The order generated by a column of the $SU(n)_k$ S-matrix is locally a numerical semigroup ring (Part I)

Algebraic number theory · orders generated by character values

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At a glance

What the paper contributes

A column μ of the modular S-matrix of $SU(n)_k$, normalised by its first entry, generates a ring A_μ of character values. Its field was known (Gannon–Schopieray); the paper determines the **ring** when the height $p = n + k$ is prime.

- **The decomposition** (Lemma 1, Proposition 2): $A_\mu = \mathbb{Z}[\zeta_m] \otimes C_T$, a maximal cyclotomic factor fixed by the charge under the centre, and an order C_T that depends only on the reduced spectrum $T \subset \mathbb{F}_p$; the index is a power of p.
- **The exact semigroup** (Theorem 3, Corollary 5): locally at p, C_T is a monomial ring whose valuations form a numerical semigroup S_T, written by Gauss sums and Stickelberger's theorem; the index is $p^{\varphi(m)\delta}$, δ the number of gaps.
- **Read in $\mathbb{F}_p$** (Theorem 4): when $p > (n-1)(n-2)$, and for every p when $n \le 12$, S_T is generated by e and the j with $M_{hj}(T) \neq 0$.
- **One semigroup, every property** (Corollaries 6–8, Propositions 9–10): Gorenstein ⇔ S_T symmetric (Kunz), exponent p when the conductor is small, the vacuum is maximal, level–rank keeps S_T, the conformal weight sees M_2.
- **Counts and extremes** (Propositions 11–12, §6): a closed count of the maximal columns, the largest defect $(n-1)(n-2)/2$ attained exactly by the trinomials $X^n + aX + b$, and the first non-Gorenstein column at prime height, $SU(5)_{12}$.

🚀 **Start here: Proposition 2**: the ring of a column Z[ζ_m] ⊗ C_T, and its index is a power of p.

Main results:

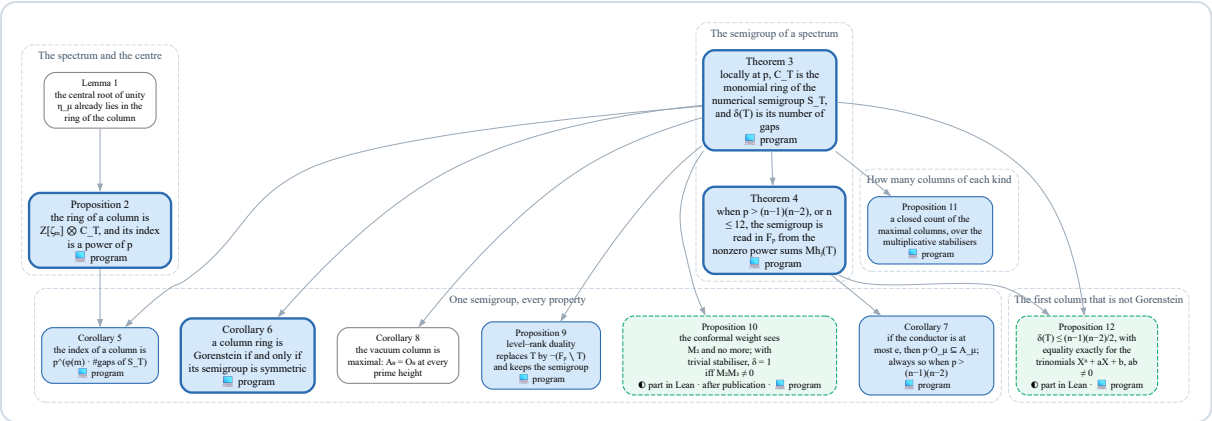
- **Proposition 2**: the ring of a column Z[ζ_m] ⊗ C_T, and its index is a power of p
- **Theorem 3**: locally at p, C_T is the monomial ring of the numerical semigroup S_T, and δ(T) is its number of gaps
- **Theorem 4**: when $p > (n-1)(n-2)$, or $n \le 12$, the semigroup is read in F_p from the nonzero power sums $Mh_j(T)$
- **Corollary 6**: a column ring is Gorenstein if and only if its semigroup is symmetric

12 results | ✅ **0** entirely in Lean | ◐ **2** partly in Lean

💻 **10** checked by a program | written proof only | (a result can carry several tags)

🗺️ Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's “Evidence & scope” block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



📐 Setting (copied from the paper, §1–§3)

(Abstract) Fix a column μ of the modular S-matrix of $\mathrm{SU}(n)_k$ and let A_μ be the ring generated by the ratios $S_{\lambda\mu}/S_{0\mu}$: the character values of $\mathrm{SU}(n)$ at the element of finite order that labels μ. *(§1)* For $\mathrm{SU}(n)$ at level k and integrable weights λ, μ,

$$\frac{S_{\lambda\mu}}{S_{0\mu}} = \mathrm{ch}_\lambda(g_\mu), \qquad g_\mu = \exp\!\left(2\pi i\,\frac{\mu+\rho}{k+n}\right).$$

Throughout, p is an odd prime, $n \geq 2$, $k = p - n \geq 1$, $\zeta_N = e^{2\pi i/N}$, and for a column μ we write $\mathcal{O}_\mu$ for the integers of the field K_μ of A_μ. *(§2)* Write $\ell_i = \sum_{j\geq i}\mu_j + n - i$ $(1 \leq i \leq n)$, $L = \sum_i \ell_i$, and fix integers u, v with $up + vn = 1$. The eigenvalues of g_μ on $\mathbb{C}^n$ are $\zeta_{np}^{n\ell_i - L}$, and

$$\zeta_{np}^{n\ell_i - L} = \eta_\mu\,\zeta_p^{\,t_i}, \qquad \eta_\mu = \zeta_n^{-uL}, \qquad t_i = \ell_i - vL \bmod p.$$

