Robust qubit–memory transfer against quasi-static frequency noise costs time and exposure: exact bounds from Dubins curves

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Statements

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At a glance

🚀 **Start here: Theorem 1**: Robustness costs exposure and time: the disk inequality and the minimal duration T₀ + arccos(|ζ|/z₀)/G.

Main results:

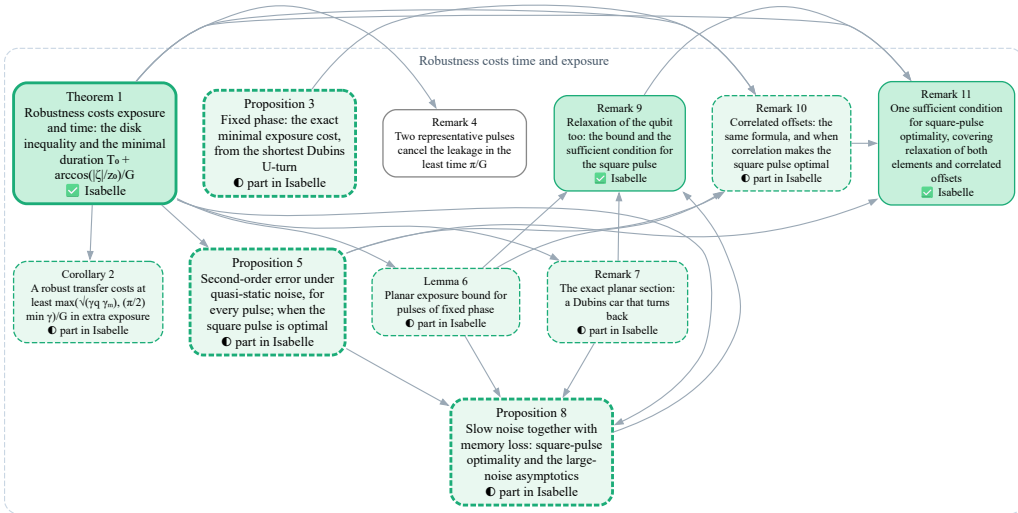
- **Theorem 1**: Robustness costs exposure and time: the disk inequality and the minimal duration T₀ + arccos(|ζ|/z₀)/G
- **Proposition 3**: Fixed phase: the exact minimal exposure cost, from the shortest Dubins U-turn
- **Proposition 5**: Second-order error under quasi-static noise, for every pulse; when the square pulse is optimal
- **Proposition 8**: Slow noise together with memory loss: square-pulse optimality and the large-noise asymptotics

11 results | ✅ **0** entirely in Lean | ◐ **7** partly in Lean | ✅ **3** in Isabelle

💻 **0** checked by a program | written proof only | (a result can carry several tags)

🗺️ Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



📐 Setting (copied from the paper, §§1–2)

A complete transfer of one excitation from a qubit to a memory mode, with no losses, a complex coupling $|g(t)| \le G$ and any detuning $D(t)$, of duration T with $U(T)|a\rangle \propto |b\rangle$. The *exposures* and the *leakage* are the entries of $K = \int_0^T U^\dagger(t) P_a U(t)\, dt = \begin{pmatrix} A & \bar{\zeta} \\ \zeta & B \end{pmatrix}$, $A + B = T$: A and B are the times spent in the qubit and in the memory, and ζ is the first-order leakage under a static shift Δ_0 of the qubit level ($U(T) \mapsto U(T)(I - i\Delta_0 K) + O(\Delta_0^2)$). The square (resonant) pulse has $T_0 = \pi/(2G)$, $A_0 = \pi/(4G)$ and $z_0 = 1/(2G)$. With slow Gaussian offsets of variance σ_q^2 on the qubit and memory decay at rate γ_m, $\Lambda = \sigma_q^2/(\gamma_m G)$.

2 Robustness costs time and exposure

Theorem 1. Isabelle **(for the space curve)** p. 4

> *MarAthm:popsame-setting Robust_Disk.robust_disk, Robust_Disk.robust_disk_G, Robust_Disk.exposure_bound_general, Robust_Core.angle_bound, Robust_Core.equatorial_crossing, Robust_Core.frontier_general, Memory_Robust.concavity_step, Memory_Robust.tent_integral, Memory_Robust.frontier_step, Memory_Robust.disk_from_two_bounds, Memory_Robust.bang_bang_equality, Memory_Robust.turn_wait_turn_equality, Memory_Robust.robust_cost_amgm, Memory_Robust.min_robust_A For every complete transfer without losses, with $|g| \le G$ and any detuning:*
>
> 1. *$A \ge A_0$, $B \ge A_0$ and*
>
> $$\big(|\zeta| - z_0\big)^2 \le (A - A_0)(B - A_0);$$
>
> 2. *if $|\zeta| \le z_0$, then*
>
> $$T \ge T_0 + \frac{1}{G}\arccos\frac{|\zeta|}{z_0}.$$
>
> *(b) is sharp for every $|\zeta| \le z_0$: equality holds for $g = +G$ during $(\pi + d)/(2G)$ followed by $g = -G$ during $d/(2G)$, $d = \arccos(|\zeta|/z_0)$, and for its time reverse. (a) is an equality on the branch $|\zeta| \ge z_0$: for $+G$ during $\alpha/(2G)$, no coupling and no detuning during w, and $+G$ again during $(\pi - \alpha)/(2G)$, for every $\alpha \in [0, \pi]$ and $w \ge 0$ (the square pulse is $w = 0$).*

Isabelle · ours **show**

robust_disk · Isabelle/MemoryAccessFloors/Robust_Disk.thy, line 677

```
theorem robust_disk:
  assumes "sphere_curve n n' T nu"
  shows "pi / (2 * nu) \<le> integral {0..T} (\<lambda>t. (1 + n t $ 3) / 2)"
    and "pi / (2 * nu) \<le> integral {0..T} (\<lambda>t. (1 - n t $ 3) / 2)"
    and "(sqrt ((integral {0..T} (\<lambda>t. n t $ 1))\<^sup>2 + (integral
{0..T} (\<lambda>t. n t $ 2))\<^sup>2) / 2 - 1 / nu)\<^sup>2
      \<le> (integral {0..T} (\<lambda>t. (1 + n t $ 3) / 2) - pi / (2 * nu)) *
         (integral {0..T} (\<lambda>t. (1 - n t $ 3) / 2) - pi / (2 * nu))"
```

robust_disk_G · Isabelle/MemoryAccessFloors/Robust_Disk.thy, line 694

```
theorem robust_disk_G:
  assumes "0 < G" and "sphere_curve n n' T (2 * G)"
  shows "pi / (4 * G) \<le> integral {0..T} (\<lambda>t. (1 + n t $ 3) / 2)"
    and "pi / (4 * G) \<le> integral {0..T} (\<lambda>t. (1 - n t $ 3) / 2)"
    and "(sqrt ((integral {0..T} (\<lambda>t. n t $ 1))\<^sup>2 + (integral
{0..T} (\<lambda>t. n t $ 2))\<^sup>2) / 2 - 1 / (2 * G))\<^sup>2
      \<le> (integral {0..T} (\<lambda>t. (1 + n t $ 3) / 2) - pi / (4 * G)) *
         (integral {0..T} (\<lambda>t. (1 - n t $ 3) / 2) - pi / (4 * G))"
```

exposure_bound_general · Isabelle/MemoryAccessFloors/Robust_Disk.thy, line 667

```
theorem exposure_bound_general:
  assumes "sphere_curve n n' T nu"
  shows "integral {0..T} (\<lambda>t. sqrt ((n t $ 1)\<^sup>2 + (n t $ 2)\
<^sup>2)) - 2 / nu \<le>
    2 * sqrt ((integral {0..T} (\<lambda>t. (1 + n t $ 3) / 2) - pi / (2 * nu))
*
              (integral {0..T} (\<lambda>t. (1 - n t $ 3) / 2) - pi / (2 *
nu)))"
```

angle_bound · Isabelle/MemoryAccessFloors/Robust_Core.thy, line 925

```
theorem angle_bound:
  assumes "bounded_speed_curve n n' T nu"
    and "0 \<le> s" "s \<le> T" "0 \<le> t" "t \<le> T" and "nu * \<bar>s - t\
<bar> \<le> pi"
  shows "cos (nu * \<bar>s - t\<bar>) \<le> inner (n s) (n t)"
```

equatorial_crossing · Isabelle/MemoryAccessFloors/Robust_Core.thy, line 934

```
theorem equatorial_crossing:
  assumes "sphere_curve n n' T nu"
  shows "\<exists>ts. 0 \<le> ts \<and> ts \<le> T \<and> n ts $ 3 = 0"
```

frontier_general · Isabelle/MemoryAccessFloors/Robust_Core.thy, line 942

```
theorem frontier_general:
  assumes "sphere_curve n n' T nu" and "nu * T - pi \<le> pi"
  shows "cos ((nu * T - pi) / 2) \<le>
    nu * (sqrt ((integral {0..T} (\<lambda>t. n t $ 1))\<^sup>2 + (integral
{0..T} (\<lambda>t. n t $ 2))\<^sup>2) / 2)"
```

concavity_step · Isabelle/MemoryAccessFloors/Memory_Robust.thy, line 20

```
lemma concavity_step:
  fixes s a :: real
  assumes "0 \<le> s" and "s \<le> pi" and "0 \<le> a" and "a \<le> s"
  shows "1 + cos (s / 2) \<le> cos (a / 2) + cos ((s - a) / 2)"
```

tent_integral · Isabelle/MemoryAccessFloors/Memory_Robust.thy, line 51

```
lemma tent_integral:
  fixes nu a ts :: real
  assumes "0 < nu" and "0 \<le> a" and "ts = (pi / 2 + a) / nu"
  shows "((\<lambda>t. cos (min (nu * (ts - t)) (pi / 2 + nu * t)))
has_integral
           (2 * cos (a / 2) - 1) / nu) {0..ts}"
```

frontier_step · Isabelle/MemoryAccessFloors/Memory_Robust.thy, line 143

```
lemma frontier_step:
  fixes G nu s zeta z0 T T0 :: real
  assumes "0 < G" and "nu = 2 * G" and "T0 = pi / (2 * G)" and "z0 = 1 / (2 *
G)"
    and "0 \<le> s" and "s \<le> pi" and "0 \<le> zeta" and "zeta \<le> z0"
    and "T = (pi + s) / nu" and "cos (s / 2) \<le> zeta / z0"
  shows "T0 + arccos (zeta / z0) / G \<le> T"
```

disk_from_two_bounds · Isabelle/MemoryAccessFloors/Memory_Robust.thy, line 170

```
lemma disk_from_two_bounds:
  fixes G nu zeta z0 Q X Y :: real
  assumes "0 < G" and "nu = 2 * G" and "z0 = 1 / (2 * G)"
    and "\<bar>2 * zeta - 2 / nu\<bar> \<le> Q - 2 / nu" and "Q - 2 / nu \<le>
2 * sqrt (X * Y)"
    and "0 \<le> X" and "0 \<le> Y"
  shows "(zeta - z0)\<^sup>2 \<le> X * Y"
```

bang_bang_equality · Isabelle/MemoryAccessFloors/Memory_Robust.thy, line 187

```
lemma bang_bang_equality:
  fixes G d T T0 A0 z0 zeta A B :: real
  assumes "0 < G" and "0 \<le> d" and "d \<le> pi / 2"
    and "T0 = pi / (2 * G)" and "A0 = pi / (4 * G)" and "z0 = 1 / (2 * G)"
    and "T = (pi + 2 * d) / (2 * G)" and "zeta = cos d / (2 * G)"
    and "A = A0 + (d - sin d) / (2 * G)" and "B = A0 + (d + sin d) / (2 * G)"
  shows "A + B = T"
    and "T = T0 + arccos (zeta / z0) / G"
    and "(zeta - z0)\<^sup>2 \<le> (A - A0) * (B - A0)"
```

turn_wait_turn_equality · Isabelle/MemoryAccessFloors/Memory_Robust.thy, line 220

```
lemma turn_wait_turn_equality:
  fixes G L alpha z0 A0 zeta A B :: real
```

```
  assumes "0 \<le> L" and "z0 = 1 / (2 * G)" and "A0 = pi / (4 * G)"
    and "zeta = z0 + L * sin alpha / 2"
    and "A = A0 + L * (1 + cos alpha) / 2" and "B = A0 + L * (1 - cos alpha) /
2"
  shows "(zeta - z0)\<^sup>2 = (A - A0) * (B - A0)" and "A + B = 2 * A0 + L"
```

`robust_cost_amgm` · `Isabelle/MemoryAccessFloors/Memory_Robust.thy`, line 237

```
lemma robust_cost_amgm:
  fixes G gq gm x y :: real
  assumes "0 < G" and "0 \<le> gq" and "0 \<le> gm" and "0 \<le> x" and "0 \
<le> y"
    and "1 / (4 * G\<^sup>2) \<le> x * y"
  shows "sqrt (gq * gm) / G \<le> gq * x + gm * y"
```

`min_robust_A` · `Isabelle/MemoryAccessFloors/Memory_Robust.thy`, line 263

```
lemma min_robust_A:
  fixes G d A A0 zeta :: real
  assumes "0 < G" and "d = pi / 2" and "A0 = pi / (4 * G)"
    and "A = A0 + (d - sin d) / (2 * G)" and "zeta = cos d / (2 * G)"
  shows "A = (pi - 1) / (2 * G)" and "zeta = 0"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Theorem 1, p. 4, doi:10.5281/zenodo.24250503

↳ Used in: Corollary 2 (p. 6), 4 (p. 10), Proposition 5 (p. 10), Lemma 6 (p. 14), 7 (p. 17), Proposition 8 (p. 19), 10 (p. 26), 11 (p. 29)

Proof

Choose a time t_e with $n_z(t_e) = 0$ and put $\mathbf{e} = \mathbf{n}(t_e)$, $x(t) = \mathbf{n}(t) \cdot \mathbf{e}$. Since $\mathbf{e}$ is equatorial, $\int x\,dt = \mathbf{e} \cdot \int \mathbf{n}_\perp \le 2|\zeta|$. The angle $\psi(t)$ between $\mathbf{n}(t)$ and $\mathbf{e}$ is at most $\lambda|t - t_e|$, and, because the end points are the poles, at most $\pi/2 + \lambda t$ and $\pi/2 + \lambda(T - t)$; in particular $\lambda t_e \ge \pi/2$ and $\lambda(T - t_e) \ge \pi/2$.

(b) Put $\alpha = \lambda t_e - \pi/2$ and $\beta = \lambda(T - t_e) - \pi/2$, so $\alpha + \beta = \lambda T - \pi =: \Theta$, and assume $\Theta \le \pi$ (otherwise there is nothing to prove). The bound $\bar\psi(t) = \min(\lambda|t - t_e|,\ \pi/2 + \lambda t,\ \pi/2 + \lambda(T - t))$ is at most $\pi/2 + \max(\alpha, \beta)/2 \le \pi$, where the cosine decreases, so $x = \cos\psi \ge \cos\bar\psi$; integrating this piecewise linear angle on each side of t_e gives

$$2|\zeta| \;\ge\; \int_0^T x\,dt \;\ge\; \frac{2}{\lambda}\Big(\cos\frac{\alpha}{2} + \cos\frac{\beta}{2} - 1\Big) \;\ge\; \frac{2}{\lambda}\cos\frac{\Theta}{2},$$

the last step because $\alpha \mapsto \cos(\alpha/2) + \cos((\Theta - \alpha)/2)$ is concave on $[0, \Theta]$ and so is smallest at an end point. With $\lambda = 2G$ this reads $\cos(G(T - T_0)) \le |\zeta|/z_0$, which is (b).

(a) On $|t - t_e| \le \pi/(2\lambda)$ the speed bound gives $x(t) \ge \cos(\lambda|t - t_e|) \ge 0$, so $\int x_+ \ge 2/\lambda$. With $q = |\mathbf{n}_\perp| \ge |x|$ and $Q = \int q$, this gives $Q + 2|\zeta| \ge \int(q + x) \ge 2\int x_+ \ge 4/\lambda$, and with $2|\zeta| \le Q$ we get $\big|2|\zeta| - 2/\lambda\big| \le Q - 2/\lambda$. Let $\theta = \arccos n_z$, the spherical distance of n from the north pole, which is λ-Lipschitz and hence absolutely continuous (the formal proof avoids θ near the poles with a comparison argument, so that curves through a pole are covered), and $h = 1 - \dot\theta/\lambda \ge 0$; the fundamental theorem of calculus gives $\int(1 \pm \cos\theta)\dot\theta = \pi$ and $\int \sin\theta\,\dot\theta = 2$ whether or not θ is monotone, so $A - \pi/(2\lambda) = \frac12\int(1 + \cos\theta)h$, $B - \pi/(2\lambda) = \frac12\int(1 - \cos\theta)h$ and $Q - 2/\lambda = \int \sin\theta\, h$. Cauchy–Schwarz gives $Q - 2/\lambda \le 2\sqrt{(A - \pi/(2\lambda))(B - \pi/(2\lambda))}$, and the two inequalities give (a).

The equality cases are direct computations: for the bang–bang pulse of (b), $T = (\pi + 2d)/(2G)$, $|\zeta| = \cos d/(2G)$, $A = A_0 + (d - \sin d)/(2G)$ and

> $B = A_0 + (d + \sin d)/(2G)$; its time reverse exchanges the two values of $A - A_0$ and $B - A_0$.

An unknown static frequency offset leaves a first-order error measured by the leakage ζ. Reducing it has a price that no pulse escapes: the exposures A, B must exceed A_0 enough to satisfy $(|\zeta| - z_0)^2 \le (A - A_0)(B - A_0)$, and the duration must exceed T_0 by $\arccos(|\zeta|/z_0)/G$. A fully robust transfer ($\zeta = 0$) therefore takes at least π/G, twice the resonant swap.

Evidence & scope

- **Literature:** Companion paper: this statement shares the setting of paper A (Exact limits on writing a superconducting quantum memory), Theorem 4 (silent link, not printed in the paper). `https://karlesmarin.github.io/blueprints/memory-floors/`

Corollary 2. p. 6

> *Memory_Robust.robust_cost_amgm, Robust_Cost_Branches.robust_cost_min_branch, Robust_Cost_Branches.robust_cost_both_branches, Robust_Cost_Branches.robust_cost_equal_rates A complete transfer with $\zeta = 0$ satisfies*
>
> $$\gamma_q(A - A_0) + \gamma_m(B - A_0) \;\ge\; \frac{1}{G}\max\Big(\sqrt{\gamma_q\gamma_m},\; \frac{\pi}{2}\min(\gamma_q, \gamma_m)\Big).$$
>
> *With relaxation only (no dephasing), to first order in the rates the write channel loses $(\gamma_q A + \gamma_m B)/3$, and the square pulse, which does not cancel ζ, reaches the floor $(\gamma_q + \gamma_m)A_0/3$; so **cancelling the leakage costs at least a third of the right-hand side above that floor, whatever the pulse.** For equal rates γ the bound is $\pi\gamma/(2G)$ and is attained by the bang–bang pulse $(3\pi/(4G), \pi/(4G))$; the branch $\sqrt{\gamma_q\gamma_m}/G$ is not attained by pulses of fixed phase and no detuning (see above).*

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

`robust_cost_amgm` · `Isabelle/MemoryAccessFloors/Memory_Robust.thy`, line 237

