Quadratic persistence of Segre varieties and a linear gap between biquadratic SOS rank and Zarankiewicz bounds

Real algebraic geometry · sums of squares on projective varieties

Carles Marín

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At a glance

What the paper contributes

The quadratic persistence of the Segre varieties $\mathbb{P}^{m-1} \times \mathbb{P}^2$ and $\mathbb{P}^{m-1} \times \mathbb{P}^3$, exactly, and a block bound in every width. Through the Pythagoras-number bound of Blekherman, Sinn, Smith and Velasco this shows that the sum-of-squares rank $\mathrm{BSR}(m,n)$ of biquadratic forms is not captured by the graph parameters z_2, z_{2A} used to bound it: the difference grows linearly in m in every width $n \geq 4$.

- $\mathrm{qp}(\mathbb{P}^{m-1} \times \mathbb{P}^2) = m$ and $\mathrm{qp}(\mathbb{P}^{m-1} \times \mathbb{P}^3) = \lceil 4m/3 \rceil$ (Theorems 1 and 2); $\mathrm{qp} \leq n\lceil m/3 \rceil$ (Theorem 3).
- $\mathrm{BSR}(m,4) \geq \lfloor 8m/3 \rfloor$, above $z_2(m,4) = z_{2A}(m,4)$ for $m = 21$ and $m \geq 23$ (Corollary 4); a gap of order $\frac{n-3}{6}m$ in width n (Corollary 5).
- Rational forms whose rational length exceeds their real length by exactly three (Theorem 6).
- Exact persistence in widths five and six for $m \leq 18$, $m \leq 14$ (Proposition 7), and $8 \leq \mathrm{BSR}(4,3) \leq 9$ (Theorem 9).

🚀 **Start here: Theorem 1**: qp of $P^{m-1} \times P^2$ is m.

Main results:

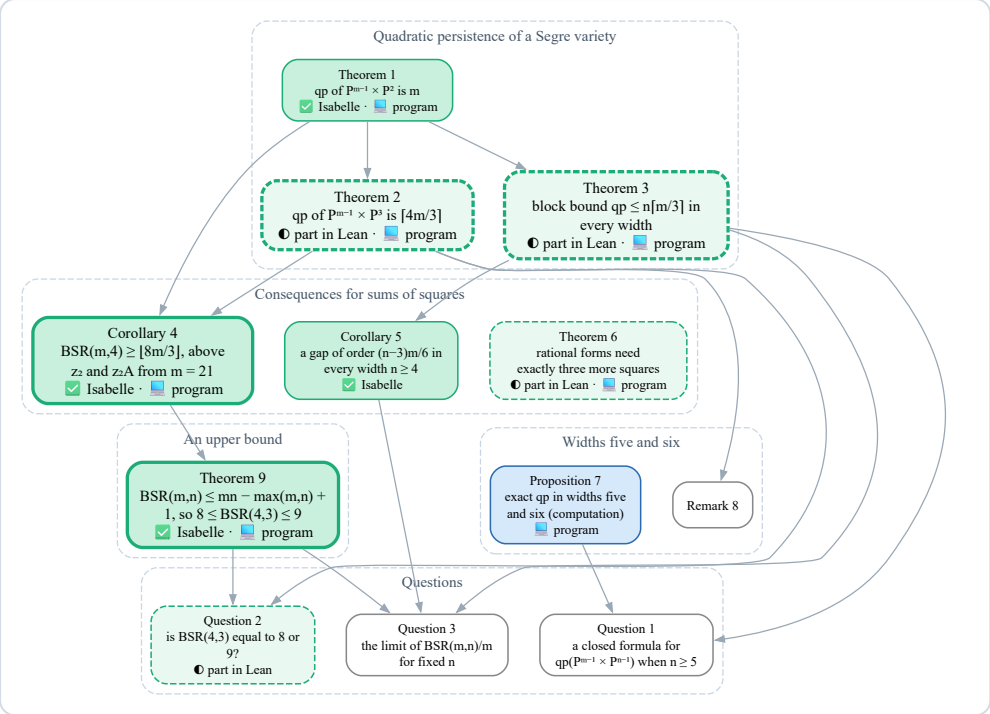
- **Theorem 2**: qp of $P^{m-1} \times P^3$ is $\lceil 4m/3 \rceil$
- **Theorem 3**: block bound $\mathrm{qp} \leq n\lceil m/3 \rceil$ in every width
- **Corollary 4**: $\mathrm{BSR}(m,4) \geq \lfloor 8m/3 \rfloor$, above z_2 and z_{2A} from $m = 21$
- **Theorem 9**: $\mathrm{BSR}(m,n) \leq mn - \max(m,n) + 1$, so $8 \leq \mathrm{BSR}(4,3) \leq 9$

12 results | ✅ **0** entirely in Lean | ◐ **4** partly in Lean | ✅ **4** in Isabelle | 💻 **7** checked by a program | written proof only

(a result can carry several tags)

🗺️ Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's “Evidence & scope” block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



📐 Setting (copied from the paper, §2)

Let $A = \mathbb{R}^m$, $B = \mathbb{R}^n$ and $X = \mathbb{P}(A) \times \mathbb{P}(B)$ in $\mathbb{P}(A \otimes B)$. Let $\mathcal{D}$ be the space of symmetric bilinear forms D on $A \otimes B$ with $D(z, z) = 0$ for every simple tensor z. It is spanned by the 2×2 minors, and identifies with $\bigwedge^2 A^* \otimes \bigwedge^2 B^*$: we write $D(a, a'; b, b')$, alternating in (a, a') and in (b, b'), and $d = \dim \mathcal{D} = \binom{m}{2}\binom{n}{2}$. A point $a \otimes b$ lies in the radical of D if and only if $D(a, x; b, y) = 0$ for all $x \in A$, $y \in B$. … $\mathrm{qp}(X)$ is the least k such that some k points of X lie in the radical of no nonzero $D \in \mathcal{D}$, and general points realize it. We use the bound

$$d \le g(s) := sc - \binom{s}{2}, \qquad s = \mathrm{qp}(X),\ c = (m-1)(n-1).$$

(§1) A biquadratic form $f(x, y)$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, is a sum of squares if $f = \sum_{k \le r} b_k^2$ with b_k bilinear; let $\ell_K(f)$ be the least such r with b_k over a field K. For $m, n \ge 2$ the invariant $\mathrm{BSR}(m, n)$ is the maximum, over real sum-of-squares biquadratic forms of size $m \times n$, of $\ell_{\mathbb{R}}(f)$.

