon} \le \sqrt{1-\varepsilon} = z(T)$; hence $x - z$ goes from 1 to ≤ 0 and vanishes at some t_c, where $x = z = s$ and $y = y_c$. There $2s^2 + y_c^2 = \tilde{n}(t_c) \ge \tilde{n}(T) \ge 1 - \varepsilon$. Then $G\int_0^T y \ge \int_0^{t_c} |\dot{z}| + \int_{t_c}^T |\dot{x}| \ge s + (s - x(T)) \ge \sqrt{2(1-\varepsilon-y_c^2)_+} - \sqrt{\varepsilon}$, and Cauchy–Schwarz gives the first inequality. For the second, the coupling changes y^2 at rate $-2(x\dot{x} + z\dot{z})$, of modulus at most $2G\,y\,(x+z) \le 2\sqrt{2}\,G\,y$, and the bus loss only decreases it; so $\dot{y} \le \sqrt{2}\,G$ where $y{>}0$. As $y(0) = 0$, $t_c \ge y_c/(\sqrt{2}G)$ and $y(t) \ge y_c - \sqrt{2}G(t_c - t)$ on $[t_c - y_c/(\sqrt{2}G), t_c]$, whence $\tilde{I} \ge y_c^3/(3\sqrt{2}\,G)$. The term $1-\varepsilon$ is the step $\tilde{I} \ge \tilde{P}/(G^2T)$ of Theorem 4.1. Finally, if $\Psi{<}2-\eta$ then $y \le (6(2-\eta)/(\sqrt{2}\tau))^{1/3} \to 0$ and the middle term tends to 2, a contradiction for large τ and small ε.

Evidence & scope

- **Program:** the lemma in 10/10 cases. `scripts/suelo_frontera_parcial.py`

Theorem 6.2. (Asymptotic floor) p. 10

Let $L^* = \inf_T \inf_{\text{controls}}(1 - P(T))$ *and* $\ell_0 = 2\sqrt{2\gamma\delta}/G$. *If* $\gamma/\delta \to 0$ *and* $\sqrt{\gamma\delta}/G \to 0$, *then*

$$\boxed{L^* \ge \frac{2\sqrt{2\gamma\delta}}{G}\,\big(1 - o(1)\big).}$$

🆕 **Classical and new:** the constant is the perturbative optimum of Wang, Yan and Chesi (NJP 2017, Eqs. (11)–(12) of the preprint) and Luo, Fang, Wang and Chesi 2024, within families of pulses; the paper adds it as a lower bound over all controls (§9).

↳ Used in: Theorem 6.3 (p. 12)

Proof

Fix a control and T, and let $L = 1 - P$; assume $L \le \ell_0$ (otherwise there is nothing to prove). Since $\tilde{P} = e^{\gamma T} P \ge P$, $\varepsilon = 1 - \tilde{P} \le L \le \ell_0 \to 0$. By Theorem 4.1, $L \ge \delta/(G^2 T + \delta)$, so $G^2 T \ge \delta(1/\ell_0 - 1)$ and $GT \ge \sqrt{\delta/\gamma}/(2\sqrt{2})\,(1 - o(1)) \to \infty$. Lemma 6.1 gives $\tilde{I} \ge (2 - \eta)/(G^2 T)$ with $\eta \to 0$ uniformly. From $P = e^{-\gamma T}\tilde{P} \le e^{-\gamma T}(1 - \delta\tilde{I}) \le e^{-s}$ with $s = \gamma T + \delta\tilde{I}$,

$$L \ge 1 - e^{-s}, \qquad s \ge \gamma T + \frac{(2-\eta)\delta}{G^2 T} \ge \frac{2\sqrt{(2-\eta)\gamma\delta}}{G} = \ell_0(1 - o(1)),$$

and $1 - e^{-s} \ge s(1 - s/2)$, where $s \to 0$ because $1 - e^{-s} \le L \le \ell_0$ gives $s \le -\log(1 - \ell_0)$.

💡 **In plain words (commentary, not part of the paper)**

When the losses are small, Theorem 4.1 is off by a factor 2 in the bus term: the excitation must cross from A to B, at the crossing both qubits hold the same amplitude, and reaching that point and leaving it doubles the bus occupation needed (Lemma 6.1). With that factor the floor becomes $2\sqrt{2\gamma(\kappa - \gamma)}/G$, the constant that perturbative and quasi-adiabatic optimisation had found as the best within families of pulses (Wang et al. 2017; Luo, Fang, Wang and Chesi 2024). Here it is a lower bound over all controls. The lower bound has no explicit error terms and converges slowly; for present devices ($GT \approx 2$–3) Theorem 4.1 is the bound to use.

Theorem 6.3. (Attaining the constant) p. 11

Let $G>0$, $\gamma>0$, $\delta>0$, $\alpha = \sqrt{\gamma\delta}/G$ and $\beta = \gamma/\delta$, with $\alpha \le 0.01$ and $\beta \le 0.01$. Suppose the bus has a mode, an eigenmode of H_C that the couplers can address alone, whose loss rate is exactly $\kappa = \gamma + \delta$ (resonant, or with its detuning compensated by the phase of the couplings). There is a control acting on that mode, with $|g_A|, |g_B| \le G$, such that

$$\boxed{1 - P \le \frac{2\sqrt{2\gamma\delta}}{G}\left(\frac{1}{1-\alpha} + 1.5\,\beta\right).}$$

Together with Theorem 6.2, $L^* = \ell_0\,(1 + o(1))$ when $\alpha, \beta \to 0$, for this realisable one-mode problem. (For a bus whose loss is only bounded below by κ, Theorem 6.2 still holds, but the upper bound need not.)