```
lemma robust_cost_amgm:
  fixes G gq gm x y :: real
  assumes "0 < G" and "0 \<le> gq" and "0 \<le> gm" and "0 \<le> x" and "0 \
<le> y"
    and "1 / (4 * G\<^sup>2) \<le> x * y"
  shows "sqrt (gq * gm) / G \<le> gq * x + gm * y"
```

`robust_cost_min_branch` · `Isabelle/MemoryAccessFloors/Robust_Cost_Branches.thy`, line 23

```
theorem robust_cost_min_branch:
  fixes G gq gm x y :: real
  assumes "0 < G" "0 \<le> gq" "0 \<le> gm" "0 \<le> x" "0 \<le> y" "pi / (2 *
G) \<le> x + y"
  shows "pi / 2 * min gq gm / G \<le> gq * x + gm * y"
```

`robust_cost_both_branches` · `Isabelle/MemoryAccessFloors/Robust_Cost_Branches.thy`, line 39

```
theorem robust_cost_both_branches:
  fixes G gq gm x y :: real
  assumes "0 < G" "0 \<le> gq" "0 \<le> gm" "0 \<le> x" "0 \<le> y"
    and "1 / (4 * G\<^sup>2) \<le> x * y" and "pi / (2 * G) \<le> x + y"
  shows "max (sqrt (gq * gm)) (pi / 2 * min gq gm) / G \<le> gq * x + gm * y"
```

`robust_cost_equal_rates` · `Isabelle/MemoryAccessFloors/Robust_Cost_Branches.thy`, line 57

```
theorem robust_cost_equal_rates:
  fixes G g :: real
  assumes "0 < G" "0 \<le> g"
  shows "max (sqrt (g * g)) (pi / 2 * min g g) / G = pi * g / (2 * G)"
```

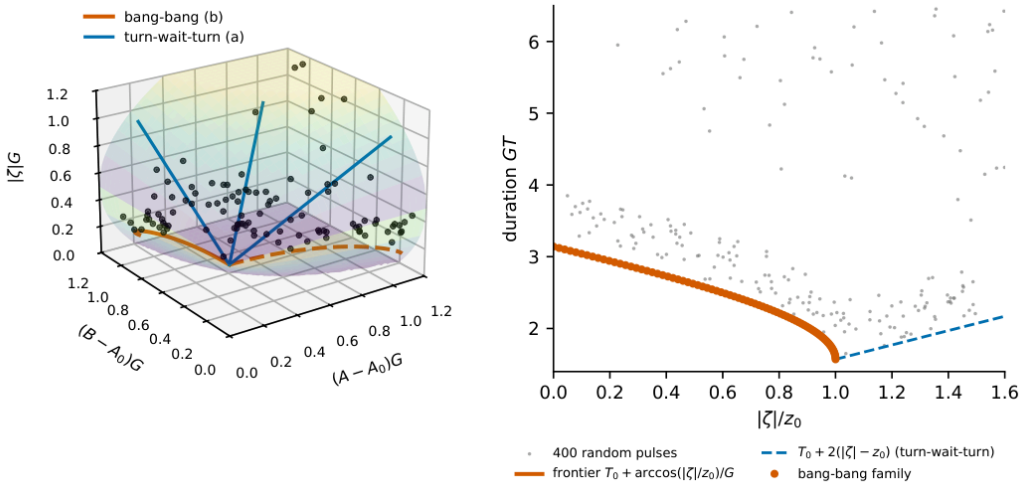
Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Corollary 2, p. 6, doi:10.5281/zenodo.24250503

↳ Not used by another result of the paper.

Proof

By (a) with $\zeta = 0$, $(A - A_0)(B - A_0) \geq 1/(4G^2)$, and $\gamma_q u + \gamma_m v \geq 2\sqrt{\gamma_q \gamma_m \, uv}$. By (b) with $\zeta = 0$, $(A - A_0) + (B - A_0) = T - 2A_0 \geq \pi/(2G)$, and $\gamma_q u + \gamma_m v \geq \min(\gamma_q, \gamma_m)(u + v)$. The bang–bang pulse has $T = \pi/G$. At first order $1 - p = \gamma_q A + \gamma_m B$, since $A = \int |a|^2$ and $B = \int |b|^2$ along the lossless transfer, and $1 - c = \frac{1}{2}(\gamma_q A + \gamma_m B)$, so $r = (3 - p - 2c)/6 = (\gamma_q A + \gamma_m B)/3$.

Figure 1



Theorem 1 as a picture. Left: in the coordinates $(A - A_0, B - A_0, |\zeta|)$ (units of $1/G$) every lossless complete transfer lies between the two sheets $|\zeta| = z_0 \pm \sqrt{(A - A_0)(B - A_0)}$ (translucent); dots: 400 random piecewise-constant pulses with complex coupling $|g| \leq G$ and detuning (those inside the box shown), none of which violates (a) or (b) (smallest margins $5 \cdot 10^{-4}$ and $2 \cdot 10^{-2}$). Orange: the bang–bang family, equality in (b); blue: turn–wait–turn pulses, equality in (a) for $|\zeta| \geq z_0$. Right: duration against $|\zeta|/z_0$, with the frontier $T_0 + \arccos(|\zeta|/z_0)/G$ of (b) and the turn–wait–turn line beyond z_0.

Proposition 3. p. 8

MarAthm:popcompares Robust_Cost.csc_derivative, Robust_Cost.closed_form_value, Robust_Cost.csc_minimum, Robust_Cost.ccc_slope, Robust_Cost.ccc_branch, Robust_Cost.equal_rates, Robust_Cost.total_minimum, Dubins_U_Turn.lstar_le_out, Dubins_U_Turn.lstar_le_lin, Dubins_U_Turn.lstar_le_ccc_a, Dubins_U_Turn.lstar_le_ccc_b, Dubins_U_Turn.ccc_b_above_lin Among complete transfers with $\zeta = 0$ by pulses $g = e^{i\varphi}u(t)$, u real, φ constant, without detuning,

$$\min\left[\gamma_q(A - A_0) + \gamma_m(B - A_0)\right] = \frac{\gamma_q + \gamma_m}{2G} \arccos \frac{|\gamma_q - \gamma_m|}{\gamma_q + \gamma_m},$$

> *attained for $\gamma_q \neq \gamma_m$, both positive, by a turn of $\pi + \phi$, a wait without coupling, and a turn of ϕ back, with $|A - B| = (\gamma_q + \gamma_m)/(2G\sqrt{\gamma_q \gamma_m})$, and for equal positive rates by the bang–bang pulse. If exactly one rate vanishes the value 0 is an infimum, approached only by ever longer waits and not attained; if both vanish the cost is identically zero.*

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

`csc_derivative` · `Isabelle/MemoryAccessFloors/Robust_Cost.thy`, line 139

```
theorem csc_derivative:
  fixes u :: real
  assumes "2 < u"
  shows "(Lcsc has_real_derivative sqrt (u\<^sup>2 - 4) / u) (at u)"
```

`closed_form_value` · `Isabelle/MemoryAccessFloors/Robust_Cost.thy`, line 225

```
theorem closed_form_value:
  fixes a b :: real
  assumes "b < a" and "0 < b"
  shows "sqrt (((a + b) / sqrt (a * b))\<^sup>2 - 4) = (a - b) / sqrt (a * b)"
    and "cost a b ((a + b) / sqrt (a * b)) = (a + b) * arccos ((a - b) / (a +
b))"
```

`csc_minimum` · `Isabelle/MemoryAccessFloors/Robust_Cost.thy`, line 435

```
theorem csc_minimum:
  fixes a b u :: real
  assumes "b < a" and "0 < b" and "2 \<le> u"
  shows "(a + b) * arccos ((a - b) / (a + b)) \<le> cost a b u"
```

`ccc_slope` · `Isabelle/MemoryAccessFloors/Robust_Cost.thy`, line 495

```
theorem ccc_slope:
  fixes u :: real
  assumes "0 \<le> u" and "u \<le> 2"
  shows "(Lccc has_real_derivative - 4 * u / sqrt ((12 - u\<^sup>2) * (4 + u\
<^sup>2))) (at u)"
    and "\<bar>- 4 * u / sqrt ((12 - u\<^sup>2) * (4 + u\<^sup>2))\<bar> \<le>
1"
    and "- 4 * u / sqrt ((12 - u\<^sup>2) * (4 + u\<^sup>2)) \<le> 0"
```

`ccc_branch` · `Isabelle/MemoryAccessFloors/Robust_Cost.thy`, line 595

```
theorem ccc_branch:
  fixes a b u :: real
  assumes "0 \<le> b" and "b \<le> a" and "0 \<le> u" and "u \<le> 2"
  shows "cost a b 2 \<le> cost a b u" and "cost a b 2 = (a + b) * pi / 2 - (a -
b)"
```

`equal_rates` · `Isabelle/MemoryAccessFloors/Robust_Cost.thy`, line 618

```
theorem equal_rates:
  fixes a u :: real
  assumes "0 < a" and "0 \<le> u"
  shows "pi * a \<le> cost a a u"
    and "cost a a 2 = pi * a"
    and "(a + a) * arccos ((a - a) / (a + a)) = pi * a"
```

`total_minimum` · `Isabelle/MemoryAccessFloors/Robust_Cost.thy`, line 652

```
theorem total_minimum:
  fixes a b u :: real
  assumes "b < a" and "0 < b" and "0 \<le> u"
  shows "(a + b) * arccos ((a - b) / (a + b)) \<le> cost a b u"
    and "cost a b ((a + b) / sqrt (a * b)) = (a + b) * arccos ((a - b) / (a +
b))"
```

`lstar_le_out` · `Isabelle/MemoryAccessFloors/Dubins_U_Turn.thy`, line 514

```
theorem lstar_le_out:
  fixes s :: real
  assumes "0 \<le> s"
  shows "Lstar s \<le> Lout s"
```

lstar_le_lin · Isabelle/MemoryAccessFloors/Dubins_U_Turn.thy, line 527

```
theorem lstar_le_lin:
  fixes s :: real
  assumes "2 \<le> s"
  shows "Lstar s \<le> Lin s"
```

lstar_le_ccc_a · Isabelle/MemoryAccessFloors/Dubins_U_Turn.thy, line 540

```
theorem lstar_le_ccc_a:
  fixes s :: real
  assumes "0 \<le> s" and "s \<le> sqrt 12"
  shows "Lstar s \<le> Lccc_a s"
```

lstar_le_ccc_b · Isabelle/MemoryAccessFloors/Dubins_U_Turn.thy, line 553

```
theorem lstar_le_ccc_b:
  fixes s :: real
  assumes "0 \<le> s" and "s \<le> sqrt 12"
  shows "Lstar s \<le> Lccc_b s"
```

ccc_b_above_lin · Isabelle/MemoryAccessFloors/Dubins_U_Turn.thy, line 478

```
theorem ccc_b_above_lin:
  fixes s :: real
  assumes "2 \<le> s" and "s \<le> sqrt 12"
  shows "Lin s \<le> Lccc_b s"
```

folded_curve · Isabelle/MemoryAccessFloors/Meridian_Unfolding.thy, line 615

```
theorem folded_curve: "sphere_curve m m' T nu"
```

folded_planar · Isabelle/MemoryAccessFloors/Meridian_Unfolding.thy, line 628

```
theorem folded_planar: "0 \<le> t \<Longrightarrow> t \<le> T \<Longrightarrow>
m t $ 2 = 0"
```

folded_displacement · Isabelle/MemoryAccessFloors/Meridian_Unfolding.thy, line 631

```
theorem folded_displacement: "integral {0..T} m = integral {0..T} n"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Proposition 3, p. 8, doi:10.5281/zenodo.24250503

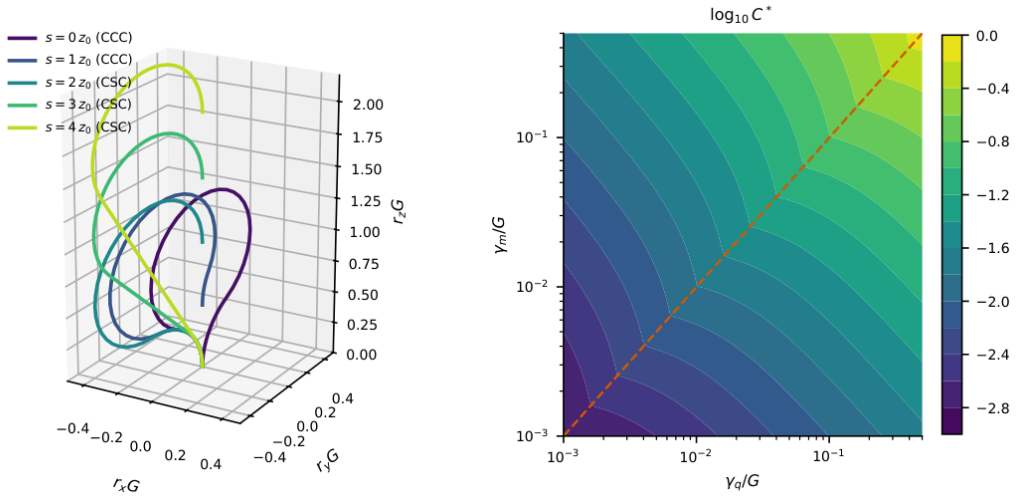
↳ Used in: 10 (p. 26)

Proof

With $A-B=s$ and $A+B\ge L(s)$, the cost is at least $f(s)=\tfrac{1}{2}[(\gamma_q+\gamma_m)L(s)+(\gamma_q-\gamma_m)s]-(\gamma_q+\gamma_m)A_0$ (in units $z_0=1$). L is continuous, and differentiable except at $|s|=2$: at $s=2$ its slope is -1 from the left and 0 from the right, and symmetrically at $s=-2$. On $|s|>2$, $L'(s)=\operatorname{sgn}(s)\sqrt{s^2-4}/|s|$, so f is convex there with its minimum where $\sqrt{s^2-4}/|s|=|\gamma_q-\gamma_m|/(\gamma_q+\gamma_m)$, i.e. $|s|=(\gamma_q+\gamma_m)/\sqrt{\gamma_q\gamma_m}$; at that point the algebraic terms cancel and $f=(\gamma_q+\gamma_m)\arctan(2/\sqrt{s^2-4})$, which is the stated arccosine. On $|s|<2$, $|L'(s)|=4|s|/\sqrt{(12-s^2)(4+s^2)}\le 1$ (equivalent to $s^2\le 4$

), so on the favourable side, where $(\gamma_q - \gamma_m)s \le 0$, f decreases towards $|s| = 2$; and since L is even, $f(s) - f(-s) = (\gamma_q - \gamma_m)s$, so the other side is no better.