The ℓ_i are distinct in $[0, p-1]$, so $T = T_\mu = \{t_1, \ldots, t_n\} \subset \mathbb{F}_p$ has n elements, and $\sum_i t_i = L(1-nv) = Lup \equiv 0$: the spectrum is *centred*. The scalar η_μ is a root of unity of order $m = m_\mu$ dividing n. *(proof of Lemma 1)* With $y_t = \zeta_p^t$, …Let $C_T = \mathbb{Z}[e_1(y_T), \ldots, e_{n-1}(y_T)] \subset \mathbb{Z}[\zeta_p]$, $H_T = \{a \in \mathbb{F}_p^\times : aT = T\}$ of order h, and $E_T = \mathbb{Q}(\zeta_p)^{H_T}$, of degree $e = (p-1)/h$. From now on $\delta(T) = v_p[\mathcal{O}_{E_T} : C_T]$. *(§3)* Let $\omega : \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$ be the Teichmüller character, and put

$$B_j(T) = \sum_{t\in T,\, t\neq 0} \omega(t)^{hj} \in \mathbb{Z}_p, \qquad b_j = v_p(B_j) \in \{0, 1, \ldots, \infty\} \qquad (1 \leq j < e).$$

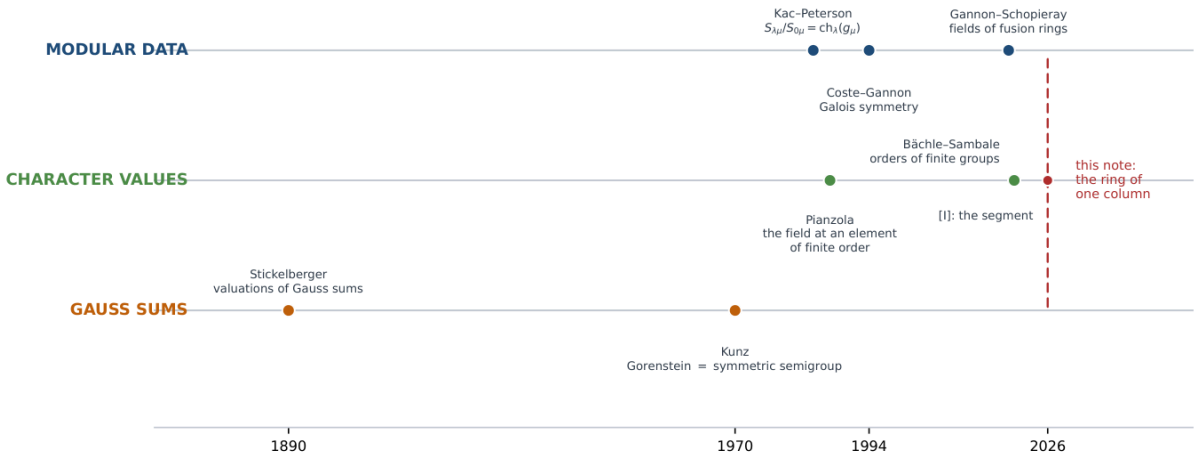
Since $\omega(t) \equiv t$, $b_j = 0$ exactly when the power sum $M_{hj}(T) = \sum_{t\in T} t^{hj}$ is nonzero in $\mathbb{F}_p$.

1 Columns since 1984, and a question of rings

No numbered statement in this section.

Its figures and tables are below; its text is in the paper, p. 1.

Figure 1



Three lanes, and this note at the crossing. *Modular data*: from Kac–Peterson's formula to the fields of Gannon and Schopieray; the question ends at a field. *Character values*: the field at an element of finite order (Pianzola), the order for finite groups (Bächle–Sambale), the order at the principal element ([10]). *Gauss sums and semigroups*: Stickelberger's valuations make the ring monomial, and Kunz reads the Gorenstein property off its semigroup.

2filaclarawhite

question	answer
What does a column generate? (§2)	the integers of $\mathbb{Q}(\zeta_m)$, fixed by the charge under the centre, tensored with the order C_T of the spectrum; the defect is a power of p. Lemma 1, Proposition 2
What is that order at p? (§3)	a monomial ring with one numerical semigroup S_T, written by Gauss sums; the index is its number of gaps. Theorem 3
Can the semigroup be read in $\mathbb{F}_p$? (§3)	yes, from the nonzero power sums of the spectrum, when $p > (n-1)(n-2)$ and for every p when $n \leq 12$. Theorem 4
What else does the semigroup say? (§4)	Gorenstein = symmetric, exponent p when the conductor is small, the vacuum is maximal, level–rank keeps the semigroup; the conformal weight sees M_2 and no more. Corollaries 6–8, Propositions 9 and 10
How many columns are maximal? (§5)	a closed count over the stabilisers: a maximal column needs a multiplicative symmetry of its spectrum, which becomes rare. Proposition 11
Where does Gorenstein first fail? (§6)	at prime height, for $n = 5$: at $\mathrm{SU}(5)_{12}$, $\mu = (0,0,2,1)$, semigroup $\langle 3,4,5\rangle$. Table 1

Colour: green = yes / holds.

2 The spectrum and the centre

Lemma 1. (the centre is in the ring) p. 3

> $\eta_\mu \in A_\mu$.

↳ Used in: Proposition 2 (p. 5)

Proof

> The complete symmetric function h_{p-n} is the character of $\mathrm{Sym}^{p-n}\mathbb{C}^n$. With $y_t = \zeta_p^t$, $\prod_{t\in T}(1-y_tX)\prod_{t\notin T}(1-y_tX) = 1-X^p$, so $\sum_j h_j(y_T)X^j = \prod_{t\notin T}(1-y_tX)/(1-X^p)$ and $h_{p-n}(y_T)$ is the top coefficient of the numerator, $(-1)^{p-n}\prod_{t\notin T} y_t = (-1)^{n-1}$, because $\sum_{t\in\mathbb{F}_p} t = 0 = \sum_{t\in T} t$. By homogeneity $\mathrm{ch}_{h_{p-n}}(g_\mu) = (-1)^{n-1}\eta_\mu^{p-n} = \pm\eta_\mu^p$, and p is invertible modulo the order of η_μ.

