

Exact zeros of weighted sums of completely multiplicative functions



Number theory · arithmetic functions

11A25

11K65

11L40

11P70

11R29

Carles Marín

Statements

Theorem A	3	Corollary 4.3, first statement — Propagation of	13
Theorem B	6	zeros along digits (first statement of Corollary	
Corollary C	6	4.3)	
Lemma 2.1	8	Corollary 4.3	14
Lemma 2.2	9	Proposition 4.4	16
Lemma 2.2(b) — Complete sequences (Lemma	10	Corollary 4.5	17
2.2 (b), the part formalized)		Proposition 4.6	17
Proposition 3.1	11	Proposition 5.1	18
Theorem 4.1	12	Corollary 5.2	19
Theorem 4.2 — Kanemitsu–Kuzumaki, Kučera ..	12	Theorem N1 (new) — No zeros at $q = 179$ and	19
		$q = 227$	

At a glance

What the paper contributes

For a completely multiplicative ε with values ± 1 , the weighted sum $S_\varepsilon(m) = \sum_{n \leq m} n \varepsilon(n)$ can vanish only when $m \equiv 0$ or $3 \pmod{4}$.

- **Every such length is a zero** of some ε , depending on m (Theorem A), even with the signs at finitely many primes prescribed in advance, once m is large (Proposition 3.1).
- **Zeros of one fixed function are rare in measure and generic in category**: the ε with infinitely many zeros form a dense G_δ of probability zero (Theorem B). For the unweighted sums the picture is the reverse: they vanish infinitely often almost surely, by a theorem of Aymone.
- **Quadratic characters give explicit zeros**. For primes $q \equiv 7 \pmod{8}$, $S_\chi((q^j-1)/2) = 0$ for every j , and Conrey's sine series of the Legendre symbol vanishes identically on an interval (Corollary C). The case $j = 1$ is classical (Lebesgue; Aebi–Cairns); the iteration in j and the null interval are the corollary's content. They come from a digit recursion in base q (Theorem 4.1) and the weighted half sum (Theorem 4.2, due to Kanemitsu–Kuzumaki and Kučera).
- **Added after submission**: for $q = 179$ and $q = 227$ the function $\chi_{q,1}$ has no zero at all (Theorem N1), by a finite certificate checked in Lean.

Main tools: a second-moment bound and Sárközy's progressions in subset sums of primes (Theorem A); the anticoncentration theorem of Sárközy and Szemerédi (Theorem B).

 **Start here: Theorem A**: every admissible length is a zero of some ε .

Main results:

- **Theorem A**: every admissible length is a zero of some ε
- **Theorem B**: infinitely many zeros: dense G_δ , but measure zero
- **Corollary C**: zeros of χ at $(q^j-1)/2$; Conrey's series vanishes on an interval
- **Theorem N1 (new)**: no zeros at all for $q = 179$ and $q = 227$