1 Introduction

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper (PDF).

2 Quadratic persistence of a Segre variety

Theorem 1. Isabelle p. 2

> $\mathrm{qp}(\mathbb{P}^{m-1} \times \mathbb{P}^2) = m$ *for* $m \geq 2$.

Isabelle · ours **show**

`segre_width3_upper` · `Isabelle/SegrePersistence/Segre_Width3.thy`, line 295 — the upper bound in coordinates: with b_i = (1, i, i²), antisymmetric blocks D_ij killing b_i and b_j all vanish, over any field of characteristic 0

```
theorem segre_width3_upper:
  fixes D :: "nat \<Rightarrow> nat \<Rightarrow> nat \
<Rightarrow> nat \<Rightarrow> 'a::field_char_0"
  assumes anti: "\<And>i j. i < m \<Longrightarrow> j < m \
<Longrightarrow> i \<noteq> j \<Longrightarrow> antisym3 (D
i j)"
    and swap: "\<And>i j k l. i < m \<Longrightarrow> j < m
\<Longrightarrow> i \<noteq> j \<Longrightarrow> D j i k l
= - D i j k l"
    and rad: "\<And>i j. i < m \<Longrightarrow> j < m \
<Longrightarrow> i \<noteq> j \<Longrightarrow> kills3 (D i
j) (bvec i)"
    and ij: "i < m" "j < m" "i \<noteq> j"
  shows "zero3 (D i j)"
```

`antisym3_two_kernel_zero` · `Isabelle/SegrePersistence/Segre_Width3.thy`, line 123 — an antisymmetric 3 × 3 matrix killing two independent vectors is zero

```
lemma antisym3_two_kernel_zero:
  fixes S :: "nat \<Rightarrow> nat \<Rightarrow>
'a::field_char_0"
  assumes A: "antisym3 S" and Ku: "kills3 S u" and Kv:
"kills3 S v" and I: "indep2 u v"
  shows "zero3 S"
```

`segre_width3_count` · `Isabelle/SegrePersistence/Segre_Width3.thy`, line 366 — the count d − g(m−1) = m − 1 of the lower bound; BSSV Corollary 3.7 itself is not formalized

```
lemma segre_width3_count:
  assumes "2 \<le> m"
  shows "3 * (m choose 2) - ((m - 1) * (2 * (m - 1)) - ((m
- 1) choose 2)) = m - 1"
```

Lean · none found **show**

Searched 2026-11-07 (excovery existe-ya: our Lean, Mathlib, LeanSearch, Isabelle/AFP with FindFacts down, GitHub, Rocq, HOL Light, Metamath): not found

🆕 **Classical and new:** the paper: the lower bound «was known» (BSSV Theorem 1.2 with Oeding–Raicu–Sam Theorem 4.1); «what is new is the equality».

↳ Used in: Theorem 2 (p. 5), Theorem 3 (p. 8), Corollary 4 (p. 10)

Proof

Take $b_i = (1, i, i^2)$ and the points $e_i \otimes b_i$. If they lie in the radical of D, then D_{ij} has b_i and b_j in its radical. These are independent, and a nonzero alternating form on $\mathbb{R}^3$ has a radical of dimension one, so $D_{ij} = 0$ and $D = 0$. Thus $\operatorname{qp} \leq m$. Here $c = 2(m-1)$ and $d = 3\binom{m}{2}$, and $d - g(m-1) = m - 1 > 0$, so (1) excludes $s \leq m - 1$. This lower bound was known: $\operatorname{qp}(X) \geq \ell(X)$ [1 , Theorem 1.2] , and the linear strand of a Segre product is nonzero up to $p = m + n - 3$ [6 , Theorem 4.1] , so $\operatorname{qp} \geq m + n - 3$; what is new is the equality.

💡 **In plain words (commentary, not part of the paper)**

Quadratic persistence asks how many points of a variety one has to pick so that no quadric through the variety is singular at all of them. For the Segre variety of $m \times 3$ matrices of rank one the answer is exactly m. Upper bound: put one point in each row, $e_i \otimes (1, i, i^2)$; a quadric singular at two such points restricts, for each pair of rows, to an alternating form on $\mathbb{R}^3$ with two independent vectors in its radical, hence zero. Lower bound: the dimension count of Blekherman, Sinn, Smith and Velasco excludes $m - 1$ points; it was already known, also from the linear strand of Oeding, Raicu and Sam, so what the note adds is the equality.

🛡️ **Evidence & scope**

- **Program:** the points e_i ⊗ (1,i,i²) checked to have full rank for the sizes of the original table. `src/supplement/recomprobar.py`

Theorem 2. 📕 p. 2

$$\operatorname{qp}(\mathbb{P}^{m-1} \times \mathbb{P}^3) = \lceil 4m/3 \rceil \text{ for } m \geq 2.$$

Lean · ours, in part show

upper bound: Descartes independence of the b_i and of the b_i ⊗ b_i, and the vanishing of the coefficients c_ij in one group and between two groups. Lower bound, for points in general position: the map M ≠ 0 with M a_i ∈ ⟨b_i⟩ (dimension count), M has rank 4, and the pulled-back Plücker form is a nonzero symmetric form vanishing on simple tensors with every a_i ⊗ b_i in its radical. Not formalized: the passage from radical conditions to the relations c_ij, that general points compute qp (BSSV Lemma 3.3) and the identification of these forms with $\mathcal{D}$

- `SegrePersistence.cross_vanish`
- `SegrePersistence.group_vanish`
- `SegrePersistence.sq_independent`
- `SegrePersistence.b_independent`

- `SegrePersistence.descartes_six`
- `SegrePersistence.signVariations_lt_card_support`
- `SegrePersistence.PluckerLower.lower_bound`
- `SegrePersistence.PluckerLower.exists_M`
- `SegrePersistence.PluckerLower.M_rank_eq_four`
- `SegrePersistence.PluckerLower.Q_simple`
- `SegrePersistence.PluckerLower.Q_radical`
- `SegrePersistence.PluckerLower.Q_ne_zero`

✅ Checked when this page was built: PluckerLower.lean, SegreBSR.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — cross_vanish, group_vanish, sq_independent, b_independent, descartes_six, signVariations_lt_card_support, lower_bound, exists_M, M_rank_eq_four, Q_simple, Q_radical, Q_ne_zero: propext, Classical.choice, Quot.sound.
Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean PluckerLower.lean SegreBSR.lean` (SHA-256 of the compiled file: `0268c8797b9dad4f`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🧮 computation (not a formal proof). Written proof; the rank certificates are a check, not part of the proof

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Theorem 1, p. 2, doi:10.5281/zenodo.24036320

🆕 **Classical and new:** the paper: «We have not found these values of qp or this comparison in the literature; our search covered the papers cited here and is not a proof of priority.»