Figure 2



Proposition 3. Left: the curve $\mathbf{r}(t) = \int \mathbf{n}$ of the shortest planar U-turns with $\zeta = 0$ for axial displacement $s = 0, 1, \ldots, 4\,z_0$ (CCC for $s{<}2z_0$, CSC beyond); each ends on the axis at height s. Right: the minimum cost $C^* = \frac{\gamma_q+\gamma_m}{2G}\arccos\frac{|\gamma_q-\gamma_m|}{\gamma_q+\gamma_m}$ for pulses of fixed phase and no detuning (colours, $\log_{10} C^*$, with C^* the dimensionless cost of Proposition 3; dashed: equal rates). Its ratio to the bound of Corollary 2 for every pulse depends only on γ_m/γ_q: it is 1 for equal rates and has maximum $(\pi/2 + 2/\pi)\arctan(2/\pi) \approx 1.25140958$, attained at $\gamma_m/\gamma_q = 4/\pi^2$ and $\pi^2/4$, where the two branches of that bound meet.

Among pulses of fixed phase and no detuning, the extra loss that full robustness forces, $\gamma_q(A - A_0) + \gamma_m(B - A_0)$ at $\zeta = 0$, is computed exactly: it is the length of the shortest Dubins U-turn of the space curve, minimised over one parameter.

Evidence & scope

- **Literature:** Companion paper: this statement is compared with paper A (Exact limits on writing a superconducting quantum memory), Theorem 4 (silent link, not printed in the paper). `https://karlesmarin.github.io/blueprints/memory-floors/`

Remark 4. p. 8

Two representative pulses cancel ζ in the least time π/G (controls related to them by a constant phase of g, or by a diagonal gauge transformation with a compensating detuning, have the same curve and exposures): the bang–bang pulse ($+G$ during $3\pi/(4G)$, $-G$ during $\pi/(4G)$), with $A = (\pi - 1)/(2G)$ and $B = (\pi + 1)/(2G)$, and its time reverse, with the two values exchanged. Both exceed A_0 in each exposure; the better one, when the qubit is the noisier element, spends $(\pi - 1)/(2G)$ in the qubit, more than the square pulse, and so worsens the dephasing of the superposition with the vacuum. **Robustness of the population is not the same as robustness of the write channel.**

↳ Not used by another result of the paper.

Proposition 5. p. 9

MarAprop:qsextendsMarAcor:channeluses Unitarity_Second_Order.unitary_coefficients, Unitarity_Second_Order.unitarity_aa, Unitarity_Second_Order.transfer_modulus,

Slow_Noise_Second_Order.second_order_algebra,Slow_Noise_Second_Order.second_order_combination, Slow_Noise_Second_Order.square_value, Slow_Noise_Second_Order.qubit_only_floor, Slow_Noise_Second_Order.min_time_robust_value, Slow_Noise_Kernel.robust_worse_than_square_kernel, Slow_Noise_Kernel.square_vs_floor_kernel, Slow_Noise_Second_Order.csc_A_excess, Slow_Noise_Kernel.bang_bang_beats_square_kernel For every complete transfer without losses, with $|g| \le G$ and any detuning, to second order in the offsets

$$r = \frac{2\,\mathbb{E}(\xi_q - \xi_m)^2\,|\zeta|^2 + \mathbb{E}(\xi_q A + \xi_m B)^2}{6}.$$

1. *For the square pulse this is the bound of , $(8+\pi^2)(\sigma_q^2+\sigma_m^2)/(96G^2)$ for uncorrelated offsets: that bound is exact at this order.*
2. *With noise on the qubit only, $r \ge \pi^2\sigma_q^2/(96G^2)$ for every pulse. The square pulse is 1.81 times this floor; the better of the two pulses of least duration that cancel ζ gives $(\pi-1)^2\sigma_q^2/(24G^2)$, $2.66\,\%$ above the square pulse. The floor is the infimum of the second-order coefficient: zero-leakage pulses that spend their extra time in the memory approach it only as $T \to \infty$, outside the regime $\sigma_q T \ll 1$ where the expansion holds (with memory loss, see Proposition 8).*
3. *With independent offsets on both elements and $\varrho = \sigma_m^2/\sigma_q^2$: for $\varrho \le 0.1264$ the bang–bang pulse of Theorem 1(b) with $d = 0.78$ does better than the square pulse (by $3.3\,\%$ at $\varrho = 0.1$ with $d = 0.848$), and for $\varrho \ge 7.91$ its time reverse does (certified by interval arithmetic, directly at 7.91). The square pulse is optimal at this order among all pulses for $0.137 \le \varrho \le 7.30$ (the upper end certified through $10/73$, whose reciprocal is 7.3) (certified by interval arithmetic over the region allowed by Theorem 1; a floating-point minimisation over that relaxed region suggests that this certificate is lost near 0.1357, which is not established as a physical threshold). (The problem is symmetric: F/ϱ is the same problem with $1/\varrho$ and A, B exchanged.)*
4. *For pulses of fixed phase and no detuning the set of ratios at which the square pulse is optimal at this order is a closed interval $[\varrho_c, 1/\varrho_c]$, and ϱ_c is the crossing of the bang–bang family:*

$$\varrho_c = \varrho_{\mathrm{bb}} := \inf\{\varrho : \min_d [F_{\mathrm{bb}}(\varrho, d) - F_{\mathrm{sq}}(\varrho)] \ge 0\} = 0.12645352248540975\ldots$$

(enclosed in an interval of width 10^{-12}; certified by interval arithmetic, using the planar exposure bound of Lemma 6). Below ϱ_{bb} the bang–bang pulse with d near $d_* = 0.7717755295532\ldots$ is strictly better, and its time reverse for the reciprocal ratios. For pulses of varying phase or with detuning, $0.1264 < \varrho < 0.137$ and $7.30 < \varrho < 7.91$ remain undecided.

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

`unitary_coefficients` · `Isabelle/MemoryAccessFloors/Unitarity_Second_Order.thy`, line 297

```
theorem unitary_coefficients:
  fixes X1 X2 :: cm and R :: "real \<Rightarrow> cm" and \<delta> K :: real
  assumes "\<delta> > 0" and "\<And>e. \<bar>e\<bar> \<le> \<delta> \
<Longrightarrow> cm_bd K (R e)"
    and "\<And>e. \<bar>e\<bar> \<le> \<delta> \<Longrightarrow> cmul (cadj (cV
X1 X2 (R e) e)) (cV X1 X2 (R e) e) = cone"
  shows "cadj X1 = X1"
    and "cadd X2 (cadj X2) = cmul X1 X1"
```

`unitarity_aa` · `Isabelle/MemoryAccessFloors/Unitarity_Second_Order.thy`, line 316

```
theorem unitarity_aa:
  fixes X1 X2 :: cm
  assumes "cadj X1 = X1" and "cadd X2 (cadj X2) = cmul X1 X1"
  shows "Im (a11 X1) = 0"
    and "2 * Re (a11 X2) = (norm (a21 X1))\<^sup>2 + (Re (a11 X1))\<^sup>2"
```

transfer_modulus · Isabelle/MemoryAccessFloors/Unitarity_Second_Order.thy, line 352

```
theorem transfer_modulus:
  fixes X1 X2 :: cm and e :: real
  assumes "cadj X1 = X1" and "cadd X2 (cadj X2) = cmul X1 X1"
  shows "(norm (1 - \<i> * of_real e * a11 X1 - (of_real e)\<^sup>2 * a11 X2))\
<^sup>2
    = 1 - e\<^sup>2 * (norm (a21 X1))\<^sup>2
      + e^3 * (2 * Re (a11 X1) * Im (a11 X2) + e * (norm (a11 X2))\<^sup>2)"
```

second_order_algebra · Isabelle/MemoryAccessFloors/Slow_Noise_Second_Order.thy, line 25

```
lemma second_order_algebra:
  fixes e f z w :: real
  assumes "0 \<le> z" and "1 - 2 * w + f\<^sup>2 = 1 - e\<^sup>2 * z\<^sup>2"
  shows "2 * w = e\<^sup>2 * z\<^sup>2 + f\<^sup>2"
```

second_order_combination · Isabelle/MemoryAccessFloors/Slow_Noise_Second_Order.thy, line 31

```
lemma second_order_combination:
  fixes Ee2 Ef2 z P C p c :: real
  assumes "P = Ee2 * z\<^sup>2" and "C = Ee2 * z\<^sup>2 + Ef2" and "p = 1 - P"
and "c = 1 - C / 2"
  shows "(3 - p - 2 * c) / 6 = (2 * (Ee2 * z\<^sup>2) + Ef2) / 6"
```

square_value · Isabelle/MemoryAccessFloors/Slow_Noise_Second_Order.thy, line 39

```
lemma square_value:
  fixes sq2 sm2 G z0 A0 :: real
  assumes "G > 0" and "sq2 \<ge> 0" and "sm2 \<ge> 0" and "z0 = 1 / (2 * G)"
and "A0 = pi / (4 * G)"
  shows "(2 * (sq2 + sm2) * z0\<^sup>2 + sq2 * A0\<^sup>2 + sm2 * A0\<^sup>2) /
6
         = (8 + pi\<^sup>2) * (sq2 + sm2) / (96 * G\<^sup>2)"
```

qubit_only_floor · Isabelle/MemoryAccessFloors/Slow_Noise_Second_Order.thy, line 48

```
lemma qubit_only_floor:
  fixes sigma2 z A G :: real
  assumes "G > 0" and "sigma2 \<ge> 0" and "A \<ge> pi / (4 * G)"
  shows "(2 * sigma2 * z\<^sup>2 + sigma2 * A\<^sup>2) / 6 \<ge> pi\<^sup>2 *
sigma2 / (96 * G\<^sup>2)"
```

min_time_robust_value · Isabelle/MemoryAccessFloors/Slow_Noise_Second_Order.thy, line 65

```
lemma min_time_robust_value:
  fixes sigma2 G A :: real
  assumes "G > 0" and "A = (pi - 1) / (2 * G)"
  shows "(2 * sigma2 * 0\<^sup>2 + sigma2 * A\<^sup>2) / 6 = (pi - 1)\<^sup>2 *
sigma2 / (24 * G\<^sup>2)"
```

robust_worse_than_square_kernel · Isabelle/MemoryAccessFloors/Slow_Noise_Kernel.thy, line 25

```
lemma robust_worse_than_square_kernel: "(pi - 1)\<^sup>2 / 24 > (8 + pi\
<^sup>2) / 96"
```

square_vs_floor_kernel · Isabelle/MemoryAccessFloors/Slow_Noise_Kernel.thy, line 39

```
lemma square_vs_floor_kernel: "(8 + pi\<^sup>2) / pi\<^sup>2 > 1.81" and "(8 +
pi\<^sup>2) / pi\<^sup>2 < 1.82"
```

csc_A_excess · Isabelle/MemoryAccessFloors/Slow_Noise_Second_Order.thy, line 85

```
lemma csc_A_excess:
  fixes u r :: real
  assumes "u > 2" and "r = sqrt (u\<^sup>2 - 4)"
  shows "0 \<le> arctan (2 / r) + (r - u) / 2"
    and "arctan (2 / r) + (r - u) / 2 \<le> 2 / (u + r)"
    and "2 / (u + r) \<le> 2 / u"
    and "arctan (2 / r) + (r - u) / 2 \<le> 2 / u"
```