Proposition 2. (the decomposition) p. 3

> $A_\mu = \mathbb{Z}[\zeta_m]\otimes C_T$, $K_\mu = \mathbb{Q}(\zeta_m)E_T$, *and*
>
> $$\boxed{[\mathcal{O}_\mu : A_\mu] = [\mathcal{O}_{E_T} : C_T]^{\varphi(m)},}$$
>
> *a power of* p.

↳ Used in: Corollary 5 (p. 9)

Proof

> The fundamental characters generate the representation ring, so $A_\mu = \mathbb{Z}[e_j(g_\mu)]$ with $e_j(g_\mu) = \eta_\mu^j e_j(y_T)$; with Lemma 1, $A_\mu = \mathbb{Z}[\eta_\mu]\cdot C_T$. As $\gcd(m,p)=1$, $\mathbb{Q}(\zeta_m)$ and E_T are linearly disjoint with coprime discriminants, so $\mathcal{O}_{K_\mu} = \mathbb{Z}[\zeta_m]\otimes\mathcal{O}_{E_T}$ and $\mathbb{Z}[\zeta_m]\cdot C_T \cong \mathbb{Z}[\zeta_m]\otimes C_T$; the index is $[\mathcal{O}_{E_T} : C_T]^{\varphi(m)}$. It remains to see that C_T is maximal at every prime $r \neq p$. The prime r is unramified in E_T, so $\mathcal{O}_{E_T}/r$ is a finite étale $\mathbb{F}_r$-algebra, whose geometric points are the e embeddings σ_a, $a \in \mathbb{F}_p^\times/H_T$, followed by $\zeta_p \mapsto \xi$, a primitive p-th root of unity in $\overline{\mathbb{F}}_r$. The image of C_T separates them: $e_j(y_T)$ goes to the coefficients of $\prod_{t\in T}(X-\xi^{at})$, and $\{\xi^{at}\} = \{\xi^{a't}\}$ forces $aT = a'T$. A

subalgebra of a finite étale algebra that separates its geometric points is everything, so $C_T + r\mathcal{O}_{E_T} = \mathcal{O}_{E_T}$, and Nakayama gives $C_T \otimes \mathbb{Z}_r = \mathcal{O}_{E_T} \otimes \mathbb{Z}_r$.

In plain words (commentary, not part of the paper)

1. The ring A_μ of one column splits into two independent factors. The centre of $\mathrm{SU}(n)$ contributes the integers $\mathbb{Z}[\zeta_m]$ of a cyclotomic field, fixed by the charge of μ; that factor is already maximal and only multiplies the index. Everything else lives in the order C_T generated by the elementary symmetric functions of ζ_p^t, $t \in T$, which depends only on the reduced spectrum $T \subset \mathbb{F}_p$. 2. Why: the fundamental characters at g_μ are $\eta_\mu^j e_j(y_T)$, and Lemma 1 puts η_μ in the ring, so $A_\mu = \mathbb{Z}[\eta_\mu] \cdot C_T$; the two fields are linearly disjoint with coprime discriminants, which turns the product into a tensor product and the index into $[\mathcal{O}_{E_T} : C_T]^{\varphi(m)}$. 3. The index is a power of p: at a prime $r \neq p$ the image of C_T separates the geometric points of the étale algebra $\mathcal{O}_{E_T}/r$, so it is all of it, and Nakayama gives maximality at r. From here on the paper studies only $\delta(T) = v_p[\mathcal{O}_{E_T} : C_T]$.

Evidence & scope

- **Program:** the index [O_μ : A_μ] = p^φ(m)δ as a lattice index in all 662 columns of SU(3)_4, SU(4)_3, SU(3)_8, SU(8)_3, SU(5)_6, SU(6)_5. `nivel_rango.py`

3 The semigroup of a spectrum

Theorem 3. (the exact semigroup) p. 4

Let $R = C_T \otimes \mathbb{Z}_p$ and let $\theta = g_1^h$, with g_1 the Gauss sum below, a uniformizer of $E_T \otimes \mathbb{Q}_p$ on which the Galois group acts by a character. Then R is the monomial ring

$$R = \sum_{s \in S_T} \mathbb{Z}_p\, \theta^s, \qquad \boxed{S_T = \langle\, e,\ \ j + e\, b_j \ :\ 1 \le j < e,\ b_j < \infty \,\rangle,}$$

the valuations of its nonzero elements form the numerical semigroup S_T, and $\delta(T) = \#(\mathbb{N}_0 \setminus S_T)$, the number of gaps.

Classical and new: built from Stickelberger's valuations of Gauss sums [Sti90; Rob01, Cor. 1] and Selmer's formula for the genus [Sel77].

↳ Used in: Theorem 4 (p. 7), Corollary 5 (p. 9), Corollary 6 (p. 9), Corollary 8 (p. 11), Proposition 9 (p. 11), Proposition 10 (p. 11), Proposition 11 (p. 13), Proposition 12 (p. 13)

Proof

Power sums. By Newton's identities e_j is a polynomial in $p_1, \dots, p_j$ with denominators dividing $j!$, invertible in $\mathbb{Z}_p$ as $j \le n < p$; and every power sum is

an integral polynomial in the e_j. Hence $R = \mathbb{Z}_p[p_r(y_T) : r \geq 1] = \mathbb{Z}_p[\sigma_a(\alpha) : a \in \mathbb{F}_p^\times]$ with $\alpha = \sum_{t\in T} \zeta_p^t$.

Eigenvectors. R is stable under $\mathrm{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p) \cong \mathbb{F}_p^\times$, and $p-1$ is a unit, so the projectors $P_r = \frac{1}{p-1}\sum_a \omega(a)^{-r}\sigma_a$ act on R, and R is generated as a $\mathbb{Z}_p$-algebra by the $P_r\alpha$. Substituting $b = at$,

$$P_r\alpha = \frac{1}{p-1}\, B'_r\, g_r, \qquad g_r = \sum_{b\in\mathbb{F}_p^\times} \omega(b)^{-r}\zeta_p^b, \qquad B'_r = \sum_{t\in T,\, t\neq 0} \omega(t)^r,$$

and $B'_r = 0$ unless $h \mid r$, because T is H_T-stable. By Stickelberger's theorem $v_\pi(g_r) = r$ for $0 \leq r < p-1$, with $\pi = \zeta_p - 1$ [15 , Cor. 1, p. 168] .