↳ Used in: Corollary 4 (p. 10), 8 (p. 16), 2 (p. 18)

Proof

Upper bound. Write $B = E \oplus \langle w \rangle$ with $\dim E = 3$, split $\{1, \dots, m\}$ into consecutive groups of three, the last possibly shorter, and in a basis of E put $b_i = (1, i, i^3)$. Any three b_i are independent, and any six of the tensors $b_i \otimes b_i$ are independent: their coordinates are the monomials $1, t, t^3$, respectively $1, t, t^2, t^3, t^4, t^6$, at distinct positive nodes, and a nonzero combination of k monomials has at most $k-1$ positive roots by Descartes' rule. Take the $m + \lceil m/3 \rceil = \lceil 4m/3 \rceil$ integer points

$$e_i \otimes b_i \quad (1 \le i \le m), \qquad \Big(\sum_{j \in S} e_j\Big) \otimes w \quad \text{for each group } S.$$

Suppose they lie in the radical of D. As above, D_{ij} has b_i, b_j in its radical, so D_{ij} factors through the plane $B/\langle b_i, b_j \rangle$ and

$D_{ij} = c_{ij}\,\mathrm{vol}(b_i, b_j, \cdot, \cdot)$ for a fixed volume form, with $c_{ij} = c_{ji}$ and $c_{ii} = 0$. The point of the group S gives, for every i and y, $\sum_{j\in S} c_{ij}\,\mathrm{vol}(b_i, b_j, w, y) = 0$, that is,

$$\sum_{j\in S} c_{ij} b_j \in \langle b_i \rangle, \tag{1}$$

since b_i and the sum lie in E and $w \notin E$. Inside a group, (2) and the independence of its vectors give $c_{ij} = 0$. For two groups $U \neq V$ put $T = \sum_{i\in U, j\in V} c_{ij}\, b_i \otimes b_j$. By (2) for $i \in U$ and $S = V$, $T = \sum_{i\in U} \lambda_i\, b_i \otimes b_i$; by (2) for $j \in V$ and $S = U$, with $c_{ij} = c_{ji}$, $T = \sum_{j\in V} \mu_j\, b_j \otimes b_j$. The at most six tensors $b_k \otimes b_k$, $k \in U \cup V$, are independent, so $T = 0$, and since the $b_i \otimes b_j$ with $i \in U$, $j \in V$ are independent, all c_{ij} vanish. Hence $D = 0$.

Lower bound. For $m = 2, 3$ the claim is $\mathrm{qp} = 3, 4$, which follows from $\mathbb{P}^1 \times \mathbb{P}^3$, a rational normal scroll, so of minimal degree [1 , Theorem 3.8] , and Theorem 1 by symmetry. Let $m \geq 4$ and $k = \lceil 4m/3 \rceil - 1$, so $m<k$ and $3k<4m$. For general points $a_i \otimes b_i$, $i \leq k$, the conditions $Ma_i \in \langle b_i \rangle$ on $M\colon A \to B$ are at most $3k$ linear equations in $4m$ unknowns, so there is $M \neq 0$. If $\operatorname{rank} M = r<4$, at most $m - r$ of the general a_i lie in $\ker M$, and for the other indices $b_i \in \operatorname{im} M$, which allows at most r of the general b_i; so $k \leq m$, a contradiction. Thus M is onto, and $L(a \otimes b) = Ma \wedge b$ maps $A \otimes B$ onto $\bigwedge^2 B$, as decomposable vectors span it. Fix $\bigwedge^4 B \cong \mathbb{R}$ and put $Q(z, z') = L(z) \wedge L(z')$. It is nonzero, since the Plücker form on $\bigwedge^2 B$ is nondegenerate, it vanishes on simple tensors because $Ma \wedge b$ is decomposable, so $Q \in \mathcal{D}$, and $L(a_i \otimes b_i) = 0$ puts the k points in its radical. So k general points do not suffice. The bound $m + n - 3 = m + 1$ from [1 , 6] is weaker for $m \geq 4$, which is why a direct argument is needed.

In plain words (commentary, not part of the paper)

For $m \times 4$ the answer is $\lceil 4m/3 \rceil$: one point per row plus one more point for every group of three rows. The extra point of a group lies in the fourth direction w and forces linear relations between the coefficients c_{ij} that remain after the row points; inside a group they vanish at once, and between two groups the symmetric tensor $T = \sum c_{ij} b_i \otimes b_j$ would lie in two spaces of squares that meet only in 0, because any six of the tensors $b_i \otimes b_i$, with $b_i = (1, i, i^3)$, are independent (Descartes' rule of signs). The points are integers, so the bound needs no genericity. Fewer points never suffice: with $\lceil 4m/3 \rceil - 1$ general points there is a nonzero linear map M with

$Ma_i \in \langle b_i \rangle$; it must be onto, and the Plücker quadric pulled back through $a \otimes b \mapsto Ma \wedge b$ is singular at all the points.

Evidence & scope

- **Program:** the integer points of the proof: radical map of rank d modulo 1000003 for $2 \le m \le 30$, and rank below d without the last point. `src/supplement/comprobar_explicito.py`
- **Program:** with $\lceil 4m/3 \rceil - 1$ random points the rank never reaches d, $m \le 30$ (a control, not a proof of the lower bound). `src/supplement/check_width4_random.py`

Theorem 3. p. 3

> *For* $m, n \ge 3$, $\mathrm{qp}(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}) \le \min\{n\lceil m/3 \rceil,\ m\lceil n/3 \rceil\}$.

Lean · ours, in part show

the mixed component: propagation along the cyclic graph on Z/n kills every c_ijh from the vertex equations (mixto); the extraction of the coefficients of t^p over the n-th roots of unity; the internal 3 × 3 component. Not formalized: the reduction to blocks and the splitting of Λ²A*, the factorization M^ij = (a_i × a_j) ⊗ v_ij, the derivation of (mixto) from the radical conditions, and the Zariski-open passage to rational points

- `SegrePersistence.BlockBound.graph_propagation`
- `SegrePersistence.BlockBound.coeff_extraction`
- `SegrePersistence.BlockBound.internal_vanish`

✅ Checked when this page was built: BlockBound.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — graph_propagation, coeff_extraction, internal_vanish: propext, Classical.choice, Quot.sound.

Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean BlockBound.lean` (SHA-256 of the compiled file: `ca87e51bc88d679e`…; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box).
Lean: 3 results, standard axioms only

Proof: in this note · **Formalized:** Lean 4, by the author (public formalizations repository) · **Covers:** part: the mixed component: propagation along the cyclic graph on Z/n kills every c_ijh from the vertex equations (mixto); the extraction of the coefficients of t^p over the n-th roots of unity; the internal 3 × 3 component. Not formalized: the reduction to blocks and the splitting of Λ²A*, the factorization M^ij = (a_i × a_j) ⊗ v_ij, the derivation of (mixto) from the radical conditions, and the Zariski-open passage to rational points · **Cite:** Theorem 3, p. 3, doi:10.5281/zenodo.24036320

↳ Used in: Corollary 5 (p. 12), 1 (p. 18), 3 (p. 19)

Proof

> By symmetry it suffices to prove $\mathrm{qp} \le sn$ for $m = 3s$; for other m, project $\mathbb{R}^{3\lceil m/3 \rceil}$ onto A: if π is onto, $\pi(a) \otimes b$ is in the

radical of D exactly when $a \otimes b$ is in the radical of $\pi^* D$, and π^* is injective. Split $A = U_1 \oplus \cdots \oplus U_s$ with $\dim U_l = 3$ and put n points in each block, $a_t^{(l)} \otimes b_t^{(l)}$ with $a_t^{(l)} \in U_l$. Since $\bigwedge^2 A^* = \bigoplus_l \bigwedge^2 U_l^* \oplus \bigoplus_{l<l'} U_l^* \otimes U_{l'}^*$, and the radical conditions of the points of U_l only involve the components containing U_l^*, the map $D \mapsto$(radical conditions) is injective as soon as it is injective on each internal component $\bigwedge^2 U_l^* \otimes \bigwedge^2 B^*$, using the points of U_l, and on each mixed component $U_l^* \otimes U_{l'}^* \otimes \bigwedge^2 B^*$, using the points of U_l and $U_{l'}$. Each of these is a nonempty Zariski open condition once it holds for one choice of points, so it suffices to exhibit one choice per component.

Internal. Take points $a_t \otimes b_t$ with $b_1, \dots, b_n$ a basis of B and any three a_t independent. In the dual basis write $D = \sum_{t<u} D^{tu} \otimes (b_t^* \wedge b_u^*)$ with $D^{tu} \in \bigwedge^2 U^*$. The point $a_t \otimes b_t$ forces $D^{tu}(a_t, \cdot) = 0$ for all u, so D^{tu} has a_t, a_u in its radical and vanishes, as in Theorem 1.

Mixed. Let U, V be two blocks. In U take $a_i \otimes e_i$, $i \in \mathbb{Z}/n$, for a basis e_i of B and any three a_i independent; in V, over $\mathbb{C}$, take $c_t \otimes f_t$ with $c_t = (1, t, t^2)$ and $f_t = \sum_i t^i e_i$, for the n-th roots of unity t. Write the component as $\sum_{i<j} M^{ij} \otimes (e_i^* \wedge e_j^*)$ with M^{ij} bilinear on $U \times V$ and $M^{ji} = -M^{ij}$. The points of U give $M^{ij}(a_i, \cdot) = M^{ij}(a_j, \cdot) = 0$, so $M^{ij} = (a_i \times a_j) \otimes v_{ij}$, where $a_i \times a_j$ is the covector $x \mapsto \det(a_i, a_j, x)$ and $v_{ij} = v_{ji} \in V^*$. Write $v_{ij}(1, t, t^2) = \sum_{h=0}^{2} c_{ijh} t^h$. The points of V give, for each j and each t, $\sum_{i \neq j} \sum_h c_{ijh} t^{i+h} (a_i \times a_j) = 0$; as t runs over the n-th roots of unity, the coefficient of each power t^p modulo $t^n - 1$ vanishes:

$$\sum_{h=0}^{2} c_{p-h,j,h}\,(a_{p-h} \times a_j) = 0 \qquad (j, p \in \mathbb{Z}/n,\ p - h \neq j). \tag{2}$$

The unknown $c_{ijh} = c_{jih}$ appears only in the equations with $i + j + h = \lambda$ fixed, so the system splits by λ. For fixed λ consider the graph on $\mathbb{Z}/n$ with an edge $\{i, \lambda - h - i\}$ for each $h \in \{0, 1, 2\}$ and each i, loops removed; the edge carries $c_{i,\lambda-h-i,h}$, and (3) at the vertex j involves exactly its incident edges, with pairwise independent covectors $a_k \times a_j$. So at a vertex where at most two unknowns are not yet known to vanish, all of them vanish. The reflections $i \mapsto \lambda - h - i$ for $h = 0, 1$ compose to $i \mapsto i + 1$, so the graph is connected, and $2i \equiv \lambda - h$ has a solution for some $h \in \{0, 1\}$, which gives a vertex with a removed loop and at most two edges. Starting

> there and moving along edges, every vertex has at most two unknowns left, so all c_{ijh} vanish. The relevant minors are therefore nonzero polynomials, and general rational points satisfy all the finitely many open conditions at once.

💡 **In plain words (commentary, not part of the paper)**

In any width $n \geq 3$, cut the m rows into blocks of three and put n points in each block: at most $n\lceil m/3 \rceil$ points. The quadrics split into pieces living on one block or on a pair of blocks, and the points of a block only see the pieces that contain it, so it is enough to kill each piece with one good choice of points. Inside a block the alternating argument of Theorem 1 works. For a pair of blocks, a second block with points $(1, t, t^2) \otimes (1, t, \ldots, t^{n-1})$ at the n -th roots of unity turns the conditions into a system that splits by $i + j + h$; each piece is a connected cycle-like graph with a vertex of degree two, and the unknowns vanish one after another along it. This is only an upper bound for qp, but it is enough for a gap that grows linearly in m (Corollary 5).