17 results

✓ 4 formalized in Lean

🔗 4 with a part in Lean

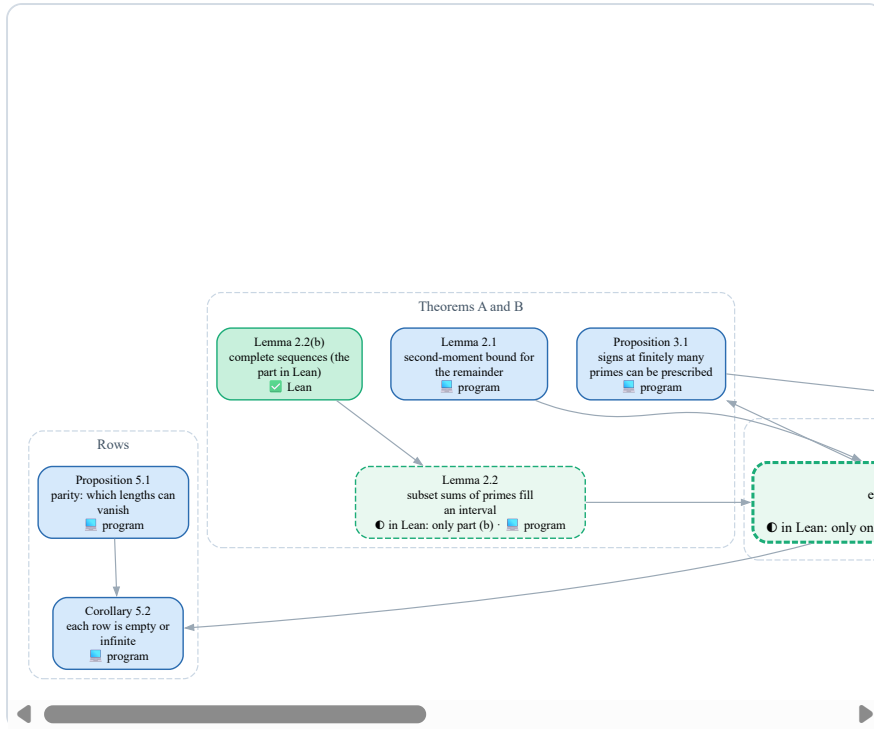
💻 11 checked by a program

written proof only

(a result can carry several tags)

Map of the results (click a box to go to it)

Dependencies in the written proof: an arrow $A \rightarrow B$ means that this proof of B uses A . Every result rests on its written proof; the tags record what was checked besides, and each statement's “Evidence & scope” block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



Setting (copied from the paper)

For a completely multiplicative $\varepsilon : \mathbb{N} \rightarrow \{\pm 1\}$, $S_\varepsilon(m) = \sum_{n \leq m} n \varepsilon(n)$. Changing one sign changes $S_\varepsilon(m)$ by an even amount, so $S_\varepsilon(m) \equiv m(m+1)/2 \pmod{2}$, and a zero is possible only when $m \equiv 0$ or $3 \pmod{4}$; call these lengths *admissible*. A prime $q \in (m/2, m]$ divides no $n \leq m$ other than q itself, so its sign enters $S_\varepsilon(m)$ once, with weight q . For a set Q of such primes, $S_\varepsilon(m) = R_Q + \sum_{q \in Q} q \varepsilon(q)$, where R_Q depends only on the signs outside Q ; $P_Q = \sum_{q \in Q} q$. For an odd prime q and $s = \pm 1$ let $\chi_{q,s}$ be the completely multiplicative function equal to the Legendre symbol $(\cdot | q)$ on the integers prime to q , with $\chi_{q,s}(q) = s$. For $\chi = \chi_{q,s}$, $C(r) = \sum_{r' \leq r} \chi(r')$, $T(r) = \sum_{r' \leq r} r' \chi(r')$ ($0 \leq r < q$), $T_q = T(q-1)$, and $R_q = \{0 \leq r < q : T_q + q C(r) = 0 = T(r)\}$.

1 Statements

Theorem A. p. 2

For every $m \geq 3$ with $m \equiv 0$ or $3 \pmod{4}$ there is a completely multiplicative $\varepsilon : \mathbb{N} \rightarrow \{\pm 1\}$ with

$$\sum_{n \leq m} n \varepsilon(n) = 0.$$

NEW Classical and new: new. With the weight n the picture of the unweighted sums (Aymone) is reversed. The proof uses Sárközy's progressions in subset sums (Lemma 2.2(c), a cited theorem).

↳ Used in: [Proposition 3.1](#) (p. 11), [Corollary 5.2](#) (p. 19)

Proof

Table 1 summarises the three ranges.

Up to 20000. For each admissible $m \leq 20000$ a function ε with $S_\varepsilon(m) = 0$ is exhibited and the sum is recomputed exactly: for all but four lengths by starting from $\varepsilon(2) = \varepsilon(3) = -1$ (with at most two signs at primes ≤ 7 changed), balancing the primes of $(\sqrt{m}, m/2]$ and solving (2) exactly with the primes of $(m/2, m]$; the lengths 12, 15, 19, 23 by exhaustive search.