The uniformizer. Put $\theta = g_1^h$. As $\sigma_a g_r = \omega(a)^r g_r$, θ is fixed by H_T, lies in E_T, and has π-valuation h: it is a uniformizer of $E_T \otimes \mathbb{Q}_p$, which is totally ramified of degree e. For the same reason $g_{hj}/\theta^{\,j}$ and $\theta^{\,e}/p$ are fixed by all of Gal, lie in $\mathbb{Q}_p$, and have valuation 0: they are units of $\mathbb{Z}_p$. So $P_{hj}\alpha = u_j\, p^{b_j}\theta^{\,j} = u'_j\,\theta^{\,j+eb_j}$ with $u_j, u'_j \in \mathbb{Z}_p^\times$, and p is a unit times θ^e.

Monomial. R is generated over $\mathbb{Z}_p$ by unit multiples of θ^s, s in the generators of S_T, so $R = \sum_{s\in S_T} \mathbb{Z}_p\theta^s$ inside $\mathcal{O}_{E_T} \otimes \mathbb{Z}_p = \mathbb{Z}_p[\theta]$, whose $\mathbb{Z}_p$-basis is $1, \theta, \ldots, \theta^{e-1}$. For each residue i modulo e let $w_i = i + ec_i$ be the least element of S_T in that class (the Apéry set of S_T with respect to e). Then $R = \bigoplus_{i<e} \mathbb{Z}_p\, p^{c_i}\theta^i$ up to units, the valuations of $R \setminus 0$ are exactly S_T, and $v_p[\mathbb{Z}_p[\theta] : R] = \sum_i c_i$, which is the number of gaps of S_T by Selmer's formula [6] .

In plain words (commentary, not part of the paper)

1. Locally at p, the order of a spectrum is as simple as it could be: it is spanned by the powers θ^s of one uniformizer θ, for s in a numerical semigroup S_T. The semigroup is generated by the ramification index e and by the numbers $j + e\,b_j$, where b_j is the p-adic valuation of the power sum $B_j(T)$ of Teichmüller lifts. Its number of gaps is the defect $\delta(T)$. 2. Why: by Newton's identities the ring is generated by the Galois conjugates of $\alpha = \sum_{t\in T}\zeta_p^t$; the Galois eigenprojectors have coefficients in $\mathbb{Z}_p$ and send α to a multiple $B'_r g_r$ of a Gauss sum, and Stickelberger's theorem gives $v_\pi(g_r) = r$. With $\theta = g_1^h$ each eigencomponent is a unit times $\theta^{\,j+eb_j}$, so the ring is monomial; the least elements of S_T in each class modulo e (the Apéry set) give the index, which by Selmer's formula is the number of gaps. 3. The paper notes that the splitting into eigenlines is automatic (a Galois-stable ring in a tamely ramified extension); what the theorem adds is which powers of p multiply them.

Evidence & scope

- **Program:** δ(T) as a Hermite-normal-form index equals g(S_T) in every centred spectrum with $3 \leq n \leq 6$, $5 \leq p \leq 31$ (2414 orbits), and for n = 7, 8, 9 below the bound (11 331 orbits). `censo.py`

Theorem 4. (the semigroup of moments) p. 4

If $p>(n-1)(n-2)$ (equivalently $p \geq (n-1)(n-2)$, as p is odd), then $S_T = \Gamma_T$ and

$$\boxed{\delta(T) = \#\big(\mathbb{N}_0 \setminus \langle e,\ j:\ M_{hj}(T) \neq 0\rangle\big).}$$

The same holds for every p when $n \leq 12$.

🆕 **Classical and new:** the conductor bound uses Schur's bound on the Frobenius number, as given by Brauer [Bra42].

↳ Used in: Corollary 7 (p. 10), Proposition 12 (p. 13)

Proof

If the conductor $c(\Gamma_T)$ is at most $e+1$, every $j+eb_j$ with $b_j \geq 1$ is $\geq e+1 \geq c$ and already lies in Γ_T. Let $V = \{t^h : t \in T \setminus 0\}$, a set of $d \leq n/h$ elements of the group $\mu_e \subset \mathbb{F}_p^\times$ of h-th powers; each $v \in V$ is hit h times, so $M_{hj}(T) = h\,p_j(V)$. With $A(Y) = \prod_{v\in V}(1-vY)$ and $B(Y) = \prod_{v\in\mu_e\setminus V}(1-vY)$, $A(Y)B(Y) = 1-Y^e$. Newton's identities $p_j + a_1 p_{j-1} + \cdots + j a_j = 0$, with $j<e<p$ invertible, show by induction that the indices $j<e$ with $p_j(V) \neq 0$ and those with $a_j \neq 0$ generate the same semigroup, and likewise for the coefficients of $1/A = B/(1-Y^e)$, which are those of B below e. So Γ_T is generated by e and a set G of integers at most $D = \min(d, e-d)$: the support of A or that of B, whichever has the smaller degree. The elements of G have no common factor $g>1$: every exponent of A is below e, hence a combination of G alone, so A would be a polynomial in Y^g; its roots, and so V, would be stable under μ_g, and as $T \setminus 0$ is the full preimage of V under $t \mapsto t^h$, T would be stable under $\mu_{gh} \supsetneq H_T$. By Schur's bound on the Frobenius number [4] , $c(\Gamma_T) \leq c(\langle G\rangle) \leq (D-1)(D-2)$, which is 0 if $D \leq 2$; otherwise $3 \leq D \leq d \leq n/h$ and $(D-1)(D-2) \leq (n/h-1)(n/h-2) \leq (n-1)(n-2)/h \leq p/h \leq e+1$ when $p \geq (n-1)(n-2)$.