🛡️ **Evidence & scope**

- **Program:** the specialization of the proof (roots of unity in F_p, p ≡ 1 mod n) for m = 6 and 3 ≤ n ≤ 8: full rank, and rank below d without one point. `src/supplement/comprobar_bloques.py`
- **Program:** random points with n⌈m/3⌉ points give full rank in 8 sizes up to (15,5). `src/supplement/check_blocks_random.py`

3 Consequences for sums of squares

Corollary 4. ✅ Isabelle 📕 p. 4

> $\mathrm{BSR}(m, 3) \geq 2m$ *and* $\mathrm{BSR}(m, 4) \geq \lfloor 8m/3 \rfloor$. *For* $m \geq 15$,
>
> $$\mathrm{BSR}(m, 4) - z_2(m, 4) \;\geq\; \left\lfloor \frac{8m}{3} \right\rfloor - \left\lfloor \frac{5m + 6}{2} \right\rfloor \;\geq\; \frac{m - 22}{6}.$$
>
> *The last expression is positive for* $m \geq 23$*, and the middle one equals* $56 - 55 = 1$ *at* $m = 21$ *and* 0 *at* $m = 22$*. The same bounds hold with* z_{2A} *in place of* z_2*.*

Isabelle · ours show

`gap_pos` · `Isabelle/SegrePersistence/Zarankiewicz_Cells.thy`, line 114 — the arithmetic: ⌊8m/3⌋ > ⌊(5m+6)/2⌋ for m ≥ 23

```
theorem gap_pos: "23 \<le> m \<Longrightarrow> (5 * m + 6)
div 2 < 8 * m div 3"
  for m :: nat
```

gap_21 · Isabelle/SegrePersistence/Zarankiewicz_Cells.thy, line 126 — strict at m = 21 (56 > 55); gap_22: tie at m = 22

```
lemma gap_21: "(5 * 21 + 6) div 2 < 8 * 21 div (3::nat)"
```

gap_linear · Isabelle/SegrePersistence/Zarankiewicz_Cells.thy, line 138 — 6(⌊8m/3⌋ − ⌊(5m+6)/2⌋) ≥ m − 22

```
theorem gap_linear:
  fixes m :: nat
  shows "int m - 22 \<le> 6 * (int (8 * m div 3) - int ((5
* m + 6) div 2))"
```

cell_count · Isabelle/SegrePersistence/Zarankiewicz_Cells.thy, line 172 — the cell count (4m + z(m,4))/2 with z(m,4) = m + 6; Chen–Chen and the cell bound of Qi–Cui are cited, not formalized

```
theorem cell_count: "(4 * m + (m + 6)) div 2 = (5 * m + 6)
div (2::nat)"
```

Lean · none found **show**

Searched 2026-11-07 (excovery existe-ya: our Lean, Mathlib, LeanSearch, Isabelle/AFP with FindFacts down, GitHub, Rocq, HOL Light, Metamath): not found

🆕 **Classical and new:** the paper: BSR and z_2 «already differ» at 4×4 (Qi–Cui); «what is new here is a gap that grows linearly».

↳ Used in: Theorem 9 (p. 16)

Proof

The first two bounds are [1 , Theorem 1.3] with Theorems 1 and 2; the first was already known as $z_2(m,3) = 2m$ [5] . The difference uses [2 , Theorem 1.1] . For z_{2A}, for $m \geq 15$,

$$\left\lfloor \frac{5m+6}{2} \right\rfloor = z_2(m,4) \leq z_{2A}(m,4) \leq \left\lfloor \frac{4m+z(m,4)}{2} \right\rfloor = \left\lfloor$$

by [2 , Theorem 1.1] , the cell bound [7 , Proposition 2.5] and $z(m,4) = m+6$ for $m \geq 6$.

💡 **In plain words (commentary, not part of the paper)**

$\mathrm{BSR}(m,n)$ is the largest number of squares a sum of squares of bilinear forms can need. Blekherman, Sinn, Smith and Velasco turn persistence into a lower bound: $\mathrm{BSR} \geq mn - \mathrm{qp}$. With Theorem 2 this gives $\mathrm{BSR}(m,4) \geq \lfloor 8m/3 \rfloor$. The groups around Qi bound BSR from below by graph parameters, the second order Zarankiewicz number z_2 and its augmented version z_{2A}; in width four Chen and Chen found $z_2(m,4) = \lfloor (5m+6)/2 \rfloor$, and a counting argument of Qi and Cui caps $z_{2A}(m,4)$ at the same value. Slope $8/3$ beats slope $5/2$: from $m = 21$ on (with a tie at $m = 22$) the persistence bound is larger, by at least $(m-22)/6$. That BSR and z_2 differ somewhere was known from 4×4 (Qi and Cui); what is new is a gap that grows.

Evidence & scope

- **Program:** the two language tabs: qp(24,4) = 32, BSR(24,4) ≥ 64, z_2(24,4) = 63, cell bound 63, first crossing 21 and tie at 22; with controls that must fail. `lab/comprueba_pestanas.py`
- **Literature:** z_2(m,4) = ⌊(5m+6)/2⌋ for m ≥ 15 (Chen–Chen Thm 1.1) and the cell bound (Qi–Cui Prop. 2.5), read in full. `arXiv:2609.11093v2, arXiv:2610.04448v1`

Read this theorem in another mathematical language

Read this theorem in:

Commentary, not part of the paper. Take $m = 24$. Theorem 2 gives $\mathrm{qp}(\mathbb{P}^{23} \times \mathbb{P}^3) = \lceil 96/3 \rceil = 32$, so some 24×4 biquadratic sum of squares needs at least $96 - 32 = 64$ squares of real bilinear forms.

Commentary, not part of the paper. The graph parameters stop at 63 for $m = 24$: Chen and Chen give $z_2(24,4) = \lfloor (5 \cdot 24 + 6)/2 \rfloor = 63$, and the cell bound caps $z_{2A}(24,4)$ at $(96 + z(24,4))/2 = (96+30)/2 = 63$. The first m where slope $8/3$ wins is 21; at 22 they tie.

Corollary 5. Isabelle p. 4

> *For every fixed $n \geq 4$, $\mathrm{BSR}(m,n) - z_{2A}(m,n) \geq \frac{n-3}{6}\, m - O_n(1)$, and the same holds for z_2.*

Isabelle · ours **show**

`c4_free_card_le` · `Isabelle/SegrePersistence/Zarankiewicz_Cells.thy`, line 95 — a C4-free support of {0..<m} × {0..<n} has at most m + n(n−1)/2 cells

```
theorem c4_free_card_le:
  assumes sub: "E \<subseteq> {0..<m} \<times> {0..<n}" and
C: "c4_free E"
  shows "card E \<le> m + n * (n - 1) div 2"
```

`sum_choose_two_le` · `Isabelle/SegrePersistence/Zarankiewicz_Cells.thy`, line 44 — the count Σ_i C(d_i, 2) ≤ C(n, 2) behind it

```
lemma sum_choose_two_le:
  assumes sub: "E \<subseteq> {0..<m} \<times> {0..<n}" and
C: "c4_free E"
  shows "(\<Sum>i\<in>{0..<m}. card (row E i) choose 2) \
<le> n choose 2"
```

