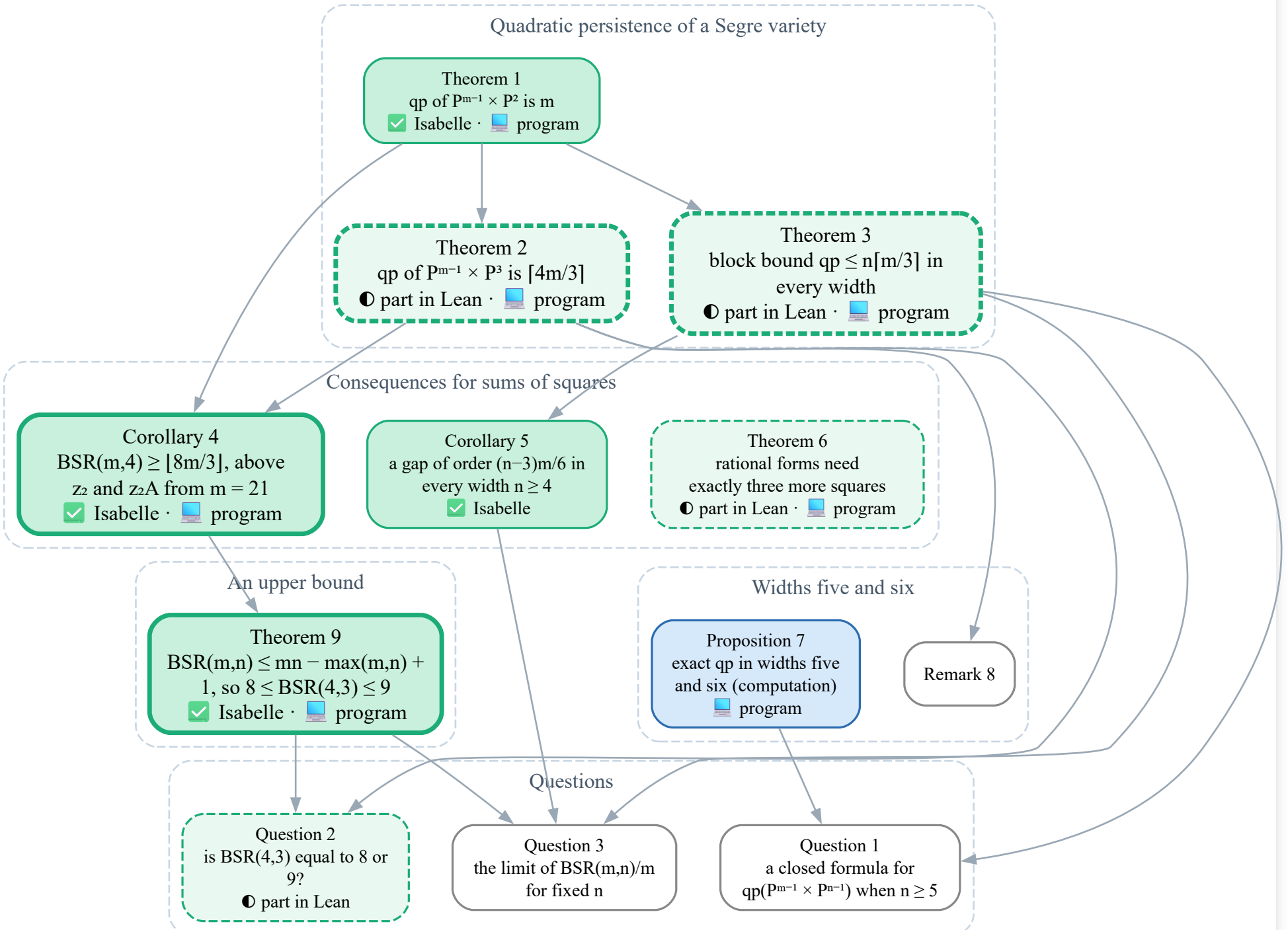
Map of *Quadratic persistence of Segre varieties and a linear gap between biquadratic SOS rank and Zarankiewicz bounds*

Real algebraic geometry · sums of squares on projective varieties

12 results | **0** entirely in Lean | **4** partly in Lean | **4** in Isabelle | **7** checked by a program | written proof only

(a result can carry several tags)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



What the paper contributes

The quadratic persistence of the Segre varieties $\mathbb P^{m-1}\times\mathbb P^2$ and $\mathbb P^{m-1}\times\mathbb P^3$, exactly, and a block bound in every width. Through the Pythagoras-number bound of Blekherman, Sinn, Smith and Velasco this shows that the sum-of-squares rank $\mathrm{BSR}(m,n)$ of biquadratic forms is not captured by the graph parameters z_2, z_{2A} used to bound it: the difference grows linearly in m in every width $n\ge4$.

- $\mathrm{qp}(\mathbb P^{m-1}\times\mathbb P^2)=m$ and $\mathrm{qp}(\mathbb P^{m-1}\times\mathbb P^3)=\lceil4m/3\rceil$ (Theorems 1 and 2); $\mathrm{qp}\le n\lceil m/3\rceil$ (Theorem 3).
- $\mathrm{BSR}(m,4)\ge\lfloor8m/3\rfloor$, above $z_2(m,4)=z_{2A}(m,4)$ for $m=21$ and $m\ge23$ (Corollary 4); a gap of order $\frac{n-3}{6}m$ in width n (Corollary 5).
- Rational forms whose rational length exceeds their real length by exactly three (Theorem 6).
- Exact persistence in widths five and six for $m\le18$, $m\le14$ (Proposition 7), and $8\le\mathrm{BSR}(4,3)\le9$ (Theorem 9).

🚀 **Start here: Theorem 1**: qp of $P^{m-1} \times P^2$ is m.

Main results:

- **Theorem 2**: qp of $P^{m-1} \times P^3$ is $\lceil 4m/3 \rceil$
- **Theorem 3**: block bound $qp \le n\lceil m/3 \rceil$ in every width
- **Corollary 4**: $BSR(m,4) \ge \lfloor 8m/3 \rfloor$, above z_2 and z_2A from $m = 21$
- **Theorem 9**: $BSR(m,n) \le mn - \max(m,n) + 1$, so $8 \le BSR(4,3) \le 9$

🧭 Fields: where the paper lives and what it connects

Tools taken from other branches

- **Multilinear algebra** → quadrics on the Segre variety as $\Lambda^2A^* \otimes \Lambda^2B^*$: radicals of alternating forms, Plücker quadric, Descartes rule for independence — Theorem 1, Theorem 2, Theorem 3
- **Quadratic forms over Q** → Witt cancellation and the fact that 7 is a sum of four, not three, rational squares — Theorem 6
- **Exact computation** → ranks of the radical map modulo primes with integer points, as certificates — Proposition 7, Theorem 2

What the results serve

- **SOS rank of biquadratic forms and Zarankiewicz numbers** ← a linear gap between BSR and z_2, z_2A in every width $n \geq 4$, and $8 \leq \mathrm{BSR}(4,3) \leq 9$ — Corollary 4, Corollary 5, Theorem 9