From 5000 to 10^8 . Cut $[5000, 10^8)$ into 104 blocks $[a_j, b_j]$,

$$a_1 = 5000, \quad b_j = \min(\lfloor 11a_j/10 \rfloor, 10^8 - 1), \quad a_{j+1} = b_j + 1,$$

and let Q be the primes of $(b_j/2, a_j]$, which lie in $(m/2, m]$ for every m in the block; the other primes of $(m/2, m]$ are among the random signs of Lemma 2.1. Each block carries a certificate: a prime $q_0 \in Q$ and disjoint pairs in Q as in Lemma 2.2(b), with $2H - q_0 + 2 \geq b$. Every other prime of Q is at most $a \leq b$, so Lemma 2.2(a) extends the interval to $[\alpha, P_Q - \alpha]$ with $\alpha = U + q_0 - 1$. The block is proved for all its m once $D = P_Q - 2\alpha$ exceeds $B_*(b) \geq B_*(m)$, which follows from the integer inequality $90D^2 > b^3(130 + 7e)$ with $e = \lceil \log_2 b \rceil$ (as $\log b < 7e/10$). All 104 certificates pass; the bound $b^3(130 + 7e)/(90D^2)$ on $B_*(b)^2/D^2$ is at most 0.445, reached at the first block.

From 10^8 on. Let $M \geq 10^8$, $L = \log M$, Q the N primes of $(M/2, M]$. By Dusart's bounds $x/(\log x - 1) \leq \pi(x)$ for $x \geq 5393$ and $\pi(x) \leq x/(\log x - 1.1)$ for $x \geq 60184$ [Dus10, Thm. 6.9, (6.6)], $N \geq M/(2L)$ and $P_Q \geq NM/2 \geq M^2/(4L)$. Let $X \subset Q$ have $n = \lfloor N/8 \rfloor \geq M/(17L)$ elements and take $\ell = 4M$: the hypothesis of Lemma 2.2(c) follows from $n^2 \geq 48M \log_2(4M/n)$, which holds once $M \geq 13872L^2 \log_2(68L)$, that is from $M \approx 4.4 \cdot 10^7$. This gives a progression $a + js$, $0 \leq j < \ell$, of step $s \leq 4M/n \leq 68L$ whose terms are sums of distinct elements of X , so $a \leq 24M^2/n$. Add a set Y of $s - 1$ further primes of Q . (i) As $s \leq 68L < M/2$, every prime of Q is a unit modulo s . (ii) The subset sums of $s - 1$ units meet every class modulo s : adding a unit u to a set $A \ni 0$ of classes gives $A \cup (A + u)$, which is strictly larger unless $A + u = A$, and then A is invariant under the generator u of $\mathbb{Z}/s\mathbb{Z}$, so $A = \mathbb{Z}/s\mathbb{Z}$; starting from $\{0\}$, $s - 1$ units reach all s classes. (iii) So for each class c there is $y_c \in \Sigma(Y)$, $y_c \equiv c$, with $0 \leq y_c \leq B := (s - 1)M$; an integer $x \in [a + B, a + (\ell - 1)s]$ is y_c plus the term $x - y_c$ of the progression, with $c \equiv x - a$, and $\Sigma(X \cup Y)$ contains the whole interval $[\alpha, \alpha + w] = [a + B, a + (\ell - 1)s]$, with $\alpha \leq 28M^2/n$ and $w = (\ell - 1)s - (s - 1)M \geq M$. The remaining primes, each at most M , extend it by Lemma 2.2(a); they are $N - n - s + 1 \geq \frac{7}{8}N - s$ primes above $M/2$, while $X \cup Y$ weighs at most $(\frac{1}{8}N + s)M$. (iv) As $N \geq M/(2L)$ and $s \leq 68L$, $5N > 24s$, which is $(\frac{7}{8}N - s)\frac{M}{2} > (\frac{1}{8}N + s)M$: the remaining primes carry more than half of P_Q , and the symmetry closes the interval to $[\alpha, P_Q - \alpha]$. With $K = 2\alpha \leq 56M^2/n$ and Lemma 2.1,

$$\frac{B_*(M) + K}{P_Q} \leq \frac{4}{3} \frac{L\sqrt{L+13}}{\sqrt{M}} + \frac{3808 L^2}{M} < 0.027 \quad (M \geq 10^8),$$

the right side being 0.02669 at $M = 10^8$ and both terms decreasing in M . So $|R_Q| < P_Q - K$ and a zero exists.