🆕 **Classical and new:** an explicit pulse with explicit error terms in the full lossy dynamics (§9); proved on paper.

↳ Not used by another result of the paper.

Proof

Work in the rescaled frame with real amplitudes $\tilde{a} = x$, $\tilde{c} = -iy$, $\tilde{b} = -z$, so that $\dot{x} = -g_A y$, $\dot{z} = g_B y$, $\dot{y} = g_A x - g_B z - \frac{\delta}{2} y$. Put $y_0 = \sqrt{\gamma/\delta}$, so that $\delta y_0 = \alpha G$ and $y_0^2 = \beta$. The control has three stages. (i) Ramp: $g_A = G$, $g_B = 0$ until $y = y_0$. (ii) Arc: writing $x = r\cos\theta$, $z = r\sin\theta$, take $g_A = h\sin\theta + d\cos\theta$, $g_B = h\cos\theta - d\sin\theta$ with $d = \delta y_0/(2r)$ and $h = (G - d)/\max(\sin\theta, \cos\theta)$, until $\theta = \pi/2$. Then $g_A x - g_B z = rd = \delta y_0/2$, so $\dot{y} = 0$ and $y \equiv y_0$; $r\dot{\theta} = y_0 h$; and $|g_A|, |g_B| \le G$ because $h\max(\sin\theta, \cos\theta) + d \le G$. (iii) Unload: $g_A = 0$, $g_B = G$ until $y = 0$.

During all three stages $y \le y_0$, so $\delta\tilde{I} \le \delta y_0^2 T = \gamma T$, and therefore $1 - P = 1 - e^{-\gamma T}\tilde{P} \le \gamma T + (1 - \tilde{P}) \le \gamma T + \delta\tilde{I} \le 2\gamma T$ (at the end $x = y = 0$, so $\tilde{P} = \tilde{n}(T) \ge 1 - \delta\tilde{I}$). It remains to bound T. Let $t_* = 0.02/\gamma$ and run the protocol until it ends or until t_*, whichever comes first. On that interval $\gamma t \le 0.02$, and the exact norm balance of the one-mode problem, $\tilde{n}(t) = 1 - \delta\int_0^t y^2 \ge 1 - \gamma t$, gives $x^2 + z^2 \ge 1 - \beta - \gamma t \ge 0.97$ in stages (i)–(ii) and $z \ge 0.97$ in stage (iii); so r stays away from zero and the control is well defined. In (i), $\dot{y} \ge Gx - \frac{\delta}{2}y_0 \ge G(0.97 - \alpha/2) \ge 0.96\,G$, so this stage lasts at most $y_0/(0.96G)$; in (iii), $\dot{y} \le -Gz \le -0.97G$, at most $y_0/(0.97G)$. In (ii), $r \le 1$ and $d \le \delta y_0/1.9 \le \alpha G$, so $\dot{\theta} \ge y_0 G(1-\alpha)/\max(\sin\theta, \cos\theta)$ and the arc lasts at most $\int_0^{\pi/2} \max(\sin\theta, \cos\theta)\, d\theta/(y_0 G(1-\alpha)) = \sqrt{2}/(y_0 G(1-\alpha))$. These speed bounds limit the time spent in each stage even before it ends. Hence the protocol ends by $\overline{T} = \sqrt{2}/(y_0 G(1-\alpha)) + 2.1\, y_0/G$, and $\gamma\overline{T} \le \sqrt{2}\,\alpha/(1-\alpha) + 2.1\alpha\beta \le 0.0145 < 0.02$; if it had not ended by t_* we would have $t_* \le \overline{T} < t_*$, a contradiction. So $T \le \overline{T}$ and

$$2\gamma T \le \frac{2\sqrt{2}\,\sqrt{\gamma\delta}}{G(1-\alpha)} + \frac{4.2\,\gamma y_0}{G} = \ell_0\Big(\frac{1}{1-\alpha} + \frac{4.2}{2\sqrt{2}}\,\beta\Big),$$

and $4.2/(2\sqrt{2}) < 1.5$.