bang_bang_beats_square_kernel · Isabelle/MemoryAccessFloors/Slow_Noise_Kernel.thy, line 86

```
lemma bang_bang_beats_square_kernel:
  "F_cost_k (1 / 10) (cos (106 / 125)) (pi / 2 + 106 / 125 - sin (106 / 125))
            (pi / 2 + 106 / 125 + sin (106 / 125))
   < F_cost_k (1 / 10) 1 (pi / 2) (pi / 2)"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Proposition 5, p. 9, doi:10.5281/zenodo.24250503

↳ Used in: Proposition 8 (p. 19), 10 (p. 26), 11 (p. 29)

Proof

A static shift changes $U(T)$ into $U(T)(I - i\Xi_1 - \Xi_2 + \dots)$, with $\Xi_1 = \xi_q K + \xi_m (T - K)$, Ξ_2 quadratic in the offsets and a remainder of third order, bounded uniformly for small offsets. With $U(T)|a\rangle = e^{i\phi}|b\rangle$ the transferred amplitude is $t = e^{i\phi}\big(1 - i(\xi_q A + \xi_m B) - (\Xi_2)_{aa} + \dots\big)$ and the leaked one is $(\xi_q - \xi_m)\zeta$ to first order; unitarity, $|t|^2 = 1 - (\xi_q - \xi_m)^2|\zeta|^2$, fixes $2\,\mathrm{Re}(\Xi_2)_{aa} = (\xi_q - \xi_m)^2|\zeta|^2 + (\xi_q A + \xi_m B)^2$. For each fixed pulse and duration, write $\xi = \epsilon X$ with X of fixed law and finite second moments: the averaged formula below then holds with a remainder $o(\epsilon^2)$; a cubic averaged remainder needs a further moment assumption. Averaging, the first-order terms vanish, $p = 1 - \mathbb{E}(\xi_q - \xi_m)^2|\zeta|^2$ and $c = |\mathbb{E}t| = 1 - \frac{1}{2}\mathbb{E}\big[(\xi_q - \xi_m)^2|\zeta|^2 + (\xi_q A + \xi_m B)^2\big]$, and $r = (3 - p - 2c)/6$ gives the formula. (i) is $|\zeta| = z_0$, $A = B = A_0$. (ii): $A \geq A_0$ by Theorem 1(a); the minimum-time pulses have $A = (\pi \mp 1)/(2G)$; and the zero-leakage pulses $C_{\pi+\phi}SC_{\phi}$ (a turn of $\pi + \phi$, a wait, a turn of ϕ back) have $A - A_0 = O(1/(G^2 T))$. (iii): the lower bound minimises $2(\sigma_q^2 + \sigma_m^2)|\zeta|^2 + \sigma_q^2 A^2 + \sigma_m^2 B^2$ over the region allowed by Theorem 1, $A, B \geq A_0$, (a), (b) and $T \geq T_0 + 2(|\zeta| - z_0)$. The difference g between a point of this region and the square pulse is linear in ϱ, so it suffices to show $g \geq 0$ at $\varrho = 0.137$ and at $\varrho = 1$; the region is symmetric in A and B, and dividing by ϱ gives $1 \leq \varrho \leq 1/0.137$. For $|\zeta| \geq z_0$ every term of g is non-negative; g increases with $A - A_0$ and $B - A_0$, so the latter can be taken at its least admissible value; near $|\zeta| = z_0$, $\arccos(|\zeta|/z_0) \geq \sqrt{2(1 - |\zeta|/z_0)}$ gives $g \geq 0$ by a one-line estimate; the rest of the region, a rectangle in $(|\zeta|, A - A_0)$, is covered by 551 accepted leaf boxes (1101 nodes of the subdivision evaluated; 554 leaves at $\varrho = 10/73$) on each of which an interval lower bound of g is non-negative (at $\varrho = 0.137$ computed twice, with mpmath and with Arb in SageMath; at the exact rational $\varrho = 10/73$, which gives the decimal endpoint 7.30, with the Python certificate). The other side evaluates one bang–bang pulse, $d = 0.78$: its difference with the square pulse is linear in ϱ and negative at $\varrho = 0$ and $\varrho = 0.1264$ (enclosed by intervals), and its time reverse covers $\varrho \geq 1/0.1264 \approx 7.9114$; at $\varrho = 7.91$ the reversed pulse is checked directly (difference $-7.6 \cdot 10^{-4}$, slope -0.74, both enclosed). The analytic patch and the subdivided region share the same exactly represented endpoint, rational parameters are enclosed directly, and every coefficient used as a lower bound is rounded downwards.

(iv): for every pulse P the difference $g_P(\varrho)$ with the square pulse is affine in ϱ, so the set $\bigcap_P \{\varrho > 0 : g_P(\varrho) \geq 0\}$ is a closed interval; it contains 1 and is invariant

under $\varrho \mapsto 1/\varrho$, hence it is $[\varrho_c, 1/\varrho_c]$. For pulses of fixed phase and no detuning the curve $\mathbf{r}$ is planar, and the region of Theorem 1 can be replaced by the bound of Lemma 6: for $|\zeta|<z_0$ and $A - A_0$ below its bang–bang value, $B - A_0$ is at least that of the family; symmetrically with A and B exchanged. Since F grows with A and with B, the minimum over that branch lies on the family, a two-parameter set covered by intervals (605 accepted leaves, in the variables (ϕ, w), which avoid the singularity at $\phi \to 0$). The rest is covered by the relaxed region (22 322 leaves in all at $\varrho = 0.12645395$). By linearity in ϱ it suffices to add $\varrho = 1$, and the symmetry gives the upper end: $\varrho_c \le 0.12645395$. At 0.126453 the bang–bang witness $d = 0.7717755$ is strictly better (difference $-3.0 \cdot 10^{-6}$, enclosed), and its difference increases with ϱ, so it is better on all of $[0, 0.126453]$ and $\varrho_c>0.126453$; `certificar_umbral_plano.py`. The identification $\varrho_c = \varrho_{\mathrm{bb}}$ is a second certificate (`certificar_rho_c_igual_bb.py`). A Krawczyk operator in (ϱ, d) proves that $\{h = 0,\ \partial_d h = 0\}$, $h = F_{\mathrm{bb}} - F_{\mathrm{sq}}$, has a unique zero in a box, with enclosure widths below $6 \cdot 10^{-41}$ for ϱ_{bb} and $4 \cdot 10^{-40}$ for d_* (regular Jacobian: $\partial_\varrho h = 5.80$, $\partial_d^2 h = 2.54$). Since $h(\varrho, 0) = 0$ always, at the crossing the global minimum of h is zero and is attained both at $d = 0$ and at the unique nonzero minimiser $d = d_*$; throughout the enclosing interval E of ϱ, the critical branch $d(\varrho)$ is the unique critical point in $U = [d_* \pm 0.01]$, and above the crossing it need not be the global minimiser of h. On $N = U \times [0, 0.02]$ in the family variables (ϕ, w) the Hessian of g is positive definite (by intervals), uniformly for $\varrho \in E$. At $w = 0$ the exact identities $g_\phi = h_d$ and $g_w = h_d/2$ make the full gradient vanish at $(d(\varrho), 0)$; since N is convex, Taylor's formula gives $g \ge h(\varrho, d(\varrho))$ on N, including its edge $w = 0$. Outside N, the analytic near-square patch and the large-wait tail are treated separately; on the remaining retained branch boxes the interval lower bound is at least $5 \cdot 10^{-7}$ at both endpoints of E, hence throughout E by affinity (equality at the square pulse is allowed in the analytic patch). The reflected branch is certified independently. For the middle region put $x_0 = d - \sin d$ and $y_0 = d + \sin d$ (in the units of Lemma 6): if $x_0 \le a \le y_0$, monotonicity in b and Theorem 1(b) reduce the cost to $g(a, 2d - a)$, which increases in a for $\varrho \le \frac{1}{4}$; if $a \ge y_0$, then $g(a, b) \ge g(y_0, x_0) \ge g(x_0, y_0)$ because $\varrho \le 1$; so the middle region is bounded below by $h(\varrho, d)$ without subdivision. Hence the square pulse is optimal at ϱ_{bb}; since $h(\varrho_{\mathrm{bb}}, d_*) = 0$ and $\partial_\varrho h>0$ at every positive d, the same nonzero bang–bang pulse is strictly better for every $\varrho<\varrho_{\mathrm{bb}}$. Controls (a neighbourhood too small, a ratio 10^{-4} lower, an off-centre Krawczyk box) fail.

With slow random offsets ξ_q, ξ_m on both elements, the average error of a transfer at second order is $r = [2\,\mathbb{E}(\xi_q - \xi_m)^2|\zeta|^2 + \mathbb{E}(\xi_q A + \xi_m B)^2]/6$ for every pulse: the pulse enters only through ζ, A and B, and the bounds of the theorem decide when the square pulse is optimal.

Evidence & scope

- **Literature:** Companion paper: this statement extends paper A (Exact limits on writing a superconducting quantum memory), Proposition 14 (silent link, not printed in the paper). `https://karlesmarin.github.io/blueprints/memory-floors/`
- **Literature:** Companion paper: this statement uses paper A (Exact limits on writing a superconducting quantum memory), Corollary 9 (silent link, not printed in the paper). `https://karlesmarin.github.io/blueprints/memory-floors/`

Lemma 6. (planar exposure bound) p. 11

Planar_Exposure_Bound.planar_exposure_bound, Planar_Exposure_Bound.planar_exposure_bound_inner, Planar_Exposure_Bound.arc_bounds, Planar_Exposure_Bound.arc_attains, Planar_Exposure_Bound.a_z_strict_mono,

Planar_Exposure_Bound.theta_z_antimono, Planar_Exposure_Bound.multi_turn_dominated, Memory_Floors_Pieces.a_z_deriv, Memory_Floors_Pieces.b_z_deriv, Memory_Floors_Pieces.a_z_range, Memory_Floors_Pieces.odd_turns_integral For a pulse of fixed phase and no detuning, put $a = 2G(A - A_0)$, $b = 2G(B - A_0)$, $z = |\zeta|/z_0 \in [0,1)$ and $d = \arccos z$, and for $0<\phi \le d$

$$a_z(\phi) = \phi - (1+z)\tan\tfrac{\phi}{2}, \qquad b_z(\phi) = \phi + (1-z)\cot\tfrac{\phi}{2}.$$

Then $a_z' = (\cos\phi - z)/(1+\cos\phi) \ge 0$, $b_z' = (z - \cos\phi)/(1-\cos\phi) \le 0$, a_z maps $(0, d]$ onto $(0, d - \sin d]$, and every such pulse with $0<a \le d - \sin d$ has

$$b \;\ge\; b_z\big(a_z^{-1}(a)\big),$$

and symmetrically with a and b exchanged. This is the type-$(1,0,-1)$ family of the remark below, written at fixed $|\zeta|$.

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

`planar_exposure_bound` · `Isabelle/MemoryAccessFloors/Planar_Exposure_Bound.thy`, line 433

```
theorem planar_exposure_bound:
  fixes z a b :: real
  assumes z: "0 \<le> z" "z < 1" and a: "0 < a" "a \<le> arccos z - sin (arccos
z)"
    and th: "arccos z \<le> (a + b) / 2"
    and arc: "a < arccos z - sin (arccos z) \<Longrightarrow>
              \<exists>phi. 0 < phi \<and> phi < arccos z \<and> a - b = x3 ((a
+ b) / 2) phi
                       \<and> y3 ((a + b) / 2) phi \<le> 2 * z"
  shows "\<exists>beta. 0 < beta \<and> beta \<le> arccos z \<and> a_z z beta =
a
                \<and> (\<forall>beta'. 0 < beta' \<longrightarrow> beta' \<le>
arccos z \<longrightarrow> a_z z beta' = a \<longrightarrow> beta' = beta)
                \<and> b_z z beta \<le> b"
```

`planar_exposure_bound_inner` · `Isabelle/MemoryAccessFloors/Planar_Exposure_Bound.thy`, line 391

```
theorem planar_exposure_bound_inner:
  fixes z a b :: real
  assumes z: "0 \<le> z" "z < 1" and a: "0 < a" "a < arccos z - sin (arccos z)"
    and arc: "\<exists>phi. 0 < phi \<and> phi < arccos z \<and> a - b = x3 ((a
+ b) / 2) phi
                       \<and> y3 ((a + b) / 2) phi \<le> 2 * z"
  shows "\<exists>beta. 0 < beta \<and> beta \<le> arccos z \<and> a_z z beta =
a
                \<and> (\<forall>beta'. 0 < beta' \<longrightarrow> beta' \<le>
arccos z \<longrightarrow> a_z z beta' = a \<longrightarrow> beta' = beta)
                \<and> b_z z beta \<le> b"
```

`arc_bounds` · `Isabelle/MemoryAccessFloors/Planar_Exposure_Bound.thy`, line 358

```
theorem arc_bounds:
  fixes z phi theta :: real
  assumes "0 < phi" and "phi < pi" and "y3 theta phi \<le> 2 * z"
  shows "theta_z z phi \<le> theta" and "a_z z phi \<le> theta + x3 theta phi /
2"
```

`arc_attains` · `Isabelle/MemoryAccessFloors/Planar_Exposure_Bound.thy`, line 335

```
theorem arc_attains:
  fixes z phi :: real
  assumes "0 \<le> z" and "z < 1" and "0 < phi" and "phi \<le> arccos z"
  defines "th \<equiv> theta_z z phi"
  shows "th + x3 th phi / 2 = a_z z phi" and "th - x3 th phi / 2 = b_z z phi"
    and "y3 th phi = 2 * z" and "0 \<le> th - phi"
```

a_z_strict_mono · Isabelle/MemoryAccessFloors/Planar_Exposure_Bound.thy, line 181

```
theorem a_z_strict_mono:
  fixes z p1 p2 :: real
  assumes "0 \<le> z" and "z < 1" and "0 < p1" and "p1 < p2" and "p2 \<le>
arccos z"
  shows "a_z z p1 < a_z z p2"
```

theta_z_antimono · Isabelle/MemoryAccessFloors/Planar_Exposure_Bound.thy, line 219

```
theorem theta_z_antimono:
  fixes z p1 p2 :: real
  assumes "0 \<le> z" and "z < 1" and "0 < p1" and "p1 \<le> p2" and "p2 \<le>
arccos z"
  shows "theta_z z p2 \<le> theta_z z p1"
```

multi_turn_dominated · Isabelle/MemoryAccessFloors/Planar_Exposure_Bound.thy, line 463

```
theorem multi_turn_dominated:
  fixes z :: real and k :: nat
  assumes "0 \<le> z" and "z < 1" and "1 \<le> k"
  shows "arccos z - sin (arccos z) < real k * pi" and "arccos z + sin (arccos
z) < real k * pi"
```

a_z_deriv · Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy, line 423

```
theorem a_z_deriv:
  fixes z phi :: real
  assumes "- pi < phi" and "phi < pi"
  shows "(a_z z has_real_derivative (cos phi - z) / (1 + cos phi)) (at phi)"
```

b_z_deriv · Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy, line 443

```
theorem b_z_deriv:
  fixes z phi :: real
  assumes "0 < phi" and "phi < pi"
  shows "(b_z z has_real_derivative (z - cos phi) / (1 - cos phi)) (at phi)"
```

a_z_range · Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy, line 609

```
theorem a_z_range:
  fixes z y :: real
  assumes "0 \<le> z" and "z < 1" and "0 < y" and "y \<le> arccos z - sin
(arccos z)"
  shows "\<exists>phi. 0 < phi \<and> phi \<le> arccos z \<and> a_z z phi = y"
```

odd_turns_integral · Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy, line 658

```
theorem odd_turns_integral:
  fixes k :: nat
  shows "((\<lambda>alpha. (cos (alpha / 2))\<^sup>2) has_integral ((2 * real k
+ 1) * pi / 2))
         {0..(2 * real k + 1) * pi}"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Lemma 6, p. 11, doi:10.5281/zenodo.24250503

↳ Used in: Proposition 8 (p. 19), 9 (p. 22), 10 (p. 26)

Proof

With fixed phase and no detuning the Bloch direction moves on one meridian, $\mathbf{n} = (\sin\alpha, 0, \cos\alpha)$ with $|\dot{\alpha}| \le 2G$, and $\mathbf{r}$ is the path of a planar Dubins car of heading α; A and B are the times weighted by $\cos^2(\alpha/2)$ and $\sin^2(\alpha/2)$; the axial displacement is $A - B$, and the magnitude of the lateral displacement is $2|\zeta|$. After normalisation, $t_f = \pi + a + b$, $x = a - b$ and $y = \pm 2z$. The write ends at the south

pole, so the final heading is an odd multiple of π. (1) Final headings $\pm(2k+1)\pi$ with $k \geq 1$: for $f \geq 0$ and an absolutely continuous angle, $2G\int_0^T f(\alpha)\,dt \geq \int_0^T f(\alpha)|\dot\alpha|\,dt \geq |\int_0^T f(\alpha)\dot\alpha\,dt|$, so with the primitives of $\cos^2(\alpha/2)$ and $\sin^2(\alpha/2)$, $A, B \geq A_0 + k\pi/(2G)$, i.e. $a, b \geq k\pi$, which exceeds the bang–bang values $d \mp \sin d$ at the same $|\zeta|$ in both exposures; such pulses are dominated by the bang–bang pulse (F grows with A and B). Equality $a = b = k\pi$ forces a full-speed monotone turn and gives $|\zeta| = z_0$, outside the sector $z<1$; the square pulse has the same leakage and smaller exposures. (2) Heading π (by reflecting the control if necessary). Set $\theta = (t_f - \pi)/2 = (a+b)/2$; Theorem 1(b) gives $\theta \geq d$. At heading π the classification of [14 , Lemma 3.1 and Theorem 11.1] gives their Case 1 for $\pi<t_f<3\pi$ and Case 3 for $t_f \geq 3\pi$; the non-simply connected Case 2 does not occur. The lower boundary on the negative-x side is the $U_3 = (1, 0, -1)$ arc

$$x_3 = -2\sin\phi - 2(\theta-\phi)\cos\phi, \qquad y_3 = 2\cos\phi - 2(\theta-\phi)\sin\phi.$$

In Case 1 the U_6 branch satisfies $a \geq \theta - \sin\theta \geq d - \sin d$ and does not enter the sector under consideration; in Case 3 it contributes no additional boundary arc. U_1 has $y \geq 2$, and U_2 lies on the opposite axial side. The retained U_3 arc includes $0 \leq \phi \leq d$: in Case 1 this follows from $\theta \geq d$, and in Case 3 its first intersection with its reflected arc occurs after $\phi = \pi/2$. Since x_3 is increasing on the relevant arc, every reachable point in the strict sector $a<d-\sin d$ lies above a unique U_3 point at the same x, with $0<\phi<d$. (3) Both lateral signs are covered directly: $y = \pm 2z \leq 2z$ and $y \geq y_3$ imply

$$\theta \;\geq\; \theta_z(\phi) := \phi + \frac{\cos\phi - z}{\sin\phi},$$

and consequently $a \geq a_z(\phi)$ (along the arc the wait is $w = (\cos\phi - z)/(G\sin\phi)$ and the exposures are the closed forms $a_z(\phi)$, $b_z(\phi)$ of the remark, simplified). (4) Let $\beta = a_z^{-1}(a)$. The strict monotonicity of a_z gives $\phi \leq \beta$, while $\theta_z'(\phi) = \cos\phi\,(z - \cos\phi)/\sin^2\phi \leq 0$ for $0<\phi \leq d$; therefore $b = 2\theta - a \geq 2\theta_z(\beta) - a = b_z(\beta)$. At $a = d - \sin d$ the same inequality follows from $\theta \geq d$, and $a = 0$ is excluded by the disk bound for $z<1$. Equality is attained by U_3 with the stated non-negative wait. Finally, time reversal combined with $\alpha \mapsto \pi - \alpha$ exchanges a and b, giving the symmetric assertion. Steps (1), (3) and (4) are elementary; the passage from the sections of [14] , which are stated at fixed duration and unwound final heading, to exposures is our written derivation, not a lemma of that source. The closed forms were checked against piecewise pulses with exact integrals and against the parametrisation of the remark (to $2\cdot 10^{-13}$, `verificar_v289.py`), and the inequality with both lateral signs on 7061 random turn–wait–turn pulses in the sector (`verificar_v291.py`).