💡 In plain words (commentary, not part of the paper)

Changing one sign changes $S_\varepsilon(m) = \sum_{n \leq m} n \varepsilon(n)$ by an even amount, so a zero is only possible when $m \equiv 0$ or $3 \pmod{4}$ (the *admissible* lengths). The theorem says parity is the only obstruction. The key observation: a prime $q \in (m/2, m]$ divides no other $n \leq m$, so its sign enters the sum once, with weight q . Choose the other signs so that the rest of the sum is small (Lemma 2.1), and show that the subset sums of these large primes fill a long interval (Lemma 2.2); then the large primes can cancel the rest exactly. The function depends on m ($S_\varepsilon(4) - S_\varepsilon(3) = 4$). The proof runs in three overlapping ranges (Table 1 of the paper): explicit functions up to 20000, 104 block certificates on $[5000, 10^8)$, and Sárközy's progressions from 10^8 on.

🔍 Evidence & scope

Scope at a glance (each row is one range; details below).

Range of m	How it is established	Status
$3 \leq m \leq 20000$	an explicit ε for each of the 10000 admissible lengths, the sum recomputed exactly	proved by exact computation, rechecked by an independent program
$5000 \leq m < 10^8$	104 block certificates applying Lemmas 2.1 and 2.2(a,b)	certificates checked exactly by program; the lemmas: written proofs, (b) also in Lean
$m \geq 10^8$	Lemmas 2.1 and 2.2(c) (a cited theorem) and Dusart's prime bounds	written proof; its inequality is evaluated on a grid, which illustrates but does not prove

- **Not formalized in Lean** as a whole. One ingredient, Lemma 2.2(b), is, in a slightly more general form (see Lemma 2.2(b)).
- **Range $3 \leq m \leq 20000$, checked by program `II_ruta_A_exacta.py`** (supplement): an explicit ε with $S_\varepsilon(m) = 0$ for each of the 10000 admissible lengths, the sum recomputed; 9996 by the route of the proof, $m = 12, 15, 19, 23$ by exhaustive search. Output `II_ruta_A_exacta_OUT.txt`; witnesses stored in `II_testigos_A.txt`.
- **Range $5000 \leq m < 10^8$, checked by program `II_certificados_A.py`**: 104 blocks, 2128 pairs, maximum of B_*^2/D^2 bound 0.4449. Output `II_certificados_A_OUT.txt`.
- **Independent checker `II_comprobador.py`** (standard library only, imports none of the other programs, reads only the stored files): 10000 witnesses recomputed exactly; 104 blocks contiguous from 5000 to 99999999; every hypothesis of Lemma 2.2(a,b) and $90D^2 > b^3(130 + 7e)$ in integers; maximum 0.444939. Verdict PASS, output `II_comprobador_OUT.txt`.

- **Range $m \geq 10^8$: proof on paper**, resting on a cited theorem (Lemma 2.2(c)) and Dusart’s prime bounds. Its numerical inequalities are evaluated by `II_cola_analitica.py`: $(B_* + K)/P_Q$ bound 0.026689 at $M = 10^8$; on a grid of 641 points from 10^8 to 10^{40} the hypothesis holds and the bounds decrease. Output `II_cola_analitica_OUT.txt`. This is a grid evaluation, not a proof for every M ; the paper’s argument for all M is monotonicity.

Theorem B.

On $\Omega = \{\pm 1\}^{\mathcal{P}}$, the sign patterns on the primes, let $\mathcal{Z}_\infty$ be the set of ε with $S_\varepsilon(m) = 0$ for infinitely many m . With the product topology, $\mathcal{Z}_\infty$ is a dense G_δ . With independent fair signs, $\Pr(\mathcal{Z}_\infty) = 0$.

 **Classical and new:** new; the measure half uses the anticoncentration theorem of Sárközy and Szemerédi.

↳ Not used by another result of the paper.

Proof

The set U_M of ε with a zero at some admissible $m \geq M$ is open, each condition involving finitely many signs, and dense by Proposition 3.1; $\mathcal{Z}_\infty = \bigcap_M U_M$ is a dense G_δ by the Baire category theorem, the space of all ε being compact and metrizable. For the measure, condition on every sign outside the primes Q of $(m/2, m]$: by (2), $S_\varepsilon(m) = R_Q + \sum_{q \in Q} q \varepsilon(q)$ with distinct weights, and the theorem of Sárközy and Szemerédi [SS65], [Ngu12, Thm. 1.1] bounds the probability of any single value by $O(|Q|^{-3/2}) = O((\log m/m)^{3/2})$. The series converges, and the first Borel–Cantelli lemma gives $\Pr(\mathcal{Z}_\infty) = 0$.