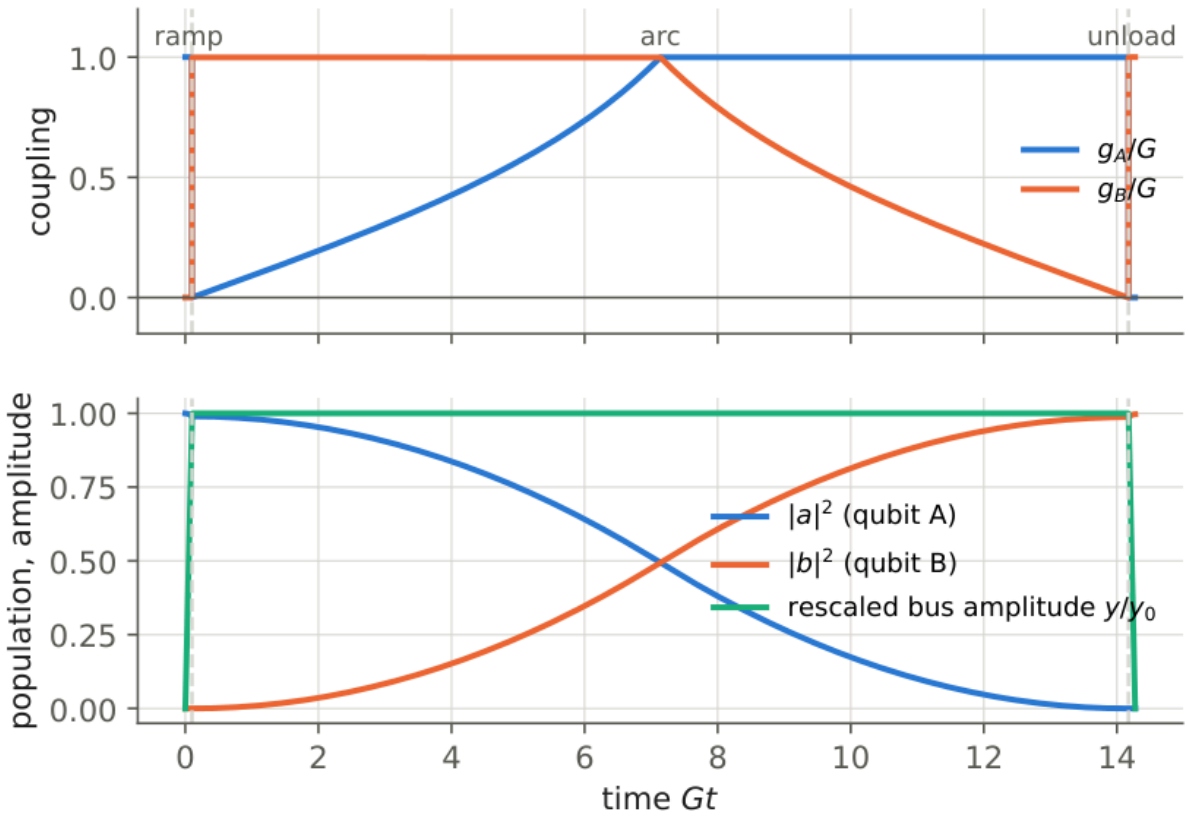
💡 **In plain words (commentary, not part of the paper)**

The constant is reached by an explicit pulse in three stages: a ramp fills the bus up to the rescaled amplitude $y_0 = \sqrt{\gamma/\delta}$, where the two losses balance; an arc turns the excitation from A to B while keeping $y = y_0$ fixed; an unload empties the bus into B. The control can be computed in advance and applied in open loop. For $\alpha = \sqrt{\gamma\delta}/G \le 0.01$ and $\beta = \gamma/\delta \le 0.01$ the paper proves $1 - P \le \ell_0[1/(1-\alpha) + 1.5\beta]$, and integrating the pulse in the full lossy dynamics gives 1.00004–1.0044 times ℓ_0 in six cases, below that bound. Proved on paper; the lab of Figure 4 runs the pulse.

🛡️ **Evidence & scope**

- **Program:** the pulse integrated in the original lossy dynamics: $(1 - P)/\ell_0 = 1.00004$–1.0044 in 6/6 cases, below the bound. `scripts/alcanzable_explicito.py`

Figure 4



The pulse of Theorem 6.3 for $G = 1$, $\gamma = 10^{-4}$, $\kappa = 0.0101$ ($\alpha = 0.001$, $\beta = 0.01$), integrated in the original lossy dynamics. Top: the couplings in the three stages (ramp, arc, unload; dashed lines). Bottom: the populations of the end qubits and the rescaled bus amplitude y/y_0, which stays at 1 during the arc; the physical amplitude is $e^{-\gamma t/2} y_0$. The final $1 - P$ is $1.0035\,\ell_0$.

7 Dephasing

Lemma 7.1. Isabelle **(Coherence is an amplitude)** p. 12

> *Under H1–H3, let ρ evolve from $(|0\rangle + |A\rangle)/\sqrt{2}$ and set $v_j = 2\rho_{j0}$ for j in the single-excitation sector. Then v obeys the no-jump dynamics (2) with the loss rates replaced by $\Gamma_j = 1/T_{1,j} + 2\gamma_{\varphi,j} = 2/T_{2,j}$, and $v(0) = (1, 0, \ldots, 0)$.*

Isabelle · ours **show**

`nojump_form` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Coherence.thy`, line 218 — the vacuum coherences obey the no-jump equations, with the decay rate of each element given by its relaxation and dephasing rates (any excitation-conserving Hamiltonian, n levels)

```
theorem nojump_form:
  assumes H0: "\<And>k. H k 0 = 0"
    and r0: "\<And>j. r j \<ge> 0" and d0: "\<And>j. d j \<ge> 0"
    and j1: "1 \<le> j" and jn: "j \<le> n"
  shows "2 * gen n H r d rho j 0
    = - \<i> * (\<Sum>k\<in>{1..n}. (H j k - (\<i>/2) * of_real (Gam r d j) * (if j
= k then 1 else 0))
                              * (2 * rho k 0))"
```

`gamma_T2` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Coherence.thy`, line 262 — that rate is $2/T_2$

```
theorem gamma_T2:
  fixes T1 T2 :: real
  assumes "T1 > 0" "T2 > 0"
  shows "1 / T1 + 2 * (1 / T2 - 1 / (2 * T1)) = 2 / T2"
```