Remark 7. (the exact planar section) p. 12

U_Turn_Exposure_Family.family_bang_bang,
U_Turn_Exposure_Family.family_inside_disc,
U_Turn_Exposure_Family.family_sum_above_boundary,
U_Turn_Exposure_Family.family_x_increasing,
U_Turn_Exposure_Family.family_y_decreasing,
U_Turn_Exposure_Family.family_product_tends_to_hyperbola For a pulse of fixed phase and no detuning, the duration, ζ and the axial displacement $A - B$ are the time, the lateral position and the axial position of a planar Dubins car of radius z_0 that turns back to heading π. The set of such end points at a given duration is the section of the

reachable set at heading π, described exactly by Patsko and Fedotov [14 , Lemma 3.1 and Theorem 11.1] : its boundary is made of arcs of extremal controls of four types. In exposures, the part with $|\zeta|<z_0$ and the smallest $A-A_0$ is the arc of their type $(1,0,-1)$: a turn of $\pi+\phi$ at full coupling, a wait w, and a turn of ϕ back,

$$|\zeta| = z_0\big[\cos\phi - Gw\sin\phi\big],$$
$$A-A_0 = \tfrac{1}{2G}\big[\phi - \sin\phi + Gw(1-\cos\phi)\big], \qquad B-A_0 = \tfrac{1}{2G}\big[\phi + \sin\phi + Gw(1+$$

For prescribed $z=|\zeta|/z_0\in[0,1)$ the branch is parametrised by $0<\phi\le\arccos z$ and $w=(\cos\phi-z)/(G\sin\phi)\ge 0$. At $\phi=\arccos z$ ($w=0$) it is the bang–bang pulse of Theorem 1(b); at fixed z, letting $\phi\downarrow 0$ sends $w\to\infty$ and gives $(A-A_0)(B-A_0)\to(z_0-|\zeta|)^2$, so the disk of Theorem 1(a) is approached along the branch only in that limit. This explains why the relaxation in the proof of (iii) stops at 0.137: its minimum lies where the disk and the frontier are both active, a point no pulse reaches. For $0\le|\zeta|/z_0<1$ and $0<\phi<\arccos(|\zeta|/z_0)$ the family satisfies both bounds of Theorem 1 strictly, $(A-A_0)(B-A_0)>(z_0-|\zeta|)^2$ and $(A-A_0)+(B-A_0)>2d/(2G)$ with $d=\arccos(|\zeta|/z_0)$, A increases and B decreases with ϕ, and the product tends to $(z_0-|\zeta|)^2$; these facts are formalised from the closed forms above (the chain from the control of type $(1,0,-1)$ to these forms is the written computation).

We conjecture that the same holds for every pulse, of varying phase and with detuning: a numerical search over three-dimensional pulses with exact integrals, aimed at going below the family, lands on it in five values of $|\zeta|$ and finds nothing below it beyond its constraint tolerance: over 40 starts per value the lowest margin is $-8.7\cdot10^{-8}$, at $|\zeta|/z_0=0.664$ and $2G(A-A_0)=0.019$, where the constraints are met to 10^{-8} and the frontier is steep enough for that error to move the margin by up to about 10^{-6} (`conjetura_adversario.py`). If it holds, the same planar exposure reduction, and hence the certified value of ϱ_c (and of Λ_c in Proposition 8), applies to all pulses.

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is not marked as formalized in the paper

`family_bang_bang` · `Isabelle/MemoryAccessFloors/U_Turn_Exposure_Family.thy`, line 117

```
lemma family_bang_bang:
  fixes z d :: real
  assumes "0 \<le> z" and "z < 1" and "d = arccos z"
  shows "fam_w z d = 0" and "fam_x z d = d - sin d" and "fam_y z d = d + sin d"
    and "fam_x z d + fam_y z d = 2 * d"
```

`family_inside_disc` · `Isabelle/MemoryAccessFloors/U_Turn_Exposure_Family.thy`, line 525

```
theorem family_inside_disc:
  fixes z \<phi> :: real
  assumes "0 \<le> z" and "z < 1" and "0 < \<phi>" and "\<phi> < arccos z"
  shows "(1 - z)\<^sup>2 < fam_x z \<phi> * fam_y z \<phi>"
```

`family_sum_above_boundary` · `Isabelle/MemoryAccessFloors/U_Turn_Exposure_Family.thy`, line 306

```
lemma family_sum_above_boundary:
  fixes z \<phi> :: real
  assumes "0 \<le> z" and "z < 1" and "0 < \<phi>" and "\<phi> < arccos z"
  shows "2 * arccos z < fam_x z \<phi> + fam_y z \<phi>"
```

`family_x_increasing` · `Isabelle/MemoryAccessFloors/U_Turn_Exposure_Family.thy`, line 260

```
lemma family_x_increasing:
  fixes z r s :: real
  assumes "0 \<le> z" and "z < 1" and "0 < r" and "r < s" and "s \<le> arccos
```

```
z"
  shows "fam_x z r < fam_x z s"
```

family_y_decreasing · Isabelle/MemoryAccessFloors/U_Turn_Exposure_Family.thy, line 280

```
lemma family_y_decreasing:
  fixes z r s :: real
  assumes "0 \<le> z" and "z < 1" and "0 < r" and "r < s" and "s \<le> arccos
z"
  shows "fam_y z s < fam_y z r"
```

family_product_tends_to_hyperbola · Isabelle/MemoryAccessFloors/U_Turn_Exposure_Family.thy, line 490

```
theorem family_product_tends_to_hyperbola:
  fixes z :: real
  shows "((\<lambda>t. fam_x z t * fam_y z t) \<longlongrightarrow> (1 - z)\
<^sup>2) (at_right 0)"
```

Cite: Remark 7, p. 12, doi:10.5281/zenodo.24250503

↳ Used in: Proposition 8 (p. 19), 9 (p. 22)

Proposition 8. p. 13

MarAprop:qsanswersMarAcor:channeluses Noise_Loss_Bound.noiseloss_lower_core, Noise_Loss_Bound.noiseloss_minimise_z, Noise_Loss_Bound.noiseloss_combined, Noise_Loss_Bound.noiseloss_large_zeta Let the offset act on the qubit only, centred with variance σ_q^2; let the memory decay at rate γ_m, with $\gamma_q = 0$ and no dephasing; let A, B, ζ be those of the lossless pulse, with $|g| \le G$ and any detuning, and J the leading-order error above. Put $\Lambda = \sigma_q^2/(\gamma_m G)$.

1. *The square pulse minimises J over all pulses for $\Lambda \le 7.1$, and for $\Lambda \ge 7.92$ it does not (both certified by interval arithmetic; a floating-point minimisation over the relaxed region suggests that the certificate is lost near 7.15, which is not established as a physical threshold).*
2. *As $\Lambda \to \infty$,*

$$\min J = \frac{\pi^2 \sigma_q^2}{96 G^2} + \frac{\sigma_q \sqrt{\pi \gamma_m}}{6\, G^{3/2}} + O\Big(\frac{\gamma_m}{G}\Big),$$

and the zero-leakage pulse $C_{\pi+\phi} S C_\phi$ whose wait lasts about $\sigma_q \sqrt{\pi}/(4 G^{3/2} \sqrt{\gamma_m})$ attains the first two terms.

3. *For pulses of fixed phase and no detuning the square pulse minimises J exactly for $\Lambda \le \Lambda_c$, and Λ_c is the crossing of the general turn–wait–turn family $C_{\pi+\phi} S C_\phi$, with ϕ and w free in the admissible domain:*

$$\Lambda_c = \sup\{\Lambda : \min_{\phi, w} g \ge 0\} = 7.9078851606875\ldots$$

(enclosed in an interval of width 10^{-12}; certified by interval arithmetic, using Lemma 6); above it the member with $\phi_* = 0.742069\ldots$ and a short wait $w_* = 0.257473\ldots$ (units $G = 1/2$) beats it; this critical member has $|\zeta|/z_0 \approx 0.6500698$, so it is not a zero-leakage pulse. For pulses of varying phase or with detuning, optimality remains undecided for $7.1 < \Lambda \le \Lambda_c$; for every $\Lambda > \Lambda_c$ the explicit planar competitor already beats the square pulse. Moreover, for every $\Lambda > \Lambda_c$ the minimum of J over pulses of fixed phase and no detuning is attained, in $(A, B, |\zeta|)$, by a member of the family $C_{\pi+\phi} S C_\phi$.

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

`noiseloss_lower_core` · `Isabelle/MemoryAccessFloors/Noise_Loss_Bound.thy`, line 22

```
lemma noiseloss_lower_core:
  fixes sigma2 gm A0 z0 z x y :: real
  assumes "0 \<le> sigma2" and "0 < gm" and "0 < A0" and "0 < z0" and "0 \<le>
z" and "z \<le> z0"
    and "0 \<le> x" and "0 \<le> y" and "(z0 - z)\<^sup>2 \<le> x * y"
  shows "(2 / 3) * (z0 - z) * sqrt (sigma2 * A0 * gm) \<le> sigma2 * A0 * x / 3
+ gm * y / 3"
```

`noiseloss_minimise_z` · `Isabelle/MemoryAccessFloors/Noise_Loss_Bound.thy`, line 75

```
lemma noiseloss_minimise_z:
  fixes sigma2 A0 gm z0 z s :: real
  assumes "0 < sigma2" and "0 < A0" and "0 < gm" and "0 < z0" and "0 \<le> z"
and "z \<le> z0"
    and "s = sqrt sigma2"
  shows "(2 / 3) * z0 * s * sqrt (A0 * gm) - A0 * gm / 3
         \<le> sigma2 * z\<^sup>2 / 3 + (2 / 3) * (z0 - z) * s * sqrt (A0 *
gm)"
```

`noiseloss_combined` · `Isabelle/MemoryAccessFloors/Noise_Loss_Bound.thy`, line 103

```
lemma noiseloss_combined:
  fixes sigma2 gm A0 z0 z x y A B :: real
  assumes "0 < sigma2" and "0 < gm" and "0 < A0" and "0 < z0" and "0 \<le> z"
and "z \<le> z0"
    and "0 \<le> x" and "0 \<le> y" and "(z0 - z)\<^sup>2 \<le> x * y" and "A =
A0 + x" and "B = A0 + y"
  shows "sigma2 * A0\<^sup>2 / 6 + (2 / 3) * z0 * sqrt sigma2 * sqrt (A0 * gm)
         \<le> sigma2 * (2 * z\<^sup>2 + A\<^sup>2) / 6 + gm * B / 3"
```

`noiseloss_large_zeta` · `Isabelle/MemoryAccessFloors/Noise_Loss_Bound.thy`, line 146

```
lemma noiseloss_large_zeta:
  fixes sigma2 gm A0 z0 z A B :: real
  assumes "0 \<le> sigma2" and "0 \<le> gm" and "0 < A0" and "0 < z0" and "z0 \
<le> z"
    and "A0 \<le> A" and "A0 \<le> B"
  shows "sigma2 * (A0\<^sup>2 + 2 * z0\<^sup>2) / 6 \<le> sigma2 * (2 * z\
<^sup>2 + A\<^sup>2) / 6 + gm * B / 3"
```

`symmetric_branch_bound` · `Isabelle/MemoryAccessFloors/Symmetric_Branch.thy`, line 38

```
theorem symmetric_branch_bound:
  fixes p w :: real
  assumes p0: "0 < p" and p1: "p \<le> pi / 2" and w0: "0 \<le> w"
  shows "pi * p + sin p * (pi - 2 * sin p)
           \<le> 2 * ((cos p - w * sin p / 2)\<^sup>2 - 1)
             + (p + sin p + w * (1 + cos p) / 2) * (pi + (p + sin p + w * (1 +
cos p) / 2))"
```

`middle_zone_min` · `Isabelle/MemoryAccessFloors/Middle_Zone.thy`, line 36

```
theorem middle_zone_min:
  fixes Lam A0 z x y x0 d :: real
  assumes "0 < A0" "2 / A0 < Lam" "0 \<le> x0" "x0 \<le> d" "x0 \<le> x" "x0 \
<le> y" "2 * d \<le> x + y"
  shows "Lam * (2 * z^2 + (A0 + x0)^2) / 12 + (A0 + (2 * d - x0)) / 3
           \<le> Lam * (2 * z^2 + (A0 + x)^2) / 12 + (A0 + y) / 3"
```

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** part: the named pieces are machine-checked; the statement as a whole is a written proof · **Cite:** Proposition 8, p. 13, doi:10.5281/zenodo.24250503

↳ Used in: 9 (p. 22)

Proof

Lower bound: with $x = A - A_0$, $y = B - A_0$, (a) gives $xy \geq (z_0 - |\zeta|)^2$ when $|\zeta| \leq z_0$, so $J \geq \sigma_q^2 A_0^2/6 + \gamma_m A_0/3 + \sigma_q^2|\zeta|^2/3 + \sigma_q^2 A_0 x/3 + \gamma_m y/3$; minimising $\sigma_q^2 A_0 x/3 + \gamma_m y/3$ over $xy \geq (z_0 - |\zeta|)^2$ and then over $|\zeta|$ (optimum $|\zeta| = \sqrt{A_0\gamma_m}/\sigma_q$) gives $J \geq \pi^2\sigma_q^2/(96G^2) + \frac{2}{3}z_0\sigma_q\sqrt{A_0\gamma_m}$, the term $\gamma_m A_0/3$ of the memory cancelling the one of the minimisation in $|\zeta|$; this is (ii) from below; when $|\zeta| > z_0$, $J \geq \sigma_q^2(A_0^2 + 2z_0^2)/6$, which is larger when Λ is large. Upper bound: the pulse $C_{\pi+\phi}SC_\phi$ with axial displacement $|s|$ has $A - A_0 = z_0^2/|s| + O(|s|^{-2})$ and $B = |s| + O(1)$, and the best $|s|$ is $z_0\sigma_q\sqrt{A_0/\gamma_m}$. (i): the difference between a pulse and the square pulse is linear in Λ and non-negative at $\Lambda = 0$, so it suffices to show it at $\Lambda = 7.1$, which is done as in Proposition 5(iii): an estimate near $|\zeta| = z_0$ and 518 accepted leaf boxes elsewhere (1035 nodes evaluated). The bang–bang pulse with $d = 0.85$ has a difference negative at $\Lambda = 7.92$ and decreasing in Λ.

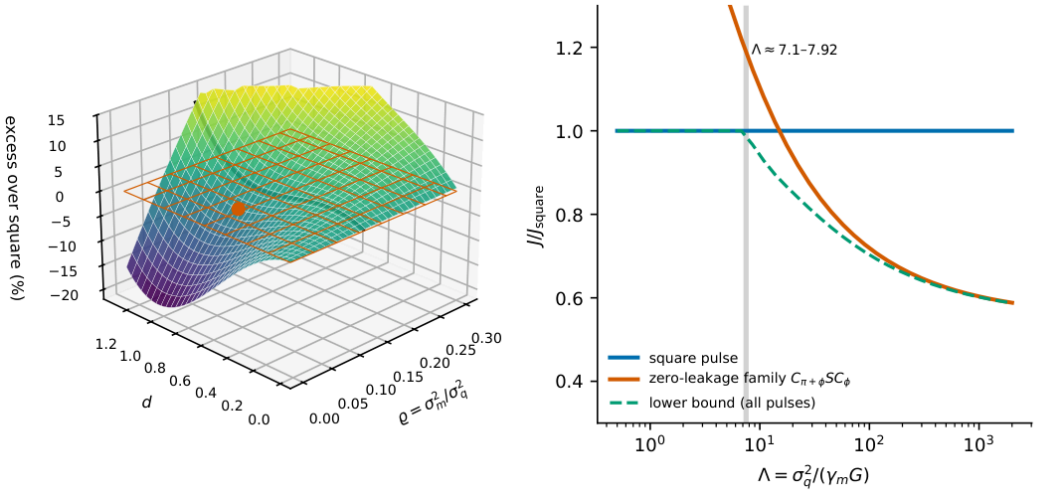
(iii): as in Proposition 5(iv), the set of $\Lambda \geq 0$ where the square pulse is optimal is an interval $[0, \Lambda_c]$ (affinity in Λ and optimality at $\Lambda = 0$), and the relaxed region is cut by Lemma 6, with the same reading of [14] . It suffices to certify $\Lambda_{\mathrm{lo}} = 7.90788$: the family in its parameters (ϕ, w) (2877 nodes; the minimum is interior, with lower bound $6 \cdot 10^{-9}$, since Λ_{lo} is $5 \cdot 10^{-6}$ below the family crossing), the symmetric branch and the rest of the relaxed region (4663 nodes in all, `certificar_umbral_plano_perdidas.py`); the same certificate at $\Lambda = 7.90789$, 7.91 and 7.92 fails, as it should. Above the enclosure the competitor is not the bang–bang pulse but the member of the family with $\phi = 0.742069$ and wait $w = 0.257473$ (units $G = 1/2$), whose difference with the square pulse is negative at $\Lambda = 7.90789$ (enclosed) and decreasing in Λ; integrated as a piecewise pulse it closes at the south pole to $3 \cdot 10^{-16}$ and reproduces the closed form to $5 \cdot 10^{-16}$. The bang–bang pulse crosses only at 7.9176. The identification with the family crossing is a second certificate (`certificar_lambda_c_identificacion.py`): a Krawczyk operator in (Λ, ϕ, w) proves that $\{g = 0,\ g_\phi = 0,\ g_w = 0\}$ has a unique zero in a box (the Λ coordinate of width below $2.7 \cdot 10^{-39}$, every coordinate below $3.3 \cdot 10^{-39}$). On the fixed rational rectangle $N = [\phi_* \pm 0.01] \times [w_* \pm 0.02]$, centred at rational decimal approximations of (ϕ_*, w_*), the Hessian satisfies $H > 0.3\, I$ uniformly for every Λ in the enclosing interval E, as verified by interval arithmetic, and the Krawczyk-enclosed critical point lies strictly inside N and the physical family domain; at its parameter value, strict convexity makes this point the minimum over N, with value zero. Outside N, after the analytic near-square patch and the non-physical boxes are separated, the retained exterior family boxes have an interval lower bound at least $1.88 \cdot 10^{-7}$ at both endpoints of E (the square pulse remains a zero in the analytic patch); the symmetric branch and the rest of the region are certified as before, at both endpoints and extended by affinity; controls fail. Λ_c is close to, but distinct from, $1/\varrho_c$: approximately 7.90788516068753 against 7.90804384366106; no relation between the two is asserted.