For $n \leq 12$, exchange A and B if necessary so that $D = \deg A \leq e/2$; then $D \leq n \leq 12$, $G = \operatorname{supp} A \setminus 0$, and $D \in G$, the coefficient of Y^D being $\pm\prod_v v \neq 0$. Moreover $e \in \langle G\rangle$: every monomial of $B = [A^{-1}]_{\leq e-D}$ has its exponent in $\langle G\rangle$, so if $e \notin \langle G\rangle$ the coefficient of Y^e in AB would be 0, not -1. Hence $\Gamma_T = \langle G\rangle$, and since $e \in \Gamma_T$ it suffices to show $c(\Gamma_T) \leq e$. Let κ be the multiplicity of Γ_T. If $\kappa = 2$ or 3, a generator $u \leq D$ prime to κ gives $c \leq c(\langle \kappa, u\rangle) \leq 2(D-1)<e$. If $\Gamma_T = \langle a, b\rangle$ has two minimal generators, then $ab \leq e$: otherwise every degree up to e has a unique expression $ia+jb$, the substitution $Y^{ia+jb} \mapsto U^iV^j$ turns $AB = 1-Y^e$ into a factorisation of a binomial $1-U^rV^s$, all of whose factors are polynomials in one monomial, while A contains both U and V; so $c = (a-1)(b-1)<e$. In general the roots of the reciprocal $F(X) = X^D A(1/X)$ are D elements of μ_e with trivial stabiliser, and for $r \in \mu_e \setminus 1$ the polynomial $F(rX) - r^D F(X)$ is nonzero of

> degree at most $D-\kappa$; counting the ordered pairs of roots with quotient r gives $D(D-1) \leq (e-1)(D-\kappa)$. Among the semigroups generated by integers at most 12 with multiplicity at least 4 and three or more minimal generators, only $\langle 6,9,11\rangle$, $\langle 6,10,11\rangle$, $\langle 7,9,12\rangle$ and $\langle 8,11,12\rangle$ are compatible with $e \geq 2D$, this inequality and $c(\Gamma_T)>e$, in sixteen triples (D,Γ_T,e). In each of them the truncation $B=[A^{-1}]_{\leq e-D}$ makes a coefficient of AB below degree e equal to a nonzero monomial in the leading coefficients of A — $-2c_6c_9$, $-2c_6c_{10}$, $-c_9^2$, $-2c_{11}c_{12}$ or $3c_8^2c_{11}$ — which contradicts $AB=1-Y^e$, since $p>e \geq 23$.

In plain words (commentary, not part of the paper)

1. The generators of Theorem 3 need p-adic valuations. This theorem says that, when $p>(n-1)(n-2)$ and for every p when $n \leq 12$, they are not needed: the semigroup S_T is generated by e and the indices $j<e$ for which the power sum $M_{hj}(T)=\sum_{t\in T} t^{hj}$ is nonzero in $\mathbb{F}_p$. The defect of a column is then computed with arithmetic modulo p alone. 2. Why: the generators read in $\mathbb{F}_p$ give a semigroup $\Gamma_T \subseteq S_T$, and the extra generators $j+eb_j$ with $b_j \geq 1$ are at least $e+1$, so they add nothing once the conductor of Γ_T is at most $e+1$. Newton's identities show that Γ_T is generated by e and the support of a polynomial of degree $D \leq n/h$; Schur's bound on the Frobenius number (as given by Brauer) bounds the conductor by $(D-1)(D-2)$, which is at most $e+1$ when $p \geq (n-1)(n-2)$. 3. For $n \leq 12$ and smaller p the proof is a finite case analysis: after general arguments only sixteen triples (D,Γ_T,e) remain, and in each a coefficient of $AB=1-Y^e$ below degree e is a nonzero monomial, a contradiction; the script `momentos_D12.py` re-enumerates them. Whether $c(\Gamma_T) \leq e$ always holds is left open in §7.

Evidence & scope

- **Program:** the sixteen residual triples (D, Γ_T, e) of the case n ≤ 12, re-enumerated, and their obstructing coefficients computed symbolically. `momentos_D12.py`
- **Program:** S_T = Γ_T in every orbit of the census (2414 + 11 331 orbits). `censo.py`

4 One semigroup, every property

Corollary 5. (the index of a column) p. 5

$$\boxed{[\mathcal{O}_\mu : A_\mu] = p^{\,\varphi(m_\mu)\,\#(\mathbb{N}_0 \setminus S_{T_\mu})}.}$$

↳ Not used by another result of the paper.

Evidence & scope

- **Program:** checked as an index of integer lattices in 662 columns. `nivel_rango.py`

Corollary 6. (Gorenstein columns) p. 5

> *A_μ is Gorenstein if and only if S_{T_μ} is symmetric.*

🆕 **Classical and new:** Kunz's theorem on the value semigroup of a one-dimensional Gorenstein ring [Kun70].

↳ Not used by another result of the paper.

Proof

> Away from p the order is maximal. At p, $A_\mu \otimes \mathbb{Z}_p$ is a product over the primes of the unramified $\mathbb{Z}[\zeta_m] \otimes \mathbb{Z}_p$ of copies of $R \otimes W$, W unramified; each is a complete local domain with residue field of the form $\mathbb{F}_{p^f}$, equal to that of its normalisation, and value semigroup S_T; Kunz's theorem [13 , Thm., p. 749] says such a ring is Gorenstein exactly when its value semigroup is symmetric.

💡 **In plain words (commentary, not part of the paper)**

1. Whether the ring of a column is Gorenstein is decided by its semigroup alone: exactly when S_T is symmetric, that is, when $s \mapsto c-1-s$ exchanges its elements and its gaps below the conductor c. 2. Why: away from p the order is maximal (Proposition 2). At p it is a product of complete local domains, each with the residue field of its normalisation and value semigroup S_T (Theorem 3), and Kunz's theorem says that such a ring is Gorenstein exactly when its value semigroup is symmetric. 3. The paper locates the first failure at prime height in §6: $\mathrm{SU}(5)_{12}$, $\mu = (0,0,2,1)$, with $S_T = \langle 3,4,5\rangle$, gaps $\{1,2\}$, not symmetric; the script `primera_no_gorenstein.py` checks the indices 17^2 and 17 as lattice indices.