`six_state_average_phase` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Fidelity.thy`, line 129 — the six-state average gives 1 − F_B = (1 − P)/6 + (1 − c)/3 (c ≥ 0 after the phase correction)

```
corollary six_state_average_phase:
  assumes "c \<ge> 0"
  shows "(Re (fid 1 0 P (of_real c)) + Re (fid 0 1 P (of_real c))
    + Re (fid s2 s2 P (of_real c)) + Re (fid s2 (- s2) P (of_real c))
    + Re (fid s2 (\<i> * s2) P (of_real c)) + Re (fid s2 (- (\<i> * s2)) P (of_real
c))) / 6
   = (3 + P + 2 * c) / 6
   \<and> 1 - (3 + P + 2 * c) / 6 = (1 - P) / 6 + (1 - c) / 3"
```

Trust base: 🔒 kernel only. Isabelle: 3 results, no oracles

🆕 **Classical and new:** elementary; the paper did not find it used to carry loss bounds over to the fidelity (§9).

↳ Used in: Proposition 7.2 (p. 15), Proposition 7.3 (p. 17)

Proof

For $j \neq 0$, every Lindblad operator L of H3 satisfies $\langle j|L = 0$ or $L^\dagger|0\rangle = 0$, so no jump term $L\rho L^\dagger$ feeds ρ_{j0}; the anticommutator contributes $-\frac{1}{2}(L^\dagger L)_{jj}\rho_{j0}$, since $(L^\dagger L)_{00} = 0$. The Hamiltonian conserves the excitation number.

💡 **In plain words (commentary, not part of the paper)**

Fidelity needs more than the population delivered: the phase must survive too. Under Markovian dephasing, the coherence of the transferred state with the vacuum obeys the same no-jump equations as the amplitude, only with each loss rate replaced by $2/T_2$, because no quantum jump feeds that coherence. So every bound of Sections 4–6 on the amplitude is also a bound on the coherence c, and through $1 - F_B = (1 - P)/6 + (1 - c)/3$ on the error that network benchmarking measures. Checked against full Lindblad simulations to 4×10^{-16}.

🛡️ **Evidence & scope**

- **Program:** both formulas against full Lindblad simulations to 4e-16 in six random controls. `scripts/coherencia_efectiva.py`, `scripts/lindblad_relevo.py`

Proposition 7.2. (Windows with a rate per element) p. 13

Let $\zeta = z(T)$. Let f_A solve $f' = -G\sqrt{1-f^2} - \frac{d_A}{2}f$, $f(0) = 1$, and w_ζ solve $w' = -G\sqrt{1-w^2} + \frac{d_B}{2}w$, $w(0) = \zeta$, both taken as the solutions that leave their initial value immediately and stopped at 0 ($w_0 \equiv 0$); equivalently $F_A(f_A(t)) = \min(t, \tau_A)$ and $K_\zeta(w_\zeta(s)) = \min(s, \tau_B(\zeta))$, with $\tau_A = F_A(0)$, $\tau_B(\zeta) = K_\zeta(0)$ and $F_A(u) = \int_u^1 dv/(G\sqrt{1-v^2} + \frac{d_A}{2}v)$ and $K_\zeta(u) = \int_u^\zeta dv/(G\sqrt{1-v^2} - \frac{d_B}{2}v)$. Let M_B solve $M' = G\sqrt{1-M^2} - \frac{d_B}{2}M$, $M(0) = 0$, and $J_B(T) = (1 - e^{-d_B T})/d_B$ ($J_B = T$ if $d_B = 0$). Put $a_A(T) = \int_0^T f_A^2$ and $b_B(T, \zeta) = \int_0^T w_\zeta^2$. Then

$$\boxed{\zeta \le M_B(T), \qquad \zeta^2 + f_A(T)^2 + d_A\, a_A(T) + d_B\, b_B(T,\zeta) + \frac{d_C\,\zeta^2}{G^2 J_B(T)} \le 1,}$$

and the physical amplitude at B is at most $e^{-r_0 T/2}\zeta$.

Isabelle · ours · part **show**

Only a part is in Isabelle: the emitter and receiver windows and the norm chain with one common excess rate; the three-rate balance, f_A(T)² and J_B are on paper

`emitter_window` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Windows.thy`, line 187 — the emitter comparison F_A(x(t)) ≤ t (d > 0, countable exceptional set)

```
theorem emitter_window:
  fixes x x' :: "real \<Rightarrow> real" and G d T :: real and S :: "real set"
  assumes T: "0 < T" and G: "0 < G" and d: "0 < d" and S: "countable S"
    and xc: "continuous_on {0..T} x"
    and rng: "\<forall>t\<in>{0..T}. 0 \<le> x t \<and> x t \<le> 1" and x0: "x 0 =
1"
    and der: "\<forall>t\<in>{0<..<T} - S. (x has_real_derivative x' t) (at t)
        \<and> - G * sqrt (1 - (x t)\<^sup>2) - d * x t \<le> x' t"
  shows "\<forall>t\<in>{0..T}. Fem G d (x t) \<le> t"
```