Lean · none found **show**

Searched 2026-11-07 (excovery existe-ya: our Lean, Mathlib, LeanSearch, Isabelle/AFP with FindFacts down, GitHub, Rocq, HOL Light, Metamath): not found

↳ Used in: 3 (p. 19)

Proof

By Theorem 3, $\mathrm{BSR}(m,n) \geq mn - n\lceil m/3 \rceil \geq mn - \frac{n(m+2)}{3}$. On the other side, $z_2 \leq z_{2A} \leq \frac{mn+z(m,n)}{2}$ by the cell bound, and a C_4-free bipartite graph with row degrees d_i has $\sum_i \binom{d_i}{2} \leq \binom{n}{2}$, so $z(m,n) = \sum_i d_i \leq m + \binom{n}{2}$. Subtracting gives $\frac{n-3}{6}m - \frac{2n}{3} - \frac{1}{2}\binom{n}{2}$.

In plain words (commentary, not part of the paper)

The same comparison in every fixed width $n \geq 4$, now with the block bound of Theorem 3: $\mathrm{BSR}(m,n) \geq mn - n\lceil m/3 \rceil$, while every graph parameter of the z_2 family is at most $(mn + m + \binom{n}{2})/2$, because a C_4-free bipartite graph has at most $m + \binom{n}{2}$ edges. The difference grows like $\frac{n-3}{6}m$. In width three the two slopes coincide, which is why the gap starts in width four.

Theorem 6. Isabelle p. 4

Let $k = \mathrm{qp}(X)$ and $e = mn - k$. There is a rational biquadratic form F with $\ell_{\mathbb{R}}(F) = e$ and $\ell_{\mathbb{Q}}(F) = e + 3$.

Lean · ours, in part **show**

the uniqueness step: two PSD Gram matrices differing by an element of $\mathcal{D}$, vanishing at points where the radical map is injective, coincide; the Gram matrix of Σ c_j u_j² (c_j > 0, u_j killing the points) is PSD, kills the points and has rank e. Uses Mathlib Matrix.PosSemidef.dotProduct_mulVec_zero_iff. Not formalized: that $\mathcal{D}$ is the space of Segre relations and the existence of the points

- `SegrePersistence.GramUnique.gramOf_unique`
- `SegrePersistence.GramUnique.eq_of_posSemidef`
- `SegrePersistence.GramUnique.mulVec_eq_zero`
- `SegrePersistence.GramUnique.gramOf_rank`
- `SegrePersistence.GramUnique.gramOf_posSemidef`

Isabelle · ours **show**

`seven_mod_8_rat` · `Isabelle/RationalGap/Rational_Gap.thy`, line 240 — the arithmetic step: a rational c ≡ 7 (mod 8), in particular 7, is not a sum of three rational squares

```
lemma seven_mod_8_rat:
  fixes x y z :: rat and c :: int
  assumes c: "c mod 8 = 7"
  shows "x^2 + y^2 + z^2 \<noteq> of_int c"
```

`seven_embedding` · `Isabelle/BSRRational/Seven_Embedding.thy`, line 366 — the embedding step: e − 1 orthonormal vectors and one of norm 7 in Q^r force r ≥ e + 3 (the role of Witt cancellation in the proof); the uniqueness of the Gram matrix is in Lean (GramUnique)

```
theorem seven_embedding:
  fixes v :: "nat \<Rightarrow> nat \<Rightarrow> rat" and
w :: "nat \<Rightarrow> rat" and e r :: nat
  assumes e: "1 \<le> e"
    and n1: "\<And>i. i < e - 1 \<Longrightarrow> dot r (v
i) (v i) = 1"
    and o1: "\<And>i j. i < e - 1 \<Longrightarrow> j < e -
1 \<Longrightarrow> i \<noteq> j \<Longrightarrow> dot r (v
i) (v j) = 0"
    and o2: "\<And>i. i < e - 1 \<Longrightarrow> dot r (v
i) w = 0"
    and nw: "dot r w w = 7"
  shows "e + 3 \<le> r"
```

✅ Checked when this page was built: GramUnique.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — gramOf_unique, eq_of_posSemidef, mulVec_eq_zero, gramOf_rank, gramOf_posSemidef: propext, Classical.choice, Quot.sound. Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean GramUnique.lean` (SHA-256 of the compiled file: `213aff5781bc109c…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🔒 kernel only — **only part of the statement is formal** (see the box). Lean: 5 results, standard axioms only

Proof: in this note · **Formalized:** Isabelle/HOL, by the author (public formalizations repository) · **Covers:** the whole statement · **Cite:** Corollary 4, p. 4, doi:10.5281/zenodo.24036320

↳ Not used by another result of the paper.

Proof

There are k independent rational simple tensors p_i with $D \mapsto (Dp_1, \dots, Dp_k)$ injective on $\mathcal{D}$, since this is a nonempty Zariski open condition and rational points are dense. The face of the forms vanishing at these points is the mechanism of [1 , Lemma 3.18] . Let $u_1, \dots, u_e$ be a rational basis of the

annihilator of the p_i, read as bilinear forms, and $F = u_1^2 + \cdots + u_{e-1}^2 + 7u_e^2$. A positive semidefinite Gram matrix G of F has $p_i^\top G p_i = F(p_i) = 0$, hence $Gp_i = 0$; the difference of two such lies in $\mathcal{D}$ and kills all p_i, so the positive semidefinite Gram matrix is unique, of rank e. Its nondegenerate part is $(e-1)\langle 1\rangle \perp \langle 7\rangle$ over $\mathbb{Q}$, so a representation by r rational squares is an embedding of this form in $r\langle 1\rangle$, and by Witt cancellation it exists exactly when 7 is a sum of $r - e + 1$ rational squares. Since 7 is a sum of four rational squares and not of three, $\ell_{\mathbb{Q}}(F) = e + 3$.

In plain words (commentary, not part of the paper)

Over the rationals the count changes. Take the rational bilinear forms that vanish at qp well-chosen points; every positive semidefinite Gram matrix of a sum of their squares must kill those points, and this pins the Gram matrix down uniquely, the face mechanism of Blekherman, Sinn, Smith and Velasco. Now weight one square by 7. Over $\mathbb{R}$ the form still needs $e = mn - \text{qp}$ squares; over $\mathbb{Q}$ a representation is an embedding of $(e-1)\langle 1\rangle \perp \langle 7\rangle$ into a sum of unit squares, and by Witt cancellation it needs 7 as a sum of rational squares, which takes four, not three. So exactly three more squares.