🛡️ **Evidence & scope**

- **Program:** the first non-Gorenstein column at prime height, SU(5)_12, μ = (0,0,2,1): [O_μ : A_μ] = 17² and [A_μ : f] = 17 as lattice indices, with the control μ = 0. `primera_no_gorenstein.py`

Corollary 7. (the exponent) 📕 p. 6

> *If $c(S_T) \le e$, then $p\,\mathcal{O}_\mu \subseteq A_\mu$ and $\mathcal{O}_\mu / A_\mu$ is elementary abelian. This holds whenever $p > (n-1)(n-2)$.*

↳ Not used by another result of the paper.

Proof

> $p\,\mathbb{Z}_p[\theta] = \theta^e \mathbb{Z}_p[\theta]$ lies in the monomial ring R exactly when every integer $\ge e$ is in S_T. For the second sentence, the proof of Theorem 4 gives $S_T = \Gamma_T$ and $c \le (n-1)(n-2)/h \le (p-1)/h = e$.

🛡️ **Evidence & scope**

- **Program:** c(S_T) ≤ e and p·O_E_T ⊆ C_T on the lattice, in the 2414 orbits of the census and in the 2644 with 3 ≤ n ≤ 8, p ≤ 23. `exponente.py`

Corollary 8. (the vacuum) p. 6

> *The column $\mu = 0$ is maximal: $A_0 = \mathcal{O}_0$ for every $\mathrm{SU}(n)_k$ with $n + k$ prime.*

Classical and new: agrees with the preceding note [I], where at prime height the column μ = 0 lies outside every family of non-maximal segments.

↳ Not used by another result of the paper.

Proof

> For $\mu = 0$, $\ell_i = n - i$ and T is a centred arithmetic progression, $-T = T$, so $h \geq 2$; for $k \geq 2$ its second moment is $M_2 = n(n^2 - 1)/12 \neq 0$ in $\mathbb{F}_p$, since $2 \leq n \leq p - 2$. Then $1 \in \Gamma_T$ and $S_T = \mathbb{N}_0$; here $h = 2$ exactly, since $M_2 \neq 0$ and $M_r = 0$ unless $h \mid r$. For $k = 1$, T is $\mathbb{F}_p$ minus a point, $C_T = \mathbb{Z}$.

Proposition 9. (level–rank) p. 6

> *Under level–rank duality $\mathrm{SU}(n)_k \leftrightarrow \mathrm{SU}(k)_n$, the spectrum of the transposed column is $T' = -(\mathbb{F}_p \setminus T)$, and $C_{T'} = \sigma_{-1}(C_T)$: the semigroup, the index δ, the conductor and the Gorenstein property are preserved. Only the central factor $\varphi(m)$ may change.*

↳ Not used by another result of the paper.

Proof

> For a partition in an $n \times k$ box, the sets $\{\lambda_i + n - i\}$ and $\{n - 1 + j - \lambda'_j\}$ are complementary in $\{0, \ldots, n + k - 1\}$ [5 , I.(1.7)] ; reducing modulo $p = n + k$ and centring gives $T' = -(\mathbb{F}_p \setminus T)$. Next, $\prod_{t \in T}(1 - y_t X) \cdot \prod_{t \notin T}(1 - y_t X) = 1 - X^p$, so the elementary symmetric functions of $y_{\mathbb{F}_p \setminus T}$ are the coefficients of a power series with integral coefficients in those of y_T, and conversely: $C_{\mathbb{F}_p \setminus T} = C_T$. Finally σ_{-1}, complex conjugation, carries $C_{\mathbb{F}_p \setminus T}$ to $C_{T'}$, and $\sigma_{-1}\theta = \omega(-1)^h \theta = \pm\theta$, so it preserves the monomial structure and S_T.

Evidence & scope

- **Program:** three level-rank pairs, every column, classes (h, gaps) in the proportion n/k. `nivel_rango.py`

Proposition 10. (the conformal weight) p. 6

> *Let $\Delta_\mu = \big(|\mu + \rho|^2 - |\rho|^2\big)/2p$ be the conformal weight of μ, with roots of square length 2, so that $|\rho|^2 = n(n^2 - 1)/12$ and $e^{2\pi i \Delta_\mu}$ is the eigenvalue of the modular matrix $\mathcal{T}$ at μ up to the central charge [11 , Thm. 13.8] . Then*

$$\boxed{M_2(T_\mu) \equiv |\mu+\rho|^2 \pmod{p},}$$

so $M_2(T_\mu)=0$ if and only if p does not divide the order of the root of unity $e^{2\pi i\Delta_\mu}e^{\pi i|\rho|^2/p}=e^{\pi i|\mu+\rho|^2/p}$. If moreover H_T is trivial, then $\delta(T)=1$ if and only if $M_2(T)M_3(T)\neq 0$; so the index of the column is $p^{\varphi(m)}$ exactly when both moments are nonzero, and the conformal weight decides the first of the two.

Lean · ours, in part **show**

added to the Lean repository after publication (21-oct-2026, written for Part II, Corollary 3): the semigroup half of the second sentence, from the generating rule with the moments abstracted to a predicate (e ≥ 4, M₁ = 0): δ ≥ 2 ↔ M₂M₃ = 0; not the congruence M₂ ≡ |μ+ρ|², nor Theorem 3 itself

- `P3b.two_le_genus_iff`

✅ Checked when this page was built: GapBound.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — two_le_genus_iff: propext, Classical.choice, Quot.sound.

Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean GapBound.lean` (SHA-256 of the compiled file: `f7f979544891ccc9`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 1 result, standard axioms only

Proof: in this note · **Covers:** not formalized · **Cite:** Lemma 1, p. 3, doi:10.5281/zenodo.22848530

↳ Not used by another result of the paper.