`receiver_window` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Windows.thy`, line 319 — the receiver comparison and ζ ≤ M_B(T) (d > 0; the barrier z ≤ c < v* as a hypothesis)

```
theorem receiver_window:
  fixes z z' :: "real \<Rightarrow> real" and G d T c :: real and S :: "real set"
  assumes T: "0 < T" and G: "0 < G" and d: "0 < d" and S: "countable S"
    and zc: "continuous_on {0..T} z"
    and nn: "\<forall>t\<in>{0..T}. 0 \<le> z t" and z0: "z 0 = 0"
    and barrier: "\<forall>t\<in>{0..T}. z t \<le> c" and cv: "c < vstar G d"
    and der: "\<forall>t\<in>{0<..<T} - S. (z has_real_derivative z' t) (at t)
        \<and> z' t \<le> G * sqrt (1 - (z t)\<^sup>2) - d * z t"
  shows "Hrec G d (z T) \<le> T"
    and "\<forall>t\<in>{0..T}. Hrec G d (z T) - Hrec G d (z t) \<le> T - t"
```

`norm_chain` · `Isabelle/AFP/Quantum_Speed_Limits/Relay_Windows.thy`, line 380 — the final norm chain with one excess rate common to both qubits; the three-rate balance, f_A² and J_B are on paper

```
corollary norm_chain:
  fixes x z :: "real \<Rightarrow> real" and T a b n \<Delta> :: real
  assumes xc: "continuous_on {0..T} x" and zc: "continuous_on {0..T} z"
    and a: "a \<le> integral {0..T} (\<lambda>t. (x t)\<^sup>2)" and b: "b \<le>
integral {0..T} (\<lambda>t. (z t)\<^sup>2)"
    and n: "n \<le> 1 - \<Delta> * integral {0..T} (\<lambda>t. (x t)\<^sup>2 + (z
t)\<^sup>2)" and D: "0 \<le> \<Delta>"
    and zn: "(z T)\<^sup>2 \<le> n"
  shows "(z T)\<^sup>2 + \<Delta> * (a + b) \<le> 1"
```

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Isabelle: 3 results, no oracles

↳ Not used by another result of the paper.

Proof

Since $y^2 \le \tilde{n} - x^2 \le 1 - x^2$, (6) gives $\dot{x} \ge -G\sqrt{1-x^2} - \frac{d_A}{2}x$, so $\frac{d}{dt}F_A(x(t)) \le 1$, $F_A(x(t)) \le t$ and $x \ge f_A$; hence $\int_0^T x^2 \ge a_A(T)$. In the same way $\dot{z} \le G\sqrt{1-z^2} - \frac{d_B}{2}z$ gives, running time backwards from T, $K_\zeta(z(t)) \le T - t$, so $z(t) \ge w_\zeta(T-t)$ and $\int_0^T z^2 \ge b_B(T,\zeta)$; and forwards from $z(0) = 0$ it gives $z(t) \le M_B(t) \le M_B(T)$. If $d_B{>}0$, M_B stays below the zero $v^* = G/\sqrt{G^2 + d_B^2/4}{<}1$ of $G\sqrt{1-v^2} - \frac{d_B}{2}v$, so the denominator of K_ζ stays positive on $[0,\zeta]$; if $d_B = 0$, $K_\zeta(u) = (\arcsin\zeta - \arcsin u)/G$

and $w_\zeta(s) = \sin([\arcsin\zeta - Gs]_+)$ explicitly. If $d_A = 0$ the same holds with $F_A(u) = \arccos(u)/G$, the singular start $x(0) = 1$ being handled by the regularisation of Lemma 5.1. Next, $\frac{d}{dt}(e^{d_B t/2} z) \le G\, e^{d_B t/2} y$, so $\zeta \le G \int_0^T e^{-d_B(T-t)/2} y \le G\sqrt{J_B(T) \int_0^T y^2}$ by Cauchy–Schwarz, i.e. $\int_0^T y^2 \ge \zeta^2/(G^2 J_B(T))$. Finally, with $x(T) \ge f_A(T)$, $f_A(T)^2 + \zeta^2 \le x(T)^2 + \zeta^2 \le \tilde{n}(T) \le 1 - d_A \int x^2 - d_B \int z^2 - d_C \int y^2$.

In plain words (commentary, not part of the paper)

Real devices do not have equal elements: in IBM's links the worse qubit dephases faster than the bus, and some qubits dephase more slowly than their bus modes. This proposition keeps a rate for each element. In the frame rescaled by the smallest rate, each element keeps its excess. The emitter cannot empty faster than its window f_A, the receiver cannot fill faster than its window run backwards from the final amplitude ζ, the bus must hold the excitation for an integrated time set by J_B, and everything lost along the way comes out of the norm. The largest ζ that survives this balance bounds the amplitude at B. This is the floor of the IBM tables, and of the lab Check your link. Its window comparisons and a one-rate version of the norm chain are machine-checked in Isabelle; the balance with three rates is proved on paper. The four IBM floors are certified with ball arithmetic.

Evidence & scope

- **Program:** the floor in pure Python with angle tables; window integrals equal to 30-digit mpmath quadrature to 8 digits. `scripts/relay_floor.py`, `scripts/prueba_arbitro.py`
- **Program:** one pipeline per IBM link (Excovery lanzadera): exact control, negative control, floor, independent replica within 0.2
- **Program:** the lab Check your link reproduces Table 1 (floor, relaxation only, minimal-time pulse) to four digits; metamorphic bench: reciprocity and dimensional scaling exact, two controls break, monotonicity, lossless limit. `lab/prueba_lab.py`, `lab/prueba_metamorfica.py`

Proposition 7.3. (Reciprocity) p. 14

Under H1–H3, with local loss and diagonal local dephasing at rates that do not depend on time, let a pulse $H(t)$ on $[0,T]$ transfer from A to B, and let $H_{\mathrm{rev}}(t) = H(T-t)^{\mathsf{T}}$ (time-reversed and transposed, the bus Hamiltonian H_C included; for real couplings and real H_C, just played backwards). Then $P_{B\leftarrow A}[H] = P_{A\leftarrow B}[H_{\mathrm{rev}}]$ and $c_{B\leftarrow A}[H] = c_{A\leftarrow B}[H_{\mathrm{rev}}]$, so F_B is the same. If each coupler keeps its own bound and the reversed and transposed pulse is realisable, H_{rev} is admissible whenever H is, and the optimal error is the same in both directions.