In plain words (commentary, not part of the paper)

Here $\Omega = \{\pm 1\}^{\mathcal{P}}$ is the set of sign patterns on the primes (each pattern is one completely multiplicative ε), and $\mathcal{Z}_\infty$ the patterns whose sum $S_\varepsilon(m)$ vanishes for infinitely many m . For a fixed function, measure and category disagree. Topologically, zeros are generic: a zero at some large m can be reached from any finite prefix of signs (Proposition 3.1), so “has a zero beyond M ” is open and dense, and Baire gives a dense G_δ . In probability, zeros are rare: once all other signs are fixed, $S_\varepsilon(m)$ is a fixed number plus $\sum_q q \varepsilon(q)$ over the primes of $(m/2, m]$, with distinct weights, and Sárközy–Szemerédi bound the chance of hitting any single value by $O((\log m/m)^{3/2})$, which is summable: Borel–Cantelli gives probability 0. Without the weight, Aymone’s theorem gives zeros infinitely often almost surely.

Evidence & scope

- **Proof on paper only.** No program checks the measure or category statement, and it is not formalized.
- The density step uses Proposition 3.1; see its guarantees.

Corollary C.

Let $q \equiv 7 \pmod{8}$ be prime and $\chi = \chi_{q,s}$ with $s = \pm 1$. Then $S_\chi(\frac{q^j-1}{2}) = 0$ for every $j \geq 1$, and Conrey’s function $f_q(x) = \sum_{n \geq 1} (\frac{n}{q}) \sin(2\pi nx)/n^2$ vanishes identically on the interval $[\frac{q-1}{2q}, \frac{q+1}{2q}]$.

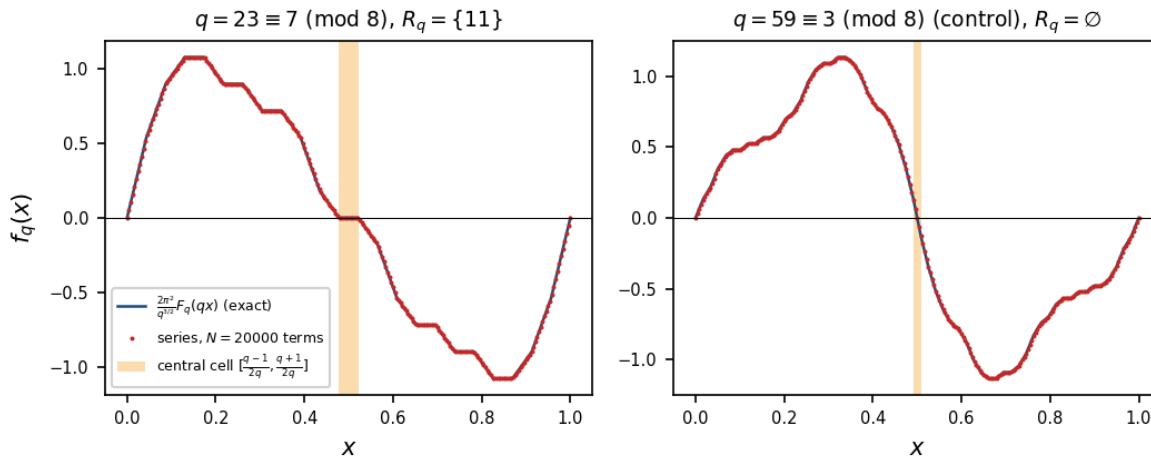
NEW Classical and new: the case $j = 1$ is classical (Lebesgue; Aebi–Cairns): the equality of the sums of quadratic residues and non-residues below $q/2$. New: the iteration in j and the null interval of Conrey's series.

↳ Not used by another result of the paper.