Evidence & scope

- **Program:** the case 3×3 exactly over Q: the radical map of the three points e_i ⊗ (1,i,i²) has rank 9 = dim $\mathcal{D}$ (unique Gram matrix; rank 7 with two points, a control), the Gram matrix of F has rank 6, and F is written explicitly as 9 rational squares; the lower bound 9 is the theory. `src/supplement/certificado_thm6.py`

4 Widths five and six

Proposition 7. p. 5

> $\text{qp}(\mathbb{P}^{m-1} \times \mathbb{P}^4) = s_0(m,5)$ *for* $5 \le m \le 18$ *and* $\text{qp}(\mathbb{P}^{m-1} \times \mathbb{P}^5) = s_0(m,6)$ *for* $6 \le m \le 14$*, with the values below.*

Lean · none found **show**

Searched 2026-11-07 (excovery existe-ya: our Lean, Mathlib, LeanSearch, Isabelle/AFP with FindFacts down, GitHub, Rocq, HOL Light, Metamath): not found

↳ Used in: 1 (p. 18)

Proof

> For each size, s_0 integer points give a radical map of rank d modulo 1000003 and modulo 1000033; a nonzero maximal minor modulo a prime is nonzero over $\mathbb{Q}$, so $\mathrm{qp} \leq s_0$. The points are in the file `qp_anchura56_cert.json`.

💡 **In plain words (commentary, not part of the paper)**

In widths five and six the note computes persistence exactly in a range: $5 \leq m \leq 18$ for $n = 5$ and $6 \leq m \leq 14$ for $n = 6$. In every case it is the smallest value the dimension count allows, s_0. The upper bound is a certificate: s_0 integer points whose radical map has full rank modulo two primes (the points are published with the site). The lower bound is the dimension count itself. In width four this pattern fails, which is why the note states these values as computations and the general formula only as a remark: if it held for all m, $\mathrm{BSR}(m, 5)/m$ would tend to at least $1 + \sqrt{6} \approx 3.45$.

🛡️ **Evidence & scope**

- **Program:** 23 sizes: s0 integer points, full rank modulo two primes; the points are published. `src/supplement/qp_anchura56_cert.py`

Remark 8. p. 5

> If $\mathrm{qp}(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}) = s_0(m, n)$ for all m, as these computations suggest for $n = 5, 6$, then $\mathrm{BSR}(m, n)/m$ is at least $1 + \sqrt{(n-1)(n-2)/2}$ asymptotically, that is $1 + \sqrt{6} \approx 3.45$ for $n = 5$ and $1 + \sqrt{10} \approx 4.16$ for $n = 6$, against $(n+1)/2$ for the cell bound. For $n = 4$ the same expression gives $1 + \sqrt{3}$, above the slope $8/3$ that Theorem 2 actually gives.

↳ Not used by another result of the paper.

5 An upper bound

Theorem 9. Isabelle p. 5

> *For $m, n \geq 2$, $\mathrm{BSR}(m, n) \leq mn - \max(m, n) + 1$. In particular $2m \leq \mathrm{BSR}(m, 3) \leq 2m + 1$ for $m \geq 3$ and $8 \leq \mathrm{BSR}(4, 3) \leq 9$.*

Isabelle · ours **show**

`bsr_upper` · `Isabelle/SegrePersistence/BSR_Upper.thy`, line 299 — BSR(m,n) ≤ mn − max(m,n) + 1 by successive orthogonal projections, with BSSV Theorem 1.4 (the m × 2 case) as an explicit hypothesis

```
theorem bsr_upper:
  assumes BSSV_1_4: "\<And>p. 2 \<le> p \<Longrightarrow>
bsr_le p 2 (p + 1)"
    and m: "2 \<le> m" and n: "2 \<le> n"
  shows "bsr_le m n (m * n - max m n + 1)"
```

`bsr_upper_4_3` · `Isabelle/SegrePersistence/BSR_Upper.thy`, line 330 — BSR(4,3) ≤ 9 (same hypothesis)

```
corollary bsr_upper_4_3:
  assumes "\<And>p. 2 \<le> p \<Longrightarrow> bsr_le p 2
(p + 1)"
  shows "bsr_le 4 3 9"
```

`bsr_upper_width_three` · `Isabelle/SegrePersistence/BSR_Upper.thy`, line 321 — BSR(m,3) ≤ 2m + 1 (same hypothesis)

```
corollary bsr_upper_width_three:
  assumes "\<And>p. 2 \<le> p \<Longrightarrow> bsr_le p 2
(p + 1)" and "3 \<le> m"
  shows "bsr_le m 3 (2 * m + 1)"
```

Lean · none found **show**

Searched 2026-11-07 (excovery existe-ya: our Lean, Mathlib, LeanSearch, Isabelle/AFP with FindFacts down, GitHub, Rocq, HOL Light, Metamath): not found

🆕 **Classical and new:** the paper: «This is the bound of [BSSV, Corollary 2.15] for the product of simplices»; for n ≥ 5 BSSV Theorem 2.1 is better.

↳ Used in: 2 (p. 18), 3 (p. 19)

Proof

> Let $m \geq n$ and $f = \sum_k b_k^2$; let $v_{ij} \in \mathbb{R}^r$ be the coefficient vector of $x_i y_j$ in $(b_1, \ldots, b_r)$. The vectors v_{ij} with $j \geq 3$ span a space W of dimension at most $m(n-2)$. After an orthogonal change of the r coordinates, W lies in the first $\dim W$ of them, and the remaining coordinates give bilinear forms in x and y_1, y_2 only. Their sum of squares is a form of size $m \times 2$, hence a sum of $m+1$ squares [1 , Theorem 1.4] . So f is a sum of $m(n-2)+m+1$ squares. For $m<n$ exchange the roles of x and y. The lower bound for $n = 3$ is Corollary 4.

💡 **In plain words (commentary, not part of the paper)**

From above: keep two columns y_1, y_2. The coefficient vectors of the other monomials span at most $m(n-2)$ directions; rotate them into the first coordinates, and what is left involves only y_1, y_2, a form of size $m \times 2$, which is a sum of $m+1$ squares. So $mn - \max(m,n) + 1$ squares always suffice. This is a covering bound of Blekherman, Sinn, Smith and Velasco (Corollary 2.15) read for the product of simplices; the note gives the direct proof and the consequence $8 \leq \mathrm{BSR}(4,3) \leq 9$, against the bound 11 used before. For $n \geq 5$ another bound of the same authors (Theorem 2.1) is better, and the note says so.