NEW **Classical and new:** proved on paper (sketch); time-independent rates, local loss and diagonal dephasing.

↳ Not used by another result of the paper.

Proof

Proof sketch. In the single-excitation block the generator is $-i(H - \frac{i}{2}\Gamma)$ with Γ diagonal, and the no-jump propagator of H_{rev} is the transpose of that of H, so $\langle A|U_{\mathrm{rev}}|B\rangle = \langle B|U|A\rangle$; this gives c by Lemma 7.1, and P under pure loss. With dephasing, the dissipative part of the Lindblad generator acts diagonally on the operator basis $|j\rangle\langle k|$, and the same transposition argument applies to the populations. Numerically,

> 30 random pulses with random relaxation and dephasing give equal P and c in both directions to 9×10^{-16}.

💡 **In plain words (commentary, not part of the paper)**

Play a pulse backwards in time (and transpose it, which for real couplings changes nothing) and it transfers from B to A exactly as well as the original transfers from A to B: same population, same coherence, same fidelity. So the direction of transfer is not a design choice, and a floor may take, at each duration, the smaller of the bounds of the two directions. Proved on paper (sketch) for local loss and diagonal dephasing at rates that do not depend on time; checked on 30 random pulses to 9×10^{-16}.

🛡️ **Evidence & scope**

- **Program:** 30 random pulses with random relaxation and dephasing: equal P and c in both directions to 9e-16. `scripts/verificar_v206.py`

9 Prior work and what is new

Remark 9.1. (Joint bound) p. 20

With a joint bound $|g_A|^2 + |g_B|^2 \le G^2$ the constant changes: in the adiabatic-elimination limit the optimum of Ref. [11] gives a loss $\approx (\pi^2/4)\,\Gamma/(g^2 T)$ in our units, against $2\delta/(G^2 T)$ for the box (Lemma 6.1). The box is the natural constraint when each coupler has its own maximal strength.

↳ Not used by another result of the paper.

The lab: check your link

Remark Lab. (Check your link (the lab of this blueprint))

Enter the maximal couplings and the coherence times T_1, T_2 of the three elements of a qubit–bus–qubit link, and, if you have them, the measured error $1 - F_B$ and the duration of your pulse. The lab computes, in your browser and with the same code as the tables of Section 8: the floor on $1 - F_B$ for every pulse (Proposition 7.2 with $f_A(T)^2$, and reciprocity, Proposition 7.3), the relaxation-only floor, the error of the minimal-time pulse (Lindblad master equation), a verdict on your measured error, and which hardware change lowers the floor most. Every result comes with its explanation. It reproduces Table 1 to four digits.

Proof: in this note · **Covers:** not formalized · **Cite:** doi:10.5281/zenodo.23836449

↳ Not used by another result of the paper.

💡 **In plain words (commentary, not part of the paper)**

Before you tune a pulse, it helps to know how far it can go. With the couplings and coherence times of your three elements, the lab tells you the best error any pulse could reach on that hardware (the floor),

what a plain pulse gives (the minimal-time pulse, simulated with the Lindblad equation), and where your measured error sits between the two. The lab does two different kinds of work. The floor is a theorem: if your measured error is below it, one of the hypotheses does not hold for your device, and the lab lists them in the order to check (couplings, coherence times at the transfer point, slow noise, the metric, the initial state). The simple pulse is a simulation inside the model: it shows that the floor can almost be reached, but it is not a proof. If you are between the two, the gap is what a better pulse could still gain; if you are on the floor, only better hardware helps, and the lab ranks single changes (doubling the coupling, a T_1 or a T_2, or a bus without pure dephasing) by how much each lowers the floor. The numbers come from the same code as Tables 1–3 of the paper.

Tables

Every table of the paper, in its order: click one to unfold it, with where the paper cites it, how its numbers were checked, and its data as CSV. The typeset table is one click away, on its page of the PDF.

Table 1. The four links (each name opens the laboratory with its values), in units of 10^{-4}, all in the metric $1 - F_B$. *Floor*: the bound for every pulse under H1–H4, with the coherence times of each element (Proposition 7.2 with $f_A(T)^2$, and reciprocity), evaluated numerically. *Certified*: a rigorous lower bound on the same floor, computed with ball arithmetic (Arb) as described in the text. *Range*: the floor when all coherence times change by $\pm 20\%$ together and $g/2\pi$ is 8 or 9 MHz. *Relaxation only*: the same floor with $T_2 = 2T_1$ for every element (no pure dephasing). *Minimal time*: $1 - F_B$ of the pulse $g_A = g_B = G$ for $\pi/(\sqrt{2}G) \approx 39$ ns, in a full Lindblad simulation with the same coherence times. *Ratio*: minimal time over floor. *Measured*: the mean transfer error of both directions reported in Ref. [IBMrelay2026] (Extended Data Table 2, with its quoted uncertainty), at their duration T_g.