The minimiser above Λ_c (units $G = 1/2$, so $z_0 = 1$ and $A_0 = \pi/2$; with $\gamma_m > 0$ we compare the normalised objective $\hat{J} = J/\gamma_m$, which has the same minimisers, and $12(\hat{J} - \hat{J}_{\mathrm{sq}}) = \Lambda Q + 4y$ with $Q = 2(z^2 - 1) + x(2A_0 + x)$, $z = |\zeta|$). At fixed z, J is strictly increasing in x and in y. Let $d = \arccos z$ and $x_0 = d - \sin d$. If $z \geq 1$, $J \geq J_{\mathrm{sq}}$. If $z < 1$ and $x \leq x_0$, Lemma 6 gives $y \geq Y(x; z)$, the y of the member of the family with the same x and z, so J is at least its value there. If $z < 1$ and $y \leq x_0$, the symmetric statement bounds J from below by a point of the mirrored branch, with $(x, y) = (y_f, x_f)$ for a member (x_f, y_f) of the family, where $12(\hat{J} - \hat{J}_{\mathrm{sq}}) = \Lambda Q_m + 4x_f$ and $Q_m = 2(z^2 - 1) + y_f(\pi + y_f)$. With $s = \sin\phi$ and $c = \cos\phi$, the closed forms of Remark 7 give $z \leq c$ and $1 - z^2 \leq (1 - z)(1 + c) = s^2 + (1 + c)ws/2$, hence

$$Q_m \;\geq\; \pi\phi + s(\pi - 2s) + w(1+c)\left(\tfrac{\pi}{2} - s\right) \;\geq\; \pi\phi + s(\pi - 2s) > 0,$$

so the mirrored branch never beats the square pulse, for any $\Lambda \geq 0$. Symmetric_Branch.symmetric_branch_bound Middle_Zone.middle_zone_min If $x, y > x_0$, Theorem 1(b) gives $x + y \geq 2d$, and by monotonicity the minimum lies on $x + y = 2d$, where $\frac{d}{dx}\hat{J}(x, 2d - x) = \frac{\Lambda}{6}(A_0 + x) - \frac{1}{3} > 0$ for $\Lambda > 2/A_0$; so it is the bang–bang point $x = x_0$, a member of the family. Since $\Lambda_c > 2/A_0$, every case is bounded by the family or by the square pulse, which the family beats above Λ_c; the family minimum exists because $J \to \infty$ as $w \to \infty$. This identifies the end point of the minimiser, not the control that reaches it. Numerically, a piecewise search with 8 to 24 segments and 252 starts per value lands on the family minimum to 10^{-8} for Λ from $\Lambda_c + 0.1$ to 1000, on the frontier of Remark 7 (`optimo_sobre_lambda_c/`).

Figure 3



Propositions 5 and 8. Left: relative excess $(F - F_{\mathrm{sq}})/F_{\mathrm{sq}}$ of the bang–bang pulse of Theorem 1(b) with parameter d over the square pulse, with $F = 2(1+\varrho)|\zeta|^2 + A^2 + \varrho B^2$ (negative: better than the square pulse; orange plane: zero; dot: $-3.3\,\%$ at $\varrho = 0.1$, $d = 0.848$). Right: leading-order error J with a lossy memory, relative to the square pulse, against $\Lambda = \sigma_q^2/(\gamma_m G)$, for the zero-leakage pulses $C_{\pi+\phi}SC_\phi$ and for the lower bound of the proof of Proposition 8, valid for every pulse: the square pulse is optimal up to $\Lambda = 7.1$ and not beyond 7.92 (grey band; for pulses of fixed phase and no detuning the exact threshold is $\Lambda_c = 7.9079$, Proposition 8(iii)); the explicit family beats it from $\Lambda \approx 15$.

When the memory also decays, robustness must be paid in exposure, which the decay punishes. Measured by $\Lambda = \sigma_q^2/(\gamma_m G)$, the square pulse stays optimal over all pulses for small Λ and stops being optimal beyond a threshold; for pulses of fixed phase and no detuning the threshold Λ_c is identified exactly, and for large noise the optimum is given to leading order.

Evidence & scope

- **Literature:** Companion paper: this statement answers a question left by paper A (Exact limits on writing a superconducting quantum memory), Proposition 14 (silent link, not printed in the paper). `https://karlesmarin.github.io/blueprints/memory-floors/`
- **Literature:** Companion paper: this statement uses paper A (Exact limits on writing a superconducting quantum memory), Corollary 9 (silent link, not printed in the paper). `https://karlesmarin.github.io/blueprints/memory-floors/`

Remark 9. Isabelle

(relaxation of the qubit; the bound and the sufficient condition) p. 15

MarAprop:qsanswers Noise_Loss_Relax.relax_bound, Noise_Loss_Relax.relax_bound_G, Noise_Loss_Relax.relax_large_zeta_margin, Noise_Loss_Relax.relax_recovers_noiseloss, Square_Condition.square_pulse_optimal, Square_Condition.square_pulse_optimal_G, Dubins_Family_Identity.dubins_identity, Dubins_Family_Identity.dubins_quantity_nonneg, Dubins_Family_Identity.dubins_D_bound, Threshold_Localisation.threshold_lower_bound, Threshold_Localisation.threshold_lower_bound_k, Threshold_Localisation.complete_square_bound, Inf_Affine_Concave.inf_affine_concave, Inf_Affine_Concave.inf_affine_monotone
With $\gamma_q{>}0$ the error is $J=\sigma_q^2(2|\zeta|^2+A^2)/6+(\gamma_q A+\gamma_m B)/3$, and the same argument, with $a=\sigma_q^2A_0+\gamma_q$ in place of $\sigma_q^2A_0$, gives

$$J\;\geq\;\frac{\sigma_q^2A_0^2}{6}+\frac{(\gamma_q+\gamma_m)A_0}{3}+\frac{2}{3}z_0\sqrt{a\gamma_m}-\frac{a\gamma_m}{3\sigma_q^2}$$

for every pulse (for $|\zeta|{>}z_0$ the difference is at least $(\sigma_q z_0-\sqrt{a\gamma_m}/\sigma_q)^2/3\geq 0$; for $\gamma_q=0$ this is the bound of the proof above), while the zero-leakage pulses $C_{\pi+\phi}SC_\phi$ exceed $\frac{2}{3}z_0\sqrt{a\gamma_m}$ only by higher-order terms. So **the price of waiting in the memory is $\frac{2}{3}z_0\sqrt{(\sigma_q^2A_0+\gamma_q)\gamma_m}$, to leading order**, when $\sqrt{a\gamma_m}\ll\sigma_q^2z_0$. Conversely, the same disk argument shows that the square pulse is optimal at this order whenever $\sqrt{a\gamma_m}\geq\sigma_q^2z_0$: minimising $\sigma_q^2u^2/3+\frac{2}{3}\sqrt{a\gamma_m}\,(z_0-u)$ over $0\leq u=|\zeta|\leq z_0$ then gives $u=z_0$, the value of the square pulse, and $|\zeta|{>}z_0$ costs more. This is a sufficient condition, not a sharp threshold; for $\gamma_q=0$ it reads $\Lambda\leq\pi$, inside the certified range of Proposition 8(i). (Checked against the grid minimisation and the explicit family in four parameter sets: the family is within $0.15\,\%$ of $\frac{2}{3}z_0\sqrt{a\gamma_m}$ and the lower bound within the stated correction.)

For pulses of fixed phase and no detuning the sufficient condition is not sharp, and the exact threshold can be certified as in Proposition 8(iii): with $r=\gamma_q/\gamma_m$ the difference $J-J_{\mathrm{sq}}$ stays linear in Λ, so $\Lambda_c(r)$ is again a minimum over the turn–wait–turn family, and interval arithmetic encloses it to 10^{-5}; a Krawczyk certificate identifies it, for each r below, with the zero, isolated uniquely in a certified box, of $\{g,g_\phi,g_w\}$ on that family, to 10^{-12} (as for $r=0$; `umbral_relajacion_cubit/`):

$r=\gamma_q/\gamma_m$	0	0.1	0.3	1	3	10	30	100	10
$\Lambda_c(r)$	7.908	7.956	8.050	8.366	9.170	11.346	15.521	24.260	67.
sufficient	3.141	3.264	3.485	4.113	5.374	8.087	12.637	21.632	64.

(the last row is the condition above, $\Lambda\leq(\pi+\sqrt{\pi^2+16r})/2$, rounded down). Given Lemma 6, $\Lambda_c(r)$ is the infimum over the members of the family with $Q{<}0$, a set that does not depend on r, of $4(y+rx)/(-Q)$, affine and non-decreasing in r since $x\geq 0$; so Λ_c **is non-decreasing and concave in r. The threshold grows with the qubit relaxation like $2\sqrt{r}$, with an exact constant**: given Lemma 6, as $r\to\infty$,

$$\Lambda_c(r)=2\sqrt{r}+\frac{\pi}{2}+\frac{4}{\sqrt{3}}+O(r^{-1/2}).$$

For the threshold $\Lambda_c^{\mathrm{all}}(r)$ over pulses of any phase and detuning two explicit bounds hold without Lemma 6: as $r\to\infty$,

$$2\sqrt{r}+\frac{\pi}{2}+O(r^{-1/2}) \;\le\; \Lambda_c^{\mathrm{all}}(r) \;\le\; 2\sqrt{r}+\frac{\pi}{2}+\frac{4}{\sqrt{3}}+O(r^{-1/2}),$$

the sharp constant of Λ_c^{all} is not identified here. The lower bound is the sufficient condition above, valid for every pulse. The upper bound needs no reading of [14] : the member of the family with $\phi=2/\sqrt{r}$ and $w=2/\sqrt{3}$ (units $G=1/2$) has $Q<0$ for large r ($Q=-2w\phi+O(\phi^2)$), so it beats the square pulse for every $\Lambda>4(y+rx)/(-Q)$, and that explicit ratio, with $h=r^{-1/2}$, has a simple pole at $h=0$ and a Laurent expansion $2\sqrt{r}+\pi/2+4/\sqrt{3}+O(r^{-1/2})$ ($\phi=2/\sqrt{r}$ minimises the leading term $\sqrt{r}(u/2+2/u)$, $u=\phi\sqrt{r}$, and $w=2/\sqrt{3}$ the next one, $w+\pi/2+4/(3w)$). Both bounds therefore hold for pulses of any phase and detuning too. At finite r the two explicit quantities, not their expansions, are the bounds: the certified values lie between $(\pi+\sqrt{\pi^2+16r})/2$ and $4(y+rx)/(-Q)$ for $10\le r\le 1000$, and approach the upper one (67.2454 against 67.2788 at $r=1000$, `cotas_asintoticas.py`).

The exact constant needs a lower bound over every member of the family that could compete, including waits that tend to zero or to infinity. Work in units $G=1/2$, with $h=r^{-1/2}$, $a=1-z$, $D=-Q=2(1-z^2)-\pi x-x^2$ and $R_r=4(y+rx)/D$ on the members with $D>0$, so that, given Lemma 6, $\Lambda_c(r)=\inf R_r$ (step (c)). (a) *Localisation.* With the closed forms of Remark 7, $xy-a^2=E:=\phi^2-2(1-\cos\phi)+w(\phi-\sin\phi)\ge 0$. Since $D\le 2(1-z^2)=4a(1-a/2)$, AM–GM gives $R_r\ge \frac{2}{h}\sqrt{1+E/a^2}/(1-a/2)$, and $D\le 4a\le 4\sqrt{xy}$ gives $R_r\ge (k+1/k)/h$ with $k=h\sqrt{y/x}$. The explicit member above has $R_r=2/h+\frac{\pi}{2}+\frac{4}{\sqrt{3}}+O(h)$, so it is enough to bound the members with $R_r\le 2/h+C$ for a fixed C; for them $a=O(h)$, $E/a^2=O(h)$ and $k=1+O(\sqrt{h})$. As $\phi\to 0$, $a=\frac{1}{2}\phi(\phi+w)[1+O(\phi^2)]$ and $E=\frac{1}{12}\phi^3(\phi+2w)[1+O(\phi^2)]$ uniformly in $w\ge 0$ (each is a sum of two non-negative terms), so $E/a^2\ge \phi/(3(\phi+w))\,[1+O(\phi^2)]$; hence $\phi/w=O(h)$, then $x=\frac{1}{4}w\phi^2[1+O(h)]$, $y=w[1+O(h)]$, $u:=\phi/h=2+O(\sqrt{h})$, and $m\le w\le M$ for constants independent of h. (b) *Expansion.* On that compact set, uniformly, $R_r=h^{-1}\big(\frac{u}{2}+\frac{2}{u}\big)+c(u,w)+O(h)$ with $c(u,w)=\frac{2-u^2/6}{w}+(u^2+4)\big(\frac{w}{8}+\frac{\pi}{16}\big)$. Since $\frac{u}{2}+\frac{2}{u}-2=\frac{(u-2)^2}{2u}$, $c(2,w)=w+\frac{4}{3w}+\frac{\pi}{2}\ge\frac{\pi}{2}+\frac{4}{\sqrt{3}}$ and c is Lipschitz in u there, completing the square in $u-2$ gives $R_r\ge 2/h+\frac{\pi}{2}+\frac{4}{\sqrt{3}}-O(h)$. (c) *From the family to the threshold.* The case analysis of Proposition 8(iii) carries over with the term $4rx$ in $12(\hat J-\hat J_{\mathrm{sq}})=\Lambda Q+4(y+rx)$: the cost still increases with y, the mirrored branch has difference $\Lambda Q_m+4x_f+4ry_f>0$, and on the middle zone $\frac{d}{dx}\hat J(x,2d-x)=\frac{\Lambda}{6}(A_0+x)+\frac{r-1}{3}>0$ for $\Lambda>4/\pi$, while below $(\pi+\sqrt{\pi^2+16r})/2\ge\pi>4/\pi$ the sufficient condition decides. So the sufficient condition misses the threshold by exactly $4/\sqrt{3}$ asymptotically. (The identity for E and the coefficients $u/2+2/u$ and $c(u,w)$ checked symbolically; $\inf R_r$ equal to the certified $\Lambda_c(r)$ to $4\cdot 10^{-15}$ at the six certified values $r=1,3,10,30,100,1000$; $\sqrt{r}\,(\Lambda_c-2\sqrt{r}-\frac{\pi}{2}-\frac{4}{\sqrt{3}})$ equal to 3.77 at $r=10^4$ and 3.76 at $r=10^6$; with $w=1$ fixed the constant is 3.907, larger, as it must be: `constante_asintotica.py`.)

