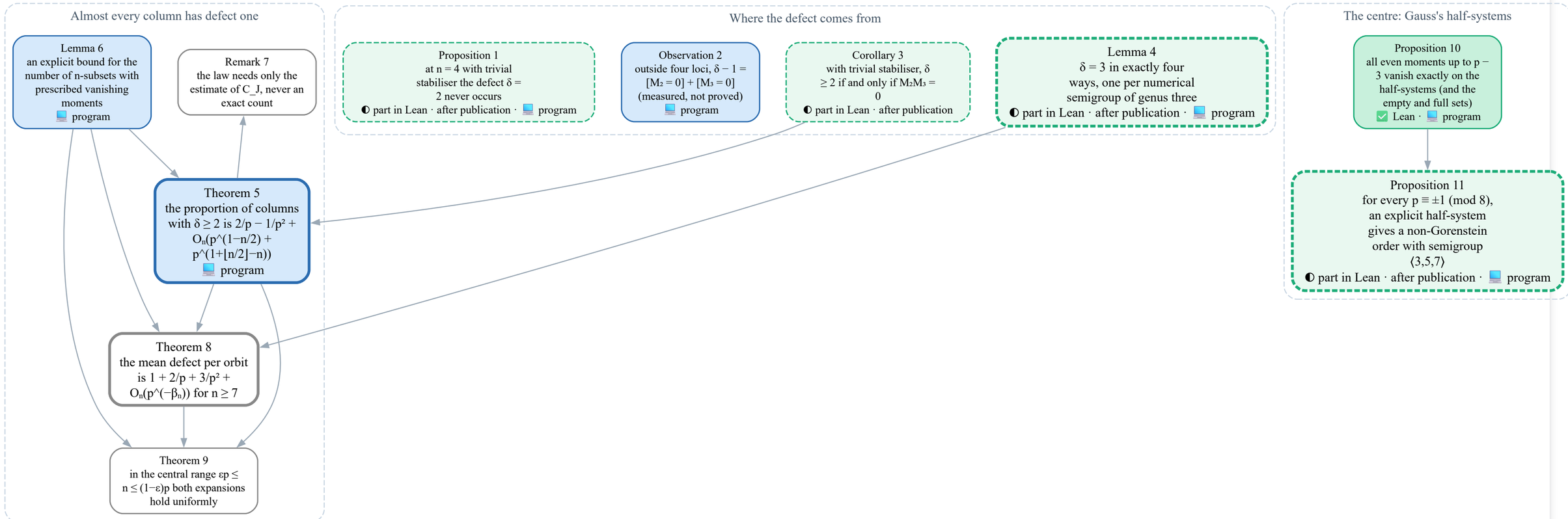
Blueprint

# Map of *The order generated by a column of the $SU(n)_k$ S-matrix is locally a numerical semigroup ring (Part II)*

Algebraic number theory · orders and discriminants of fusion rings

**11** results | **1** entirely in Lean | **4** partly in Lean | **7** checked by a program | written proof only | (a result can carry several tags)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



## What the paper contributes

Part I turned one column of the $SU(n)_k$ S-matrix into an order whose defect $\delta(T)$ is the number of gaps of a numerical semigroup $S_T$. This note asks how the columns fit together: the p-adic valuation of the discriminant of the fusion ring splits into ramification, defects and a gluing term whose only unknown piece is $\Delta(n,p) = \Sigma\, \delta$ over orbits. **That splitting is verified computationally, not proved**, as the paper says.

- **Small defects by vanishing moments** (Proposition 1, Corollary 3, Lemma 4): with trivial stabiliser, $\delta \geq 2 \Leftrightarrow M_2M_3 = 0$, and $\delta = 3$ happens in exactly four ways, one per numerical semigroup of genus three; $\delta = 2$ never occurs at $n = 4$. Observation 2 refines this and is measured.
- **The distribution law** (Theorem 5, Lemma 6, Theorem 8): for fixed $n \geq 5$ the proportion of columns with $\delta \geq 2$ is $2/p - 1/p^2$ up to an explicit error, with no factorial threshold; the mean defect per orbit is $1 + 2/p + 3/p^2$ for $n \geq 7$.
- **Uniformity in the central range** (Theorem 9, proof sketched): both expansions hold for $\varepsilon p \leq n \leq (1-\varepsilon)p$.
- **The two ends** (Propositions 10–11, §4): totally split trinomials $X^n + aX + b$ at the top (counted at $n = 5$ by an elliptic curve of conductor 50, measured); Gauss's half-systems in the middle, whose even moments vanish exactly and which give an infinite family of non-Gorenstein orders with semigroup $\langle 3,5,7 \rangle$.

🚀 **Start here: Corollary 3**: with trivial stabiliser, $\delta \geq 2$ if and only if $M_2M_3 = 0$.

**Main results:**

- **Lemma 4**: $\delta = 3$ in exactly four ways, one per numerical semigroup of genus three
- **Theorem 5**: the proportion of columns with $\delta \geq 2$ is $2/p - 1/p^2 + O_n(p^{(1-n/2)} + p^{(1+\lfloor n/2 \rfloor - n)})$
- **Theorem 8**: the mean defect per orbit is $1 + 2/p + 3/p^2 + O_n(p^{(-\beta_n)})$ for $n \geq 7$
- **Proposition 11**: for every $p \equiv \pm 1 \pmod 8$, an explicit half-system gives a non-Gorenstein order with semigroup $\langle 3,5,7 \rangle$

## 🧭 Fields: where the paper lives and what it connects

**Tools taken from other branches**

- **Character sums over finite fields** → additive orthogonality, the partition sieve of Li and Wan, and Weil's bound, with the constant made explicit — Lemma 6, Theorem 5, Theorem 9
- **Numerical semigroups** → the four semigroups of genus three, the gap bound $g_i \leq 2i - 1$, Kunz's Gorenstein criterion — Lemma 4, Theorem 8, Proposition 11

- **Symmetric functions** → Newton's recurrence for power sums, and the cascade of vanishing moments it forces — Proposition 1, Corollary 3
- **Elementary number theory** → Gauss's half-systems, Vandermonde systems on the quadratic residues, Faulhaber sums of odd powers — Proposition 10, Proposition 11

**What the results serve**

- **Discriminants of WZW fusion rings** ← an asymptotic expansion of $\Delta(n,p)$, the one term of the gluing $G(n,p)$ without a closed form (the splitting itself is measured) — Theorem 8, Theorem 9
- **Gorenstein orders** ← a natural infinite family of non-Gorenstein orders, generated by character values, with a value semigroup that does not move with $p$ — Proposition 11
- **Subsets of $F_p$ with prescribed moments** ← the counts $C_J(n,p)$, estimated here with an explicit constant, evaluated exactly in the companion series — Lemma 6