link	floor	certified	range	relaxation only	minimal time	ratio	measured (T_g)
Q240–Q247	6.06	6.03	5.06–8.51	1.71	9.02	1.49	71 ± 7 (312 ns)
Q241–Q246	8.07	8.05	6.74–11.31	1.51	11.35	1.41	113 ± 5 (296 ns)
Q242–Q245	6.37	6.34	5.31–8.95	1.84	8.83	1.39	101 ± 6 (256 ns)
Q243–Q244	10.09	10.05	8.42–14.11	1.83	14.49	1.44	114 ± 6 (312 ns)

Colour: blue shade = the number's size within its column.

p. 15 in the PDF · ⬇ CSV · cited in the opening, before §1 · cited in §8 Application to IBM's relay · cited in §8.2 Other platforms · cited in §10 Limitations · cited in §A Formula sheet: which bound to use, and when

Table 2. Floor on $1 - F_B$ ($\times 10^{-4}$) after doubling one parameter, with the coherence times of each element and $g/2\pi = 9$ MHz otherwise. "Worse qubit" is the qubit of the pair with the shorter T_2^*; doubling the T_1 of either qubit changes the floor by less than 1% and is not shown. The last column removes the pure dephasing of the bus ($T_2 = 2T_1$); doubling T_2 alone would violate $T_2 \le 2T_1$ for every bus mode here. Each change keeps the other measured times fixed, which may change the inferred dephasing rate of the same element.

link	now	$2G$	$2T_2^*$ worse qubit	$2T_2^*$ other qubit	$2T_1$ bus	bus, no pure dephasing
Q240–Q247	6.06	3.04	4.44 (Q240)	5.39	5.74	6.03
Q241–Q246	8.07	4.05	4.88 (Q241)	7.99	7.83	7.94
Q242–Q245	6.37	3.19	4.30 (Q245)	6.27	5.96	6.19
Q243–Q244	10.09	5.08	7.19 (Q244)	8.56	9.71	10.05

Colour: blue shade = the number's size within its column.

p. 16 in the PDF · ⬇ CSV · cited in §8.1 Which hardware change lowers the floor · cited in §11 Verification and supplementary files · cited in §A Formula sheet: which bound to use, and when

Table 3. Floor on $1 - F_B$ ($\times 10^{-4}$) against the coupling $g/2\pi$ (MHz), with the coherence times of each element kept fixed. 9 MHz is the present value; 22.5 MHz corresponds to $(g/2\pi)/\mathrm{FSR} = 0.15$ with the 80-cm cable.

link	9	12	15	18	22.5
Q240–Q247	6.06	4.55	3.65	3.04	2.43
Q241–Q246	8.07	6.07	4.86	4.05	3.25
Q242–Q245	6.37	4.79	3.83	3.19	2.56
Q243–Q244	10.09	7.59	6.09	5.08	4.07

Colour: blue shade = the number's size within its column.

p. 17 in the PDF · ⬇ CSV · cited in §8.1 Which hardware change lowers the floor

Table 4. Floor on $1 - F_B$ ($\times 10^{-4}$) when the coupling $g/2\pi$ rises and falls with a slope of at most S (MHz/ns), up to 9 MHz, with the coherence times of each element; in brackets, the factor over the floor without a slope limit (the first column; both directions, by reciprocity).

link	no limit	$S = 3$	$S = 1$	$S = 0.3$	$S = 0.1$
Q240–Q247	6.06	6.66 (1.10)	7.90 (1.30)	12.27 (2.02)	21.15 (3.49)
Q241–Q246	8.07	8.84 (1.10)	10.48 (1.30)	16.25 (2.01)	27.85 (3.45)
Q242–Q245	6.37	6.96 (1.09)	8.18 (1.28)	12.54 (1.97)	21.58 (3.39)
Q243–Q244	10.09	11.08 (1.10)	13.15 (1.30)	20.42 (2.02)	34.96 (3.46)

Colour: blue shade = the number's size within its column.

p. 18 in the PDF · ⬇ CSV · cited in §8.1 Which hardware change lowers the floor · cited in §11 Verification and supplementary files · cited in §A Formula sheet: which bound to use, and when

Table 5. Floors of effective one-mode models and measured errors of other transfers through a bus mode (each name opens the laboratory with its parameters). Heya *et al.*: SATD through

a 60-cm coplanar waveguide; error per state transfer from network benchmarking, which estimates $1-F_B$ under the assumptions of the benchmarking. Chang *et al.*: adiabatic transfer through a tunable lossy channel; readout-corrected process fidelity $F_p = 0.96 \pm 0.01$ from process tomography, converted to the average fidelity $(2F_p+1)/3$. Burkhart *et al.*: transfer between two 3D cavities through a bus mode at the nominal coupling used; the measured loss $1-\eta$, with η the single-photon transfer efficiency, is compared with the floor on $1-P$. *Ratio*: measured over floor, an upper bound on the factor by which a better pulse could lower the error within the model, not a measure of how far the device is from its optimum.

experiment	$G/2\pi$ (MHz)	metric	floor	measured	ratio
Heya *et al.* [Heya2025]	3.5	$1-F_B$ (10^{-4})	26.8	120 ± 10	4.5
Chang *et al.* [Chang2020]	15 (peak)	$1-F_B$ (10^{-4})	49.9	267 ± 67	5.4
Burkhart *et al.* [Burkhart2021]	0.56 (nominal)	$1-P$	0.021	0.16 ± 0.01	7.6
IBM (Table 1)	9	$1-F_B$ (10^{-4})	6.03–10.05 (cert.)	71–114	11–16