Evidence & scope

- **Program:** comparison with BSSV Theorem 2.1 for $2 \leq n \leq m \leq 20$: Theorem 2.1 strictly better exactly when $n \geq 5$ (136 pairs). `src/supplement/cota21.py`

6 The companion site

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper, p. 5.

Questions

Question 1. p. 6

Is there a closed formula for $\mathrm{qp}(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1})$ for $n \geq 5$?

↳ Not used by another result of the paper.

Question 2. Isabelle p. 6

Is $\mathrm{BSR}(4,3)$ equal to 8 or 9? Han, Jung and Park observe that all known cases satisfy equality in $\mathrm{py} \geq N + 1 - \mathrm{qp}$ [4 , p. 3] ; for $\mathbb{P}^{m-1} \times \mathbb{P}^2$ this would give $\mathrm{BSR}(m,3) = 2m$.

Lean · ours, in part show

not in the note (added 7-nov-2026): above the length-eight form Q of Xu–Cui–Qi, F_t = Q + t(x₂y₁ + x₃y₂ + x₄y₃)² is a sum of eight squares for t ≥ 1/2 although its nine-square Gram matrix is extreme of rank nine (the extremality is a computation, not formalized)

- `SegrePersistence.XCQFamily.eight_squares`

- `SegrePersistence.XCQFamily.seven_squares_half`
- `SegrePersistence.XCQFamily.scaled_identity`

Isabelle · ours **show**

`bsr_zero_4_3` · `Isabelle/SegrePersistence/BSR_Zero.thy`, line 297 — not in the note (added 7-nov-2026): a 4 × 3 sum of squares with a real zero is a sum of at most 8 squares (with BSSV Theorem 1.4 as an explicit hypothesis), so a form needing 9 must be strictly positive

```
corollary bsr_zero_4_3:
  assumes BSSV_1_4: "\<And>p. 2 \<le> p \<Longrightarrow>
bsr_le p 2 (p + 1)"
    and x0: "\<exists>i<4. x0 i \<noteq> 0" and y0: "\
<exists>j<3. y0 j \<noteq> 0"
    and Z: "sos_form 4 3 r c x0 y0 = 0"
  shows "\<exists>s\<le>8. \<exists>c'. \<forall>x y.
sos_form 4 3 r c x y = sos_form 4 3 s c' x y"
```

`bsr_zero_filter` · `Isabelle/SegrePersistence/BSR_Zero.thy`, line 240 — the general step: a real zero saves one square, BSR ≤ (m − 1) + BSR(m, n − 1)

```
theorem bsr_zero_filter:
  assumes n: "1 \<le> n" and H: "bsr_le m (n - 1) N"
    and x0: "\<exists>i<m. x0 i \<noteq> 0" and y0: "\
<exists>j<n. y0 j \<noteq> 0"
    and Z: "sos_form m n r c x0 y0 = 0"
  shows "\<exists>s\<le>(m - 1) + N. \<exists>c'. \
<forall>x y. sos_form m n r c x y = sos_form m n s c' x y"
```

✅ Checked when this page was built: XCQFamily.lean compiles with Lean 4.35.0, no errors, no sorry. Axioms — eight_squares, seven_squares_half, scaled_identity: propext, Classical.choice, Quot.sound.

Reproduce: Lean 4.35.0, Mathlib f0b4c8150aa0; in that Lean project, `lake env lean XCQFamily.lean` (SHA-256 of the compiled file: `0919be16c7e33328…`; a source page on this site may show it with comments edited, under its own hash).

Trust base: 🧮 computation (not a formal proof). Rank modulo two primes of integer points (a nonzero maximal minor mod p is nonzero over Q); lower bound by BSSV Cor. 3.7

Proof: in this note · **Covers:** not formalized · **Cite:** Proposition 7, p. 5, doi:10.5281/zenodo.24036320

↳ Not used by another result of the paper.

Question 3. p. 6

For fixed n, does $\mathrm{BSR}(m,n)/m$ have a limit, and what is it? Asymptotically it lies between $2n/3$ and $\min\{n-1, \sqrt{n(n+1)/2}\}$.

Cite: Question 3, p. 6, doi:10.5281/zenodo.24036320

↳ Not used by another result of the paper.

Data and acknowledgement

No numbered statement in this section.

It is prose only (no statement, figure or table): read it in the paper (PDF).

Tables

Every table of the paper, in its order: click one to unfold it, with where the paper cites it, how its numbers were checked, and its data as CSV. The typeset table is one click away, on its page of the PDF.

Table without a number, in the text of §4 Widths five and six

$m, n = 5$	5	6	7	8	9	10	11	12	13	14	15	16	17	18
qp	8	10	12	13	15	16	18	19	21	22	24	25	27	29
BSR $\geq$	17	20	23	27	30	34	37	41	44	48	51	55	58	61

⬇ CSV

Table without a number, in the text of §4 Widths five and six

$m, n = 6$	6	7	8	9	10	11	12	13	14
qp	12	14	15	17	19	21	23	25	26
BSR $\geq$	24	28	33	37	41	45	49	53	58

⬇ CSV

References

1

G. Blekherman, R. Sinn, G. G. Smith, M. Velasco, Sums of squares and quadratic persistence on real projective varieties, J. Eur. Math. Soc. 24 (2022), 925–965; statement numbers as in arXiv:1902.02754v2.

2

Z. Chen, Y. Chen, Recursive-line Zarankiewicz numbers with four columns, arXiv:2609.11093v2.

3

C. Cui, L. Qi, Y. Xu, The sum of squares rank of biquadratic forms and the Zarankiewicz number, arXiv:2602.07844v2.

4

J. I. Han, J. Jung, E. Park, On quadratic persistence and Pythagoras numbers of totally real projective varieties, arXiv:2506.13247v1.

5

J. Löfberg, L. Qi, Second order Zarankiewicz number, arXiv:2608.30555v3.

6

L. Oeding, C. Raicu, S. V. Sam, On the (non-)vanishing of syzygies of Segre embeddings, Algebr. Geom. 6 (2019), 571–591. 🔎 find it

7

L. Qi, C. Cui, The second-order augmented Zarankiewicz number, arXiv:2610.04448v1.

8

Y. Xu, C. Cui, L. Qi, Narrowing the gap: SOS ranks of 4×3 biquadratic forms and a lower bound of 8, arXiv:2602.21570.