Proof

The zeros $S_\chi((q^j - 1)/2) = 0$ are Corollary 4.3(b). By the same corollary $(q - 1)/2 \in R_q$, so F_q vanishes identically on the cell $[(q - 1)/2, (q + 1)/2]$, and f_q on $[\frac{q-1}{2q}, \frac{q+1}{2q}]$.

 **Figure**



Conrey's sine series f_q for $q = 23 \equiv 7 \pmod{8}$, where it vanishes identically on the central cell, and for the control $q = 59 \equiv 3 \pmod{8}$, where it does not. Exact curve from F_q and the series truncated at 20000 terms (figuras/fig_conrey_celda.py).

In plain words (commentary, not part of the paper)

For an odd prime q and $s = \pm 1$, $\chi_{q,s}$ is the completely multiplicative function equal to the Legendre symbol $(\cdot | q)$ on integers prime to q , with $\chi_{q,s}(q) = s$. In base q , $(q^j - 1)/2$ is the number with j digits all equal to $(q - 1)/2$. By the digit recursion (Theorem 4.1) the sum vanishes along any string of digits from the set R_q (Lemma 4.3, first statement), so it is enough that $(q - 1)/2 \in R_q$. That is where $q \equiv 7 \pmod{8}$ enters: then $\chi(2) = 1$, the Euler factor $\chi(2)^{-1} - 1$ of Theorem 4.2 is 0, and the weighted half sum $T(\frac{q-1}{2})$ vanishes; the class number formula, equation (3) of the paper, gives the other condition, $T_q + q C(\frac{q-1}{2}) = 0$. The second half is the same fact read through Conrey's formula $f_q(x) = \frac{2\pi^2}{q^{3/2}} F_q(qx)$ (Section 4.3 of the paper): F_q is linear on each cell $[r, r + 1]$ and vanishes on it exactly when $r \in R_q$. The case $j = 1$ is classical (Lebesgue; Aebi–Cairns).

Evidence & scope

Scope at a glance (three different checks, on three different sets of primes; details below).

What is checked	On which cases	What it shows
$(q - 1)/2 \in R_q$	the 78 primes $q \equiv 7 \pmod{8}$, $q \leq 2000$	finite, exact; for every q : written proof

		(Theorem 4.2 and the class number formula)
$S_\chi\left(\frac{q^j-1}{2}\right) = 0$	$j \leq 8$, both s , only the 7 primes $q = 7, 23, 31, 47, 71, 79, 103$	finite, exact; for every j : the propagation step, in Lean, from the membership above
$f_q = 0$ on the central cell	interior points for $q = 7, 23, 31, 47$; controls $q = 59, 83$	an illustration in floating point, not a proof; the statement: written proof (Conrey's Theorem 4)

- **Partly formalized in Lean:** the propagation step (digits in R_q give zeros) is `SegmentoII.zeros_from_digits`, which takes as hypotheses membership in R_q and the properties of χ it uses (complete multiplicativity, periodicity modulo q , $C(q-1) = 0$). The input $(q-1)/2 \in R_q$ (Theorem 4.2 and the class number formula) and the statement about f_q are **proof on paper only**; the latter rests on Conrey's Theorem 4, which the paper applies to primes $q \equiv 7 \pmod{8}$ because its proof uses only that the character is real, odd and primitive.
- **Checked by program** `II_ceros_R.py`: $(q-1)/2 \in R_q$ for all 78 primes $q \equiv 7 \pmod{8}$, $q \leq 2000$, 0 failures (`II_ceros_R_OUT.txt`).
- **Checked by** `segmento_II.py autotest`: $S_\chi\left(\frac{q^j-1}{2}\right) = 0$ for $j \leq 8$, both s , at $q = 7, 23, 31, 47, 71, 79, 103$ (`autotest_OUT.txt`).
- **Numerical illustration, not a proof:** `II_celdas_nulas_conrey.py` evaluates the truncated series at interior points of the central cell for $q = 7, 23, 31, 47$ (values below $1.4 \cdot 10^{-12}$ in absolute value, truncation scale $5 \cdot 10^{-7}$), with control cells at $q = 59, 83$ ($q \equiv 3 \pmod{8}$) where it does not vanish identically (values $\pm 6.5 \cdot 10^{-2}$ and $\pm 3.9 \cdot 10^{-2}$ off the midpoint of the cell) (`II_celdas_nulas_conrey_OUT.txt`).