Colour: blue shade = the number's size within its column.

p. 19 in the PDF | ⬇ CSV | cited in §11 Verification and supplementary files

Table without a number, in the text of §A Formula sheet: which bound to use, and when

formula	valid when	use it to
$P(T) \le B(T) = \dfrac{e^{-\gamma T}}{1+\delta/(G^2 T)}$	every pulse, every T, $\kappa \ge \gamma$	a lower bound $1-B(T)$ loss at a given duration; $1-B(T)$ over T for the link
$T_{\mathrm{bound}} = \dfrac{\sqrt{a^2+4a/\gamma}-a}{2}$, $a=\delta/G^2$	as the row above, with $\gamma>0$ and $\delta>0$	the duration that minimis bound (not the duration pulse)
$1-P \ge 1-\left(1+\dfrac{\sqrt{\gamma\delta}}{G}\right)^{-2}$	same as the first row, every T	a rigorous estimate by h first order $2\sqrt{\gamma\delta}/G$
$1-P \ge \dfrac{2\sqrt{2\gamma\delta}}{G}\,(1-o(1))$	$\alpha=\sqrt{\gamma\delta}/G \to 0$, $\beta=\gamma/\delta \to 0$	the asymptotically exact not for present devices ( )
$1-P \le \ell_0\left(\frac{1}{1-\alpha}+1.5\beta\right)$, $\ell_0 = 2\sqrt{2\gamma\delta}/G$	$\gamma,\delta>0$, $\alpha,\beta \le 0.01$; one bus mode with loss exactly κ	a guarantee for the expli arc–unload pulse (Figur
$T_{\mathrm{asym}} \sim \sqrt{2\delta/\gamma}\,/G$	$\alpha,\beta \to 0$	the leading time scale of asymptotically best puls $\sqrt{2}$ above T_{bound}
$G\displaystyle\int_0^T (x^2+z^2)\,dt \ge \frac{\pi}{4}+\Phi(z(T))$	rescaled moduli $x=\lvert\tilde a\rvert$, $z=\lvert\tilde b\rvert$, A emptied; $\to \pi/2$ for a	the occupation of the en the frame where they are for lossy end qubits use t windows of Prop. 7.2

	complete transfer, possible only without bus excess loss	
$T_{\min} = \dfrac{\pi}{\sqrt{2}\,G}$	no losses, box bound	the shortest complete tra minimal-time pulse g_A =
$G^2T \to J(T) = \displaystyle\int_0^T m^2,\ m = \min(St, S(T-t), G)$	hypotheses of the first row, plus slew rate $\|\dot g\| \le S$ and $g = 0$ at both ends	the floor with ramped pu $J = S^2T^3/12$ for $T \le 2$ $G^2T - 4G^3/(3S)$ for T
the row of Prop. 7.2 with $G \to m(t)$, $J_B = \displaystyle\int_0^T m^2 e^{-d_B(T-t)}dt$	rates per element and slew rate S	the floor of a real device ramped couplers; how fa couplers must be
$\gamma \to \dfrac{2}{T_{2q}}, \quad \kappa \to \dfrac{2}{T_{2c}}$	Markovian dephasing (H3); each bound keeps its own hypotheses, including $\kappa \ge \gamma$	the lower bounds of Theorems 4.1–6.2 applie instead of P; for one pul , so a guarantee $1 - c^2 \le$ (Thm. 6.3 at the rates $2/$ all its hypotheses) gives $1 - F_B \le L/6 + (1 - \gamma$
$1 - F_B = \dfrac{1-P}{6} + \dfrac{1-c}{3}$	one excitation, Lindblad local; P from the input $\|A\rangle$, c from $(\|0\rangle + \|A\rangle)/\sqrt{2}$, phase of B corrected (Sec. 7)	turning bounds on P and error that network bench measures
$c^2 \le \dfrac{e^{-2T/T_{2q}}}{1 + \Delta_2/(G^2T)}$, $\Delta_2 = 2/T_{2c} - 2/T_{2q}$	H3, equal qubits, bus dephases more ($T_{2c} < T_{2q}$; with several modes, the longest T_2 among all accessible modes)	the dephasing bound wh is the worse element
$\zeta_U(T) = \sup\{0 \le \zeta \le M_B(T) :\ \zeta^2 + f_A^2 + d_A a_A + d_B b_B(\zeta) + \frac{d_C \zeta^2}{G^2 J_B} \le 1\}$, amplitude $\le e^{-r_0 T/2}\zeta_U(T)$	any rates per element (r_A, r_C, r_B; $d_j = r_j - \min r$); $2/T_2$ for c, $1/T_1$ for P; f_A, a_A, b_B, M_B, J_B as in Prop. 7.2; G_A for the emitter terms,	the floor of a real device each element's own cohe times; evaluated numeric tables include f_A^2 and rec the four IBM floors are a certified)

	G_B for the receiver terms	
$P_{B\leftarrow A}[H] = P_{A\leftarrow B}[H_{\mathrm{rev}}]$, same c	local loss and diagonal dephasing; each coupler with its own bound	take, at each T, the smal of the two directions; the of transfer is not a desig

Colour: red = no / fails.

⬇ CSV

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