Proof

In the coordinates of §2, $\mu+\rho$ is $(\ell_1,\dots,\ell_n)$ modulo $(1,\dots,1)$, so $|\mu+\rho|^2=\sum_i(\ell_i-L/n)^2=\sum_i\ell_i^2-L^2/n$. With $t_i=\ell_i-vL$ and $vn\equiv 1$, $\sum_i t_i^2\equiv\sum_i\ell_i^2-2vL^2+nv^2L^2\equiv\sum_i\ell_i^2-L^2/n$. The root of unity is $e^{2\pi iQ/2np}$ with $Q=n|\mu+\rho|^2\in\mathbb{Z}$, and as $p\nmid 2n$ its order is prime to p exactly when $p\mid Q$. For the second sentence, $h=1$ gives $e=p-1\geq 4$. The only semigroup of genus one is $\langle 2,3\rangle$. By Theorem 3 an integer below e lies in S_T exactly when it is a sum of indices $j<e$ with $M_j\neq 0$; $M_1=0$, so $1\notin S_T$, and 2 and 3 lie in S_T exactly when M_2 and M_3 are nonzero, and then $S_T\supseteq\langle 2,3\rangle$ has at most one gap.

Evidence & scope

- **Program:** the congruence M₂ ≡ |μ+ρ|² in all 1820, 4368 and 1540 columns of SU(5)_12, SU(6)_11, SU(4)_19; the genus-one criterion in every orbit with p ≤ 31, n ≤ 9. `peso_conforme.py`

5 How many columns of each kind

Proposition 11. (the maximal columns) p. 7

> *Let $2 \le n \le p-2$, and for $h \mid p-1$ let $e = (p-1)/h$, $G_e \subset \mathbb{F}_p^\times$ the subgroup of order e, and $Z_{p,e}(d)$ the number of d-subsets of G_e with sum 0. The number of columns of $\mathrm{SU}(n)_{p-n}$ with $S_T = \mathbb{N}_0$ is*
>
> $$N_0(p,n) = n \sum_{\substack{h \mid p-1,\ h>1 \\ n \bmod h \in \{0,1\}}} \left[\binom{(p-1)/h}{\lfloor n/h \rfloor} - Z_{p,(p-1)/h}\big(\lfloor n/h \rfloor\big) \right].$$

↳ Not used by another result of the paper.

Proof

> $S_T = \mathbb{N}_0$ exactly when $1 \in S_T$, and a generator $j + eb_j$ equals 1 only for $j = 1$, $b_1 = 0$, that is when $M_h(T) \neq 0$. If $h = 1$ this fails, $M_1 = \sum T = 0$. For $h>1$, $T \setminus 0$ is a union of d cosets of H_T, so $n = hd$ or $hd+1$ according as $0 \notin T$ or $0 \in T$, and every such set is centred; and $M_h = h \sum_{u \in U} u$ with $U = \{t^h\} \subset G_e$ of size d. If $\sum U \neq 0$ the stabiliser is exactly of order h, since a larger one would kill M_h. So the sets with $S_T = \mathbb{N}_0$ and stabiliser of order h are counted by $\binom{e}{d} - Z_{p,e}(d)$, the families for different h are disjoint, and each centred spectrum stands for n columns.

Evidence & scope

- **Program:** the formula against enumeration: 30 of 30 pairs (p, n), p = 17, 19. `maximales.py`

6 The first column that is not Gorenstein

Proposition 12. (the largest defect) p. 8

> *For every centred T of size $n \ge 3$, $\delta(T) \le (n-1)(n-2)/2$, with equality exactly when $\prod_{t \in T}(X - t) = X^n + aX + b$ with $ab \neq 0$; then $\Gamma_T = S_T = \langle n-1, n \rangle$. Such T exist for every sufficiently large prime p.*

Lean · ours, in part show

the semigroup half: 2g ≤ (D−1)(D−2) for a semigroup generated in degrees ≤ D, equality only for ⟨D−1, D⟩, attained; not the passage to T, nor existence

- `P3a.genus_le_D`

- `P3a.eq_pair_of_genus_eq`
- `P3a.genus_pair_n`

✅ Checked when this page was built: GenusBound.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — genus_le_D, eq_pair_of_genus_eq, genus_pair_n: propext, Classical.choice, Quot.sound. Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean GenusBound.lean` (SHA-256 of the compiled file: `3c315a332e4ffcc2…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 3 results, standard axioms only

Proof: in this note · **Covers:** not formalized · **Cite:** Proposition 11, p. 7, doi:10.5281/zenodo.22848530

🆕 **Classical and new:** existence for large p from the count of Bank, Bary-Soroker and Rosenzweig [BBR15, Thm. 2.3].

↳ Not used by another result of the paper.