2 Theorems A and B

Lemma 2.1. ■ p. 3

Give the primes outside Q independent uniform signs. Then

$$\mathbb{E} R_Q^2 \leq \frac{m^3(\log m + 13)}{9},$$

so some choice has $|R_Q| \leq B_*(m) := \frac{1}{3}m^{3/2}\sqrt{\log m + 13}$.

↳ Used in: [Theorem A](#) (p. 3)

Proof

$\mathbb{E} \varepsilon(n)\varepsilon(k)$ is 1 if nk is a square and 0 otherwise. Writing $n = da^2$ with d squarefree, $\mathbb{E} S_\varepsilon(m)^2 = \sum_d b_m(d)^2$ with $b_m(d) = d \sum_{j \leq \sqrt{m/d}} j^2$, and $\mathbb{E} R_Q^2$ is the same sum

without the terms $d = q \in Q$, whose class contains only $n = q$. With $J = \lfloor \sqrt{m/d} \rfloor$, $\sum_{j \leq J} j^2 \leq J^3/3 + J^2/2 + J/6$, and $dJ^2 \leq m$, $dJ \leq \sqrt{md} \leq m$ give $b_m(d) \leq m^{3/2}/(3\sqrt{d}) + 2m/3$. Squaring and summing over $d \leq m$ with $\sum 1/d \leq 1 + \log m$, $\sum d^{-1/2} \leq 2\sqrt{m}$ and at most m terms gives $m^3((1 + \log m) + 8 + 4)/9$.

In plain words (commentary, not part of the paper)

Notation from the paper: Q is a set of primes in $(m/2, m]$, and equation (2) splits $S_\varepsilon(m) = R_Q + \sum_{q \in Q} q \varepsilon(q)$, where R_Q depends only on the signs outside Q . The lemma says that a random choice of those signs makes R_Q small on average, so some choice makes $|R_Q| \leq B_*(m) = \frac{1}{3}m^{3/2}\sqrt{\log m + 13}$. Why: for random signs, $\mathbb{E} \varepsilon(n)\varepsilon(k)$ is 1 when nk is a square and 0 otherwise. Grouping $n = da^2$ by its squarefree part d turns the second moment into a sum of squares $\sum_d b_m(d)^2$, which elementary bounds control. Its role: $B_*(m)$ is of order $m^{3/2}$ up to a logarithm, while the primes of $(M/2, M]$ weigh at least $M^2/(4 \log M)$ (the paper's bound for $M \geq 10^8$), so they have room to cancel R_Q .

Evidence & scope

- **Proof on paper only** (not formalized).
- **Checked numerically by program `II_segundo_momento.py`** (supplement): $\mathbb{E} S_\varepsilon(m)^2$ evaluated from the exact squarefree-kernel formula, in double precision, against the bound, at 24176 values of m up to 999999999; maximum ratio 0.4288, at $m = 999999999$ (`II_segundo_momento_OUT.txt`). The program measures $\mathbb{E} S_\varepsilon(m)^2$; $\mathbb{E} R_Q^2$ is the same sum with the terms $d \in Q$ removed, hence not larger.

Lemma 2.2. p. 4

Write $\Sigma(A)$ for the set of subset sums of a finite set A of positive integers.

1. If $\Sigma(A) \supseteq [\alpha, \beta]$ and $x \notin A$, $0 < x \leq \beta - \alpha + 1$, then $\Sigma(A \cup \{x\}) \supseteq [\alpha, \beta + x]$.
As $\Sigma(A)$ is symmetric under $x \mapsto P_A - x$, $\Sigma(A) \supseteq [\alpha, \beta]$ with $\beta \geq P_A/2$ implies $\Sigma(A) \supseteq [\alpha, P_A - \alpha]$.
2. (pairs) Let $q_0, u_1, v_1, \dots, u_t, v_t$ be distinct positive odd integers, $d_j = (v_j - u_j)/2$ with $1 \leq d_j \leq 1 + d_1 + \dots + d_{j-1}$, $H = \sum d_j$, $U = \sum u_j$, and $q_0 \leq 2H + 1$.
Then $\Sigma(\{q_0, u_j, v_j\}) \supseteq [U + q_0 - 1, U + 2H + 1]$.
3. (Sárközy, with Lev's constants [BW21, Thm. 4.23]) If X is a set of n positive integers with maximum x^* and $4x^* \leq \ell \leq n^2/(12 \log_2(4x^*/n))$, then $\Sigma(X)$ contains an arithmetic progression of ℓ terms and step $s \leq 4x^*/n$, each term a sum of at most $6\ell/n$ distinct elements of X .