Isabelle · ours **show**

`relax_bound` · `Isabelle/MemoryAccessFloors/Noise_Loss_Relax.thy`, line 72

```
theorem relax_bound:
  fixes s g gm A0 z0 z A B :: real
  assumes "0 < s" and "0 \<le> g" and "0 < gm" and "0 < A0" and "0 < z0" and "0
\<le> z"
    and "A0 \<le> A" and "A0 \<le> B" and "(z - z0)\<^sup>2 \<le> (A - A0) * (B
- A0)"
  shows "s * A0\<^sup>2 / 6 + (g + gm) * A0 / 3 + (2 / 3) * z0 * sqrt ((s * A0
```

```
+ g) * gm)
           - (s * A0 + g) * gm / (3 * s)
         \<le> s * (2 * z\<^sup>2 + A\<^sup>2) / 6 + (g * A + gm * B) / 3"
```

relax_bound_G · Isabelle/MemoryAccessFloors/Noise_Loss_Relax.thy, line 126

```
corollary relax_bound_G:
  fixes s g gm G z A B :: real
  assumes "0 < G" and "0 < s" and "0 \<le> g" and "0 < gm" and "0 \<le> z"
    and "pi / (4 * G) \<le> A" and "pi / (4 * G) \<le> B"
    and "(z - 1 / (2 * G))\<^sup>2 \<le> (A - pi / (4 * G)) * (B - pi / (4 *
G))"
  shows "s * (pi / (4 * G))\<^sup>2 / 6 + (g + gm) * (pi / (4 * G)) / 3
           + (2 / 3) * (1 / (2 * G)) * sqrt ((s * (pi / (4 * G)) + g) * gm)
           - (s * (pi / (4 * G)) + g) * gm / (3 * s)
         \<le> s * (2 * z\<^sup>2 + A\<^sup>2) / 6 + (g * A + gm * B) / 3"
```

relax_large_zeta_margin · Isabelle/MemoryAccessFloors/Noise_Loss_Relax.thy, line 39

```
lemma relax_large_zeta_margin:
  fixes s c z0 z :: real
  assumes "0 < s" and "0 \<le> z0" and "z0 \<le> z"
  shows "(sqrt s * z0 - c / sqrt s)\<^sup>2 / 3
         \<le> s * z\<^sup>2 / 3 - ((2 / 3) * z0 * c - c\<^sup>2 / (3 * s))"
```

relax_recovers_noiseloss · Isabelle/MemoryAccessFloors/Noise_Loss_Relax.thy, line 157

```
corollary relax_recovers_noiseloss:
  fixes sigma2 gm A0 z0 z A B :: real
  assumes "0 < sigma2" and "0 < gm" and "0 < A0" and "0 < z0" and "0 \<le> z"
    and "A0 \<le> A" and "A0 \<le> B" and "(z0 - z)\<^sup>2 \<le> (A - A0) * (B
- A0)"
  shows "sigma2 * A0\<^sup>2 / 6 + (2 / 3) * z0 * sqrt sigma2 * sqrt (A0 * gm)
         \<le> sigma2 * (2 * z\<^sup>2 + A\<^sup>2) / 6 + gm * B / 3"
```

square_pulse_optimal · Isabelle/MemoryAccessFloors/Square_Condition.thy, line 39

```
theorem square_pulse_optimal:
  fixes s g gm A0 z0 z A B :: real
  assumes "0 < s" and "0 \<le> g" and "0 < gm" and "0 < A0" and "0 < z0" and "0
\<le> z"
    and "A0 \<le> A" and "A0 \<le> B" and "(z - z0)\<^sup>2 \<le> (A - A0) * (B
- A0)"
    and "s * z0 \<le> sqrt ((s * A0 + g) * gm)"
  shows "s * (2 * z0\<^sup>2 + A0\<^sup>2) / 6 + (g + gm) * A0 / 3
         \<le> s * (2 * z\<^sup>2 + A\<^sup>2) / 6 + (g * A + gm * B) / 3"
```

square_pulse_optimal_G · Isabelle/MemoryAccessFloors/Square_Condition.thy, line 111

```
corollary square_pulse_optimal_G:
  fixes s gm G z A B :: real
  assumes "0 < G" and "0 < s" and "0 < gm" and "0 \<le> z"
    and "pi / (4 * G) \<le> A" and "pi / (4 * G) \<le> B"
    and "(z - 1 / (2 * G))\<^sup>2 \<le> (A - pi / (4 * G)) * (B - pi / (4 *
G))"
    and "s / (gm * G) \<le> pi"
  shows "s * (2 * (1 / (2 * G))\<^sup>2 + (pi / (4 * G))\<^sup>2) / 6 + (0 +
gm) * (pi / (4 * G)) / 3
         \<le> s * (2 * z\<^sup>2 + A\<^sup>2) / 6 + (0 * A + gm * B) / 3"
```

dubins_identity · Isabelle/MemoryAccessFloors/Dubins_Family_Identity.thy, line 29

```
lemma dubins_identity:
  "dub_x \<phi> w * dub_y \<phi> w - (1 - dub_z \<phi> w)\<^sup>2
     = \<phi>\<^sup>2 - 2 * (1 - cos \<phi>) + w * (\<phi> - sin \<phi>)"
```

dubins_quantity_nonneg · Isabelle/MemoryAccessFloors/Dubins_Family_Identity.thy, line 55

```
lemma dubins_quantity_nonneg:
  assumes "0 \<le> \<phi>" "0 \<le> w"
  shows "0 \<le> dub_x \<phi> w * dub_y \<phi> w - (1 - dub_z \<phi> w)\
<^sup>2"
```

dubins_D_bound · Isabelle/MemoryAccessFloors/Dubins_Family_Identity.thy, line 67

```
lemma dubins_D_bound:
  fixes a D :: real
  assumes "a = 1 - dub_z \<phi> w" "D = 2 * (1 - (dub_z \<phi> w)\<^sup>2) - pi
* dub_x \<phi> w - (dub_x \<phi> w)\<^sup>2"
    "0 \<le> dub_x \<phi> w"
  shows "D \<le> 4 * a * (1 - a / 2)"
```

threshold_lower_bound · Isabelle/MemoryAccessFloors/Threshold_Localisation.thy, line 34

```
lemma threshold_lower_bound:
  fixes x y a h r D :: real
  assumes "0 < x" "0 < y" "0 < a" "0 < h" "0 < r" "r = 1 / h\<^sup>2" "0 < D"
    "D \<le> 4 * a * (1 - a / 2)" "a < 2" "a\<^sup>2 \<le> x * y"
  shows "(2 / h) * sqrt (x * y) / (a * (1 - a / 2)) \<le> 4 * (y + r * x) / D"
```

threshold_lower_bound_k · Isabelle/MemoryAccessFloors/Threshold_Localisation.thy, line 65

```
lemma threshold_lower_bound_k:
  fixes x y h r D :: real
  assumes "0 < x" "0 < y" "0 < h" "r = 1 / h\<^sup>2" "0 < D" "D \<le> 4 * sqrt
(x * y)"
  shows "((h * sqrt (y / x)) + 1 / (h * sqrt (y / x))) / h \<le> 4 * (y + r *
x) / D"
```

complete_square_bound · Isabelle/MemoryAccessFloors/Threshold_Localisation.thy, line 91

```
lemma complete_square_bound:
  fixes h L t :: real
  assumes "0 < h" "0 \<le> L"
  shows "- (3 / 2) * L\<^sup>2 * h \<le> t\<^sup>2 / (6 * h) - L * \<bar>t\
<bar>"
```

inf_affine_concave · Isabelle/MemoryAccessFloors/Inf_Affine_Concave.thy, line 40

```
lemma inf_affine_concave:
  assumes "I \<noteq> {}" "\<forall>r\<ge>0. bdd_below ((\<lambda>i. \<alpha> i
+ \<beta> i * r) ` I)"
  shows "concave_on {0..} (inf_affine I \<alpha> \<beta>)"
```

inf_affine_monotone · Isabelle/MemoryAccessFloors/Inf_Affine_Concave.thy, line 21

```
lemma inf_affine_monotone:
  assumes "I \<noteq> {}" "\<forall>i\<in>I. 0 \<le> \<beta> i"
    "\<forall>r\<ge>0. bdd_below ((\<lambda>i. \<alpha> i + \<beta> i * r) `
I)"
  shows "monotone_on {0..} (\<le>) (\<le>) (inf_affine I \<alpha> \<beta>)"
```

Cite: Remark 9, p. 15, doi:10.5281/zenodo.24250503

↳ Used in: 11 (p. 29)

Evidence & scope

- **Literature:** Companion paper: this statement answers a question left by paper A (Exact limits on writing a superconducting quantum memory), Proposition 14 (silent link, not printed in the paper). `https://karlesmarin.github.io/blueprints/memory-floors/`

Remark 10. (correlated offsets) p. 17

MarAprop:qsextends Band_Quadratic.band_between, Memory_Floors_Pieces.square_corr, Memory_Floors_Pieces.square_corr_far, Memory_Floors_Pieces.square_corr_uncorrelated_fails, Memory_Floors_Pieces.square_corr_equal_variances,

Correlation_Monotone.six_delta_difference,
Correlation_Monotone.delta_monotone_in_C,
Correlation_Monotone.delta_nonneg_transfer The formula of Proposition 5 holds for correlated offsets: with $C = \mathbb{E}(\xi_q \xi_m)$ and $V = \sigma_q^2 + \sigma_m^2 - 2C$,
$6r = 2V|\zeta|^2 + \sigma_q^2 A^2 + 2CAB + \sigma_m^2 B^2$. At second order, the two limiting covariance structures simplify exactly. For a common offset, $\xi_q = \xi_m$, $6r = \sigma^2 T^2$, so the square pulse, of least duration, minimises this coefficient. For opposite offsets, $\xi_m = -\xi_q$ with $\sigma^2 > 0$, $6r = \sigma^2(8|\zeta|^2 + (A-B)^2)$ vanishes exactly when $\zeta = 0$ and $A = B$, i.e. when the curve $\mathbf{r}$ closes; among pulses of fixed phase and no detuning the shortest such transfer is the U-turn with $s = 0$, of duration $L(0) = \frac{7}{3}\,T_0$ (Proposition 3), the closed-curve condition of [19 , 12] . In between, for $C \geq 0$, the square pulse is optimal at second order among all pulses whenever

$$\frac{\pi}{2}\sqrt{(\sigma_q^2 + C)(\sigma_m^2 + C)} \;\geq\; \sigma_q^2 + \sigma_m^2 - 2C.$$

Indeed, with $x = A - A_0$, $y = B - A_0 \geq 0$ and $u = |\zeta|$, the difference from the square pulse is $6\Delta = 2V(u^2 - z_0^2) + 2A_0[(\sigma_q^2 + C)x + (\sigma_m^2 + C)y] + \sigma_q^2 x^2 + 2Cxy + \sigma_m^2 y^2$; the quadratic part is non-negative for $C \geq 0$, and for $u \leq z_0$ the disk of Theorem 1(a) and $ax + by \geq 2\sqrt{abxy}$ give $6\Delta \geq 2(z_0 - u)\big[2A_0\sqrt{(\sigma_q^2 + C)(\sigma_m^2 + C)} - 2Vz_0\big]$, with $A_0/z_0 = \pi/2$; for $u > z_0$ every term is non-negative. For $C = 0$ and $\sigma_q^2 + \sigma_m^2 > 0$ this sufficient condition cannot hold, since $(\pi/2)\sigma_q\sigma_m \leq (\pi/4)(\sigma_q^2 + \sigma_m^2) < \sigma_q^2 + \sigma_m^2$ (if both variances vanish it holds trivially and every pulse has zero noise cost); the content is the correlated case: for equal positive variances the condition is $\kappa \geq (4 - \pi)/(4 + \pi)$, approximately 0.1201983. For pulses of fixed phase and no detuning the band can be computed exactly. With $\kappa = C/(\sigma_q\sigma_m)$ the square pulse is optimal on $[\varrho_c(\kappa), 1/\varrho_c(\kappa)]$ and on no larger interval, where ϱ_c decreases from 0.12645 at $\kappa = 0$ through 0.0902 ($\kappa = 0.1$), 0.0476 (0.3), 0.0272 (0.5) to 0.00948 at $\kappa = 1$. At $\kappa = 0,\ 0.1,\ 0.1202\,(= 601/5000),\ 0.3,\ 0.5,\ 1$ the threshold is identified exactly, under the same reading of [14] : it is the zero of the stationarity system of the competing family isolated uniquely in a certified box, enclosed to 10^{-12} by a Krawczyk certificate, the bang–bang family for the first three values ($\varrho_c = 0.12645352249,\ 0.09019042063,\ 0.0843166534$) and the turn–wait–turn family with a positive wait for the last three ($0.047565938906,\ 0.027186606021,\ 0.009479169051$; every digit printed is fixed by the enclosure); a strictly convex neighbourhood of the critical point, $\partial g/\partial s > 0$ there, and interval bounds on the rest of the region at both ends of the enclosure make the identification exact (`banda_correlada/`). Two facts reduce the certificate to three values of ϱ: with $s = \sqrt{\varrho}$ the difference from the square pulse is $Q_x + bs + Q_y s^2$ with $b \geq 0$ at each point of the region, so optimality at s_{hi} and at $s = 1$ gives it in between; and $\varrho \mapsto 1/\varrho$ exchanges the two elements. A rational turn–wait–turn witness beats the square pulse on $(0, \varrho_{\mathrm{lo}}]$. So **correlation widens the band but never makes the square pulse optimal for every ratio**: as $\varrho \to 0$ the correlation term $C = \kappa\sigma_q\sigma_m$ vanishes and the bang–bang pulse wins again. The competitor is the bang–bang pulse up to $\kappa \approx 0.2072$ and a turn–wait–turn pulse with a positive wait beyond it (numerical). The sufficient condition above holds at equal variances exactly for $\kappa \geq (4 - \pi)/(4 + \pi)$; at $\kappa = 0$ and 0.1 it holds for no ratio, and at $\kappa = 601/5000,\ 0.3,\ 0.5,\ 1$ its band lies inside the exact one, with lower end exceeding ϱ_c by factors 11.80, 3.79, 3.01 and 2.53 (interval arithmetic, with a failing control, `banda_suficiente.py`).

The band can only widen as the correlation grows, for pulses of any phase and detuning and without Lemma 6: fix positive variances and the relaxation rates; if the square pulse minimises the leading-order error among the pulses of an admissible class at $\kappa_1 \in [0, 1]$, it does at every $\kappa_2 \in [\kappa_1, 1]$. Indeed, in the difference 6Δ above, $\partial(6\Delta)/\partial C = 4(z_0^2 - u^2) + 2[A_0(x + y) + xy]$, non-negative for $u \leq z_0$ since $x, y \geq 0$ (Theorem 1(a)), while for $u > z_0$ every term of 6Δ is non-negative at every

$C \geq 0$; relaxation adds terms that do not depend on C. So where the square pulse is optimal exactly on $[\varrho_c(\kappa), 1/\varrho_c(\kappa)]$ these bands are nested and ϱ_c is non-increasing (not shown to be strictly decreasing).

Isabelle · ours · part **show**

Only a part is in Isabelle: the named pieces are machine-checked; the statement as a whole is a written proof

band_between · Isabelle/MemoryAccessFloors/Band_Quadratic.thy, line 56

```
theorem band_between:
  fixes Qx b Qy sa s :: real
  assumes "0 \<le> sa" "sa \<le> s" "s \<le> 1" "0 \<le> b" "0 \<le> Qx + b *
sa + Qy * sa^2" "0 \<le> Qx + b + Qy"
  shows "0 \<le> Qx + b * s + Qy * s^2"
```

square_corr · Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy, line 93

```
theorem square_corr:
  fixes A0 z0 sq sm C V x y u :: real
  assumes "0 < z0" and "A0 = pi / 2 * z0" and "0 \<le> sq" and "0 \<le> sm" and
"0 \<le> C"
    and "V = sq + sm - 2 * C" and "0 \<le> V" and "0 \<le> x" and "0 \<le> y"
and "0 \<le> u" and "u \<le> z0"
    and "(z0 - u)\<^sup>2 \<le> x * y"
    and "V \<le> pi / 2 * sqrt ((sq + C) * (sm + C))"
  shows "0 \<le> 2 * V * (u\<^sup>2 - z0\<^sup>2) + 2 * A0 * ((sq + C) * x +
(sm + C) * y)
             + sq * x\<^sup>2 + 2 * C * x * y + sm * y\<^sup>2"
```

square_corr_far · Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy, line 150

```
theorem square_corr_far:
  fixes A0 z0 sq sm C V x y u :: real
  assumes "0 < z0" and "0 \<le> A0" and "0 \<le> sq" and "0 \<le> sm" and "0 \
<le> C"
    and "V = sq + sm - 2 * C" and "0 \<le> V" and "0 \<le> x" and "0 \<le> y"
and "z0 < u"
  shows "0 \<le> 2 * V * (u\<^sup>2 - z0\<^sup>2) + 2 * A0 * ((sq + C) * x +
(sm + C) * y)
             + sq * x\<^sup>2 + 2 * C * x * y + sm * y\<^sup>2"
```

square_corr_uncorrelated_fails · Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy, line 166

```
theorem square_corr_uncorrelated_fails:
  fixes sq sm :: real
  assumes "0 < sq" and "0 < sm"
  shows "pi / 2 * sqrt (sq * sm) < sq + sm"
```

square_corr_equal_variances · Isabelle/MemoryAccessFloors/Memory_Floors_Pieces.thy, line 202

```
theorem square_corr_equal_variances:
  fixes s0 kappa :: real
  assumes "0 < s0" and "0 \<le> kappa"
  shows "s0 + s0 - 2 * (kappa * s0) \<le> pi / 2 * sqrt ((s0 + kappa * s0) *
(s0 + kappa * s0))
         \<longleftrightarrow> (4 - pi) / (4 + pi) \<le> kappa"
```

six_delta_difference · Isabelle/MemoryAccessFloors/Correlation_Monotone.thy, line 22

```
lemma six_delta_difference:
  "six_delta sq sm z0 A0 C2 u x y - six_delta sq sm z0 A0 C1 u x y
   = (C2 - C1) * (4 * (z0\<^sup>2 - u\<^sup>2) + 2 * (A0 * (x + y) + x * y))"
```

delta_monotone_in_C · Isabelle/MemoryAccessFloors/Correlation_Monotone.thy, line 28

```
lemma delta_monotone_in_C:
  assumes "0 \<le> u" "u \<le> z0" "0 \<le> x" "0 \<le> y" "0 \<le> A0" "C1 \
<le> C2"
  shows "delta_fn sq sm z0 A0 C1 u x y \<le> delta_fn sq sm z0 A0 C2 u x y"
```

delta_nonneg_transfer · Isabelle/MemoryAccessFloors/Correlation_Monotone.thy, line 44

```
lemma delta_nonneg_transfer:
  assumes "0 \<le> u" "u \<le> z0" "0 \<le> x" "0 \<le> y" "0 \<le> A0" "C1 \
<le> C2"
    "0 \<le> delta_fn sq sm z0 A0 C1 u x y"
  shows "0 \<le> delta_fn sq sm z0 A0 C2 u x y"
```