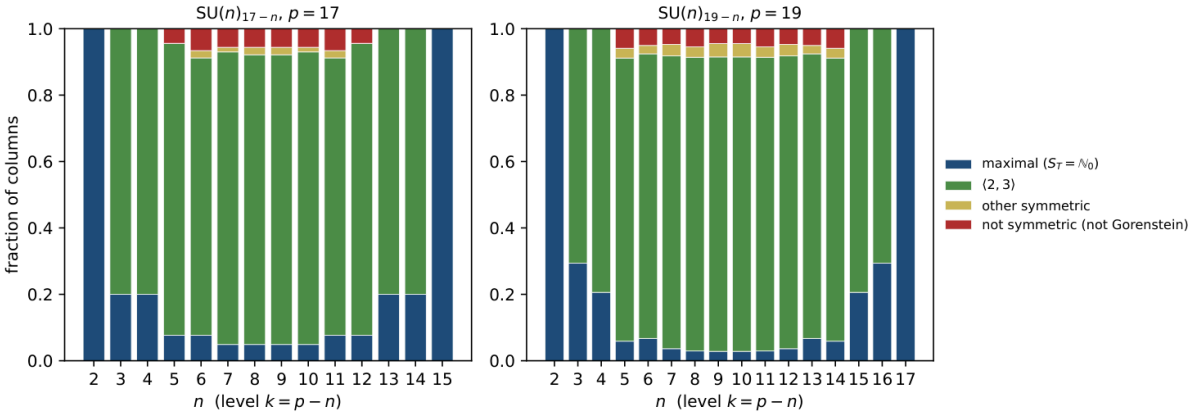
Proof

A numerical semigroup of multiplicity κ generated in degrees at most D has its Apéry elements modulo κ at most $D, 2D, \dots, (\kappa-1)D$, so its genus is at most $(\kappa-1)(D-1)/2 \le (D-1)(D-2)/2$; with $\delta(T) = g(S_T) \le g(\Gamma_T)$ and $D \le n$ this is the bound. Equality forces $D = n$, $h = 1$, $0 \notin T$ and $\kappa = n-1$, hence $\Gamma_T = \langle n-1, n\rangle$, and by Newton the only coefficients of $\prod(X-t)$ besides the leading one sit in degrees 1 and 0. Conversely, for a trinomial with n distinct roots the quotients of distinct roots are all different (if $x = ry$ with $r \neq 1$, then $a(r-r^n)y + b(1-r^n) = 0$ fixes y), so $n(n-1) \le p-2$, the conductor $(n-1)(n-2)$ is below $e = p-1$, and $S_T = \Gamma_T$. Existence: Bank, Bary-Soroker and Rosenzweig [3 , Thm. 2.3] count the polynomials of a given factorisation type in the short interval $f + \mathbb{F}_p[X]_{\le m}$; we use it with $f = X^n$, $m = 1$, which they allow when $p \nmid n(n-1)$, as here since $n<p$, and with the type $(1, \dots, 1)$. The number of pairs (a, b) for which $X^n + aX + b$ splits into linear factors is then $p^2/n! + O_n(p^{3/2})$, while $ab = 0$ or a repeated root excludes $O(p)$ of them.

Evidence & scope

- **Program:** n = 4, p ≤ 71: every split trinomial gives δ = 3 and conversely; n = 5, p ≤ 71: only p = 67, all 66 give δ = 6. `maximo.py`

Figure 2



The columns of $\mathrm{SU}(n)_{p-n}$ by the semigroup S_T of their spectrum, computed with the exact formula of Theorem 3; each centred spectrum stands for n columns. Maximal columns (blue) have $A_\mu = \mathcal{O}_\mu$; $\langle 2,3\rangle$ (green) has one gap; the other symmetric semigroups (gold) are Gorenstein with more gaps; the red ones are not Gorenstein. The bars for n and $p-n$ agree, as level–rank duality demands.

2filaclarawhite

n	$\delta(T)$ **by the moments (with** $h = \lvert H_T\rvert$**)**	**semigroups that occur**
3	0 if $h{>}1$; 1 if $h = 1$	$\mathbb{N}_0$, $\langle 2,3\rangle$
4	0 if $h{>}1$; 1 if $h = 1$, $M_2 \neq 0$; 3 if $h = 1$, $M_2 = 0$	$\mathbb{N}_0$, $\langle 2,3\rangle$, $\langle 3,4\rangle$
5, 6	≤ 4 for $p \leq 31$; up to $(n-1)(n-2)/2$ for large p	first non-symmetric: $\langle 3,4,5\rangle$, $p = 17$

Table 1: Small n. For $n = 3, 4$ the index of a column is a function of the stabiliser and of one or two moments, for every p. For $n \geq 5$ the largest number of gaps, $(n-1)(n-2)/2$, appears only for larger primes (Proposition 12).

Colour: blue shade = the number's size within its column.

7 What is open

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper, p. 9.

MSC 2020

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper (PDF).

Declaration of competing interest

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper (PDF).

Data availability

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper (PDF).

Declaration on generative AI

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper (PDF).

📊 Tables

Every table of the paper, in its order: click one to unfold it, with where the paper cites it, how its numbers were checked, and its data as CSV. The typeset table is one click away, on its page of the PDF.

Table without a number, in the text of §1 Columns since 1984, and a question of rings

question	answer
What does a column generate? (§2)	the integers of $\mathbb{Q}(\zeta_m)$, fixed by the charge under the centre, tensored with the order C_T of the spectrum; the defect is a power of p. Lemma 1, Proposition 2

What is that order at p? (§3)	a monomial ring with one numerical semigroup S_T, written by Gauss sums; the index is its number of gaps. Theorem 3
Can the semigroup be read in $\mathbb{F}_p$? (§3)	yes, from the nonzero power sums of the spectrum, when $p > (n-1)(n-2)$ and for every p when $n \leq 12$. Theorem 4
What else does the semigroup say? (§4)	Gorenstein = symmetric, exponent p when the conductor is small, the vacuum is maximal, level–rank keeps the semigroup; the conformal weight sees M_2 and no more. Corollaries 6–8, Propositions 9 and 10
How many columns are maximal? (§5)	a closed count over the stabilisers: a maximal column needs a multiplicative symmetry of its spectrum, which becomes rare. Proposition 11
Where does Gorenstein first fail? (§6)	at prime height, for $n = 5$: at $\mathrm{SU}(5)_{12}$, $\mu = (0,0,2,1)$, semigroup $\langle 3,4,5\rangle$. Table 1

Colour: green = yes / holds.

⬇ CSV

Table 1. Small n. For $n = 3, 4$ the index of a column is a function of the stabiliser and of one or two moments, for every p. For $n \geq 5$ the largest number of gaps, $(n-1)(n-2)/2$, appears only for larger primes (Proposition 12).

n	$\delta(T)$ **by the moments (with** $h = \lvert H_T \rvert$**)**	**semigroups that occur**
3	0 if $h > 1$; 1 if $h = 1$	$\mathbb{N}_0$, $\langle 2,3\rangle$
4	0 if $h > 1$; 1 if $h = 1$, $M_2 \neq 0$; 3 if $h = 1$, $M_2 = 0$	$\mathbb{N}_0$, $\langle 2,3\rangle$, $\langle 3,4\rangle$
5, 6	≤ 4 for $p \leq 31$; up to $(n-1)(n-2)/2$ for large p	first non-symmetric: $\langle 3,4,5\rangle$, $p = 17$

Colour: blue shade = the number's size within its column.

p. 8 in the PDF ⬇ CSV cited in §1 Columns since 1984, and a question of rings

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