Cite: Remark 10, p. 17, doi:10.5281/zenodo.24250503

↳ Used in: 11 (p. 29)

Evidence & scope

- **Literature:** Companion paper: this statement extends paper A (Exact limits on writing a superconducting quantum memory), Proposition 14 (silent link, not printed in the paper). https://karlesmarin.github.io/blueprints/memory-floors/

Remark 11. Isabelle **(one sufficient condition for both)** p. 18

MarAprop:qsanswers Square_Unified_Condition.square_unified, Square_Unified_Condition.square_unified_no_relax, Square_Unified_Condition.square_unified_qubit_only, Square_Unified_Condition.square_unified_uncorrelated_needs_relax The sufficient conditions of Remarks 9 and 10 are cases of one. With slow offsets on both elements (second moments σ_q^2, σ_m^2, covariance $C \geq 0$, $V = \sigma_q^2 + \sigma_m^2 - 2C \geq 0$) and Markovian relaxation of both ($\gamma_q, \gamma_m \geq 0$, no pure dephasing), the leading-order error is $6J = 2V|\zeta|^2 + \sigma_q^2 A^2 + 2CAB + \sigma_m^2 B^2 + 2(\gamma_q A + \gamma_m B)$, and the square pulse minimises it over all pulses whenever

$$Vz_0 \;\leq\; \sqrt{\big((\sigma_q^2 + C)A_0 + \gamma_q\big)\big((\sigma_m^2 + C)A_0 + \gamma_m\big)}.$$

The proof is the disk of Theorem 1(a), $(z_0 - |\zeta|)^2 \leq (A - A_0)(B - A_0)$, followed by the arithmetic–geometric mean inequality. Without relaxation it is the condition of Remark 10; with $\sigma_m = C = 0$ it is the condition of Remark 9; with $\gamma_q = \gamma_m = C = 0$ and $\sigma_q^2 + \sigma_m^2 > 0$ it cannot hold ($\frac{\pi}{2}\sqrt{\sigma_q^2\sigma_m^2} < \sigma_q^2 + \sigma_m^2$; if both variances also vanish it holds trivially and every pulse has zero leading-order cost), in agreement with the bang–bang pulse beating the square pulse outside $[\varrho_c, 1/\varrho_c]$. It is sufficient, not sharp: at $\gamma_q = \gamma_m = C = 0$ the exact band of Proposition 5 is wider. (Checked on 3620 random parameter sets meeting the condition, over the region the disk allows: minimum of $6(J - J_{\mathrm{sq}})$ equal to $3.6 \cdot 10^{-4} \geq 0$; the control with $\gamma_q = \gamma_m = C = 0$, $\varrho = 0.05$ and the bang–bang pulse of $d = 0.8$ violates the condition and gives $6(J - J_{\mathrm{sq}}) = -0.115$; `condicion_unificada.py`.)

Isabelle · ours **show**

square_unified · Isabelle/MemoryAccessFloors/Square_Unified_Condition.thy, line 52

```
theorem square_unified:
  fixes z0 A0 sq sm C gq gm x y u :: real
  assumes "0 < z0" "A0 = pi/2 * z0" "0 \<le> sq" "0 \<le> sm" "0 \<le> C" "0 \
<le> gq" "0 \<le> gm"
      "0 \<le> sq + sm - 2*C" "0 \<le> x" "0 \<le> y" "0 \<le> u"
      "u \<le> z0 \<Longrightarrow> (z0 - u)^2 \<le> x*y"
      "(sq + sm - 2*C) * z0 \<le> sqrt (((sq + C)*A0 + gq) * ((sm + C)*A0 +
gm))"
  shows "0 \<le> 2*(sq + sm - 2*C)*(u^2 - z0^2)
           + 2*(((sq + C)*A0 + gq)*x + ((sm + C)*A0 + gm)*y)
           + sq*x^2 + 2*C*x*y + sm*y^2"
```

`square_unified_no_relax` · Isabelle/MemoryAccessFloors/Square_Unified_Condition.thy, line 123

```
corollary square_unified_no_relax:
  fixes z0 A0 sq sm C x y u :: real
  assumes "0 < z0" "A0 = pi/2 * z0" "0 \<le> sq" "0 \<le> sm" "0 \<le> C"
      "0 \<le> sq + sm - 2*C" "0 \<le> x" "0 \<le> y" "0 \<le> u"
      "u \<le> z0 \<Longrightarrow> (z0 - u)^2 \<le> x*y"
      "(sq + sm - 2*C) * z0 \<le> A0 * sqrt ((sq + C) * (sm + C))"
  shows "0 \<le> 2*(sq + sm - 2*C)*(u^2 - z0^2)
           + 2*((sq + C)*A0*x + (sm + C)*A0*y) + sq*x^2 + 2*C*x*y + sm*y^2"
```

`square_unified_qubit_only` · Isabelle/MemoryAccessFloors/Square_Unified_Condition.thy, line 149

```
corollary square_unified_qubit_only:
  fixes z0 A0 sq gq gm x y u :: real
  assumes "0 < z0" "A0 = pi/2 * z0" "0 \<le> sq" "0 \<le> gq" "0 \<le> gm"
      "0 \<le> x" "0 \<le> y" "0 \<le> u"
      "u \<le> z0 \<Longrightarrow> (z0 - u)^2 \<le> x*y"
      "sq * z0 \<le> sqrt ((sq*A0 + gq) * gm)"
  shows "0 \<le> 2 * sq * (u^2 - z0^2) + 2*((sq*A0 + gq)*x + gm*y) + sq*x^2"
```

`square_unified_uncorrelated_needs_relax` · Isabelle/MemoryAccessFloors/Square_Unified_Condition.thy, line 163

```
corollary square_unified_uncorrelated_needs_relax:
  fixes z0 A0 sq sm :: real
  assumes "0 < z0" "A0 = pi/2 * z0" "0 \<le> sq" "0 \<le> sm" "0 < sq + sm"
  shows "\<not> ((sq + sm - 2*0) * z0 \<le> sqrt (((sq + 0)*A0 + 0) * ((sm +
0)*A0 + 0)))"
```

Cite: Remark 11, p. 18, doi:10.5281/zenodo.24250503

↳ Not used by another result of the paper.

Evidence & scope

- **Literature:** Companion paper: this statement answers a question left by paper A (Exact limits on writing a superconducting quantum memory), Proposition 14 (silent link, not printed in the paper). `https://karlesmarin.github.io/blueprints/memory-floors/`

Tables

Every table of the paper, in its order: click one to unfold it, with where the paper cites it, how its numbers were checked, and its data as CSV. The typeset table is one click away, on its page of the PDF.

Table without a number, in the text of §2.5 Slow noise together with memory loss

$r = \gamma_q/\gamma_m$	0	0.1	0.3	1	3	10	30	100	1000
$\Lambda_c(r)$	7.908	7.956	8.050	8.366	9.170	11.346	15.521	24.260	67.245
sufficient	3.141	3.264	3.485	4.113	5.374	8.087	12.637	21.632	64.835

⬇ CSV

Table without a number, in the text of §A Formula sheet

quantity	formula

robustness (complete lossless transfer, $\lvert g\rvert \le G$, any detuning)	$\big(\lvert\zeta\rvert - z_0\big)^2 \le (A-A_0)(B-A_0)$, $A, B \ge A_0$; $T \ge T_0 + \frac{1}{G}\arccos\frac{\lvert\zeta\rvert}{z_0}$ for $\lvert\zeta\rvert \le z_0$; $T_0 = \frac{\pi}{2G}$, $A_0 = \frac{\pi}{4G}$, $z_0 = \frac{1}{2G}$
cost of $\zeta = 0$, fixed phase, no detuning, $\gamma_q, \gamma_m > 0$	$\min\big[\gamma_q(A-A_0) + \gamma_m(B-A_0)\big] = \frac{\gamma_q+\gamma_m}{2G}\arccos\frac{\lvert\gamma_q-\gamma_m\rvert}{\gamma_q+\gamma_m}$
slow noise, second order (static offsets, no losses)	$r = \frac{2\,\mathbb{E}(\xi_q-\xi_m)^2\lvert\zeta\rvert^2 + \mathbb{E}(\xi_q A + \xi_m B)^2}{6}$; qubit only: $r \ge \frac{\pi^2\sigma_q^2}{96G^2}$
noise on both elements, fixed phase, no detuning	square pulse optimal exactly for $\varrho_c \le \sigma_m^2/\sigma_q^2 \le 1/\varrho_c$, $\varrho_c = 0.12645352248541\ldots$ (the bang–bang crossing)
correlated offsets, $C = \mathbb{E}(\xi_q\xi_m) \ge 0$	$6r = 2V\lvert\zeta\rvert^2 + \sigma_q^2A^2 + 2CAB + \sigma_m^2B^2$, $V = \sigma_q^2 + \sigma_m^2 - 2C$; square pulse optimal if $\frac{\pi}{2}\sqrt{(\sigma_q^2+C)(\sigma_m^2+C)} \ge V$; fixed phase, no detuning: exactly on $[\varrho_c(\kappa), 1/\varrho_c(\kappa)]$, $\varrho_c(0.5) = 0.0272$, $\varrho_c(1) = 0.00948$
offsets and relaxation of both elements	square pulse optimal if $Vz_0 \le \sqrt{\big((\sigma_q^2+C)A_0 + \gamma_q\big)\big((\sigma_m^2+C)A_0 + \gamma_m\big)}$
slow noise on the qubit and memory decay	$\min J = \frac{\pi^2\sigma_q^2}{96G^2} + \frac{\sigma_q\sqrt{\pi\gamma_m}}{6G^{3/2}} + O(\gamma_m/G)$ for $\gamma_m G \ll \sigma_q^2 \ll \sqrt{\gamma_m}G^{3/2}$; square pulse optimal for $\sigma_q^2 \le 7.1\,\gamma_m G$, not for $\sigma_q^2 \ge 7.92\,\gamma_m G$; fixed phase, no detuning: optimal exactly for $\sigma_q^2 \le \Lambda_c\gamma_m G$, $\Lambda_c = 7.9078851606875\ldots$; with qubit relaxation $r = \gamma_q/\gamma_m$: $\Lambda_c(1) = 8.366$, $\Lambda_c(1000) = 67.245$, $\Lambda_c(r) = 2\sqrt{r} + \pi/2 + 4/\sqrt{3} + O(r^{-1/2})$ given Lemma 6, non-decreasing and concave in r; any phase and detuning: $2\sqrt{r} + \pi/2 \lesssim \Lambda_c^{\mathrm{all}}(r) \lesssim 2\sqrt{r} + \pi/2 + 4/\sqrt{3}$ up to $O(r^{-1/2})$

⬇ CSV

References

1

Q. Ansel, S. J. Glaser and D. Sugny, Selective and robust time-optimal rotations of spin systems, J. Phys. A 54, 085204 (2021); we used arXiv:2010.12454v2, §4.

2

J. Bae, J. H. Bai, B.-Y. Lee, J.-Y. Lee and C.-H. Lee, On the reachability of 3-dimensional paths with a prescribed curvature bound, arXiv:2403.18707v3 (2025); Theorems 3 and 5, and p. 10.

3

E. Barnes, F. A. Calderon-Vargas, W. Dong, B. Li, J. Zeng and F. Zhuang, Dynamically corrected gates from geometric space curves, Quantum Sci. Technol. 7, 023001 (2022); we used arXiv:2103.16015v1, Eqs. (6)–(8) and §II.B.

4

M. Bergonzoni, S. Jandura and G. Pupillo, iSWAP gate with polar molecules: robustness criteria for entangling operations, Phys. Rev. A 112, 032621 (2025); we used arXiv:2502.21238v2.

5 D. Daems, A. Ruschhaupt, D. Sugny and S. Guérin, Robust quantum control by a single-shot shaped pulse, Phys. Rev. Lett. 111, 050404 (2013); we used arXiv:1304.4016, Eqs. (7)–(10).

6 L. E. Dubins, On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents, Amer. J. Math. 79, 497–516 (1957). 🔎 find it

7 S. Jandura, J. D. Thompson and G. Pupillo, Optimizing Rydberg gates for logical-qubit performance, PRX Quantum 4, 020336 (2023); we used arXiv:2210.06879v3, Sections II–III and Appendix A.

8 E. Kyoseva, H. Greener and H. Suchowski, Detuning-modulated composite pulses for high-fidelity robust quantum control, Phys. Rev. A 100, 032333 (2019); we used arXiv:1811.02623v1.

9 X.-J. Lu, X. Chen, A. Ruschhaupt, D. Alonso, S. Guérin and J. G. Muga, Fast and robust population transfer in two-level quantum systems with dephasing noise and/or systematic frequency errors, Phys. Rev. A 88, 033406 (2013); we used arXiv:1305.6127v1.

10 C. Marín, Exact limits on writing a superconducting quantum memory: when can pulse shaping beat the resonant swap?, companion paper, Zenodo (2026), doi:10.5281/zenodo.24250501.

11 D. Mittenhuber, Dubins' problem is intrinsically three-dimensional, ESAIM: Control Optim. Calc. Var. 3, 1–22 (1998); Theorem 1.1 and the planar and spatial shifts, pp. 2–3. 🔎 find it

12 H. Nelson and E. Barnes, Exact quantum speed limits for quantum gates from space curve geometry, Phys. Rev. A 114, 052426 (2026), doi:10.1103/28zn-ch8y; we used arXiv:2604.23031v1.

13 V. S. Patsko and A. A. Fedotov, Analytic description of a reachable set for the Dubins car, Trudy Inst. Mat. Mekh. UrO RAN 26(1), 182–197 (2020), in Russian; Eq. (2.6), §§3.3–3.4 and Theorem 1. 🔎 find it

14 V. S. Patsko and A. A. Fedotov, Three-dimensional reachable set for the Dubins car: foundation of analytical description, Commun. Optim. Theory 2022, Art. 23, 1–42, doi:10.23952/cot.2022.23; Eq. (3.5), §3.4, Property 8 (§3.7), Theorem 11.1.

15 P. Souères, A. Balluchi and A. Bicchi, Optimal feedback control for route tracking with a bounded-curvature vehicle, Proc. IEEE Int. Conf. Robotics and Automation (ICRA 2000), vol. 3, 2473–2478; Property 5 and the synthesis. 🔎 find it

16

J. M. Sullivan, Curves of finite total curvature, in *Discrete Differential Geometry*, Birkhäuser (2008); we used arXiv:math/0606007, §5 (Schur's comparison theorem and its proof).

17 H. J. Sussmann, Shortest 3-dimensional paths with a prescribed curvature bound, Proc. 34th IEEE CDC (1995) 3306–3312; Theorem 1 and the proof on p. 3312. 🔎 find it

18 L. Van Damme, Q. Ansel, S. J. Glaser and D. Sugny, Robust optimal control of two-level quantum systems, Phys. Rev. A 95, 063403 (2017); we used arXiv:1704.07653.

19 J. Zeng and E. Barnes, Fastest pulses that implement dynamically corrected single-qubit phase gates, Phys. Rev. A 98, 012301 (2018); we used arXiv:1802.10241v2.

20 J. Zeng, C. H. Yang, A. S. Dzurak and E. Barnes, Geometric formalism for constructing arbitrary single-qubit dynamically corrected gates, Phys. Rev. A 99, 052321 (2019); we used arXiv:1811.04864v2, Eqs. (1)–(8).