↳ Used in: [Theorem A](#) (p. 3)

Proof

(a) is immediate. (b) The condition on the d_j makes their subset sums exactly $\{0, \dots, H\}$; choosing one element of each pair gives $U + 2\{0, \dots, H\}$, and adding q_0 or not fills the other parity: an integer x of the interval with $x - U$ even is $U + 2k$, $k \leq H$, and one with $x - U$ odd is $U + q_0 + 2k$ with $0 \leq k \leq H$. (c) is the cited theorem: as $\log_2 y \geq \log y$ for $y \geq 1$, its hypothesis with $\log_2$ implies the hypothesis with the natural logarithm, so (c) holds whichever logarithm the source intends.

In plain words (commentary, not part of the paper)

$\Sigma(A)$ is the set of subset sums of A , and P_A the sum of all of A . The signed sums $\sum_{q \in Q} \pm q$ are exactly the numbers $P_Q - 2x$ with $x \in \Sigma(Q)$, so a long interval inside $\Sigma(Q)$ gives every value of the right parity in a long symmetric window, which is what the proof of Theorem A needs. The lemma collects three ways to get that interval. (a) An element no larger than the interval's length plus one extends it; and since $\Sigma(A)$ is symmetric under $x \mapsto P_A - x$, reaching half of P_A is enough. (b) Pairs $u_j < v_j$ whose half-gaps d_j grow slowly enough fill an interval; this builds the block certificates. (c) Sárközy's theorem, with explicit constants, gives a long progression inside $\Sigma(X)$; it covers all $m \geq 10^8$. Part (b) needs $d_1 = 1$, a pair of twin primes, which no theorem guarantees in every interval: so the certificates alone cannot cover all m , and from 10^8 on the proof uses (c), which is available from about $4.4 \cdot 10^7$.

Evidence & scope

- **Part (b) formalized in Lean**, in a slightly more general form (node Lemma 2.2(b)).
- **Part (a): proof on paper** (“immediate”); its hypotheses are checked for every prime of every block by `II_comprobador.py` (`II_comprobador_OUT.txt`, PASS).
- **Part (c): a cited theorem** (Bringmann–Wellnitz, Thm. 4.23); not checked. The inequality that verifies its hypothesis in the proof of Theorem A is evaluated by `II_cola_analitica.py` (holds from $M \approx 4.393 \cdot 10^7$).
- **Part (b) against the true subset sums**: `II_lemma_pares_directo.py`, blocks 1–8 (m from 5000 to 10725), all covered (`II_lemma_pares_directo_OUT.txt`).

Lemma 2.2(b).



Lean

(Complete sequences (Lemma 2.2(b), the part formalized))



The subset-sum statement of Lemma 2.2(b), in the form of the Lean theorems `complete_range` and `pairs_subset_sum`: a sequence with $d_j \leq 1 + \sum_{i < j} d_i$ represents every integer in $[0, \sum d_i]$ as a subset sum, and the pairs version used in the proof of Theorem A.



Checked when this page was built: [SegmentoII.lean](#) compiles with Lean 4.30.0-rc2, no errors, no sorry. Axioms — `complete_range`, `pairs_subset_sum`: `propext`, `Classical.choice`, `Quot.sound`.

Reproduce: Lean 4.30.0-rc2, Mathlib 701fb6e9c3b9; in that Lean project, `lake env lean`

`SegmentoII.lean` (SHA-256 of the compiled file: `3c5a9eac3055c58e...`; a source page on this site may show it with comments edited, under its own hash).

↳ Used in: [Lemma 2.2](#) (p. 9)

Proof

Formalized in Lean (see the linked declarations).

In plain words (commentary, not part of the paper)

A sequence $d_1, d_2, \dots$ in which each term is at most one more than the sum of the earlier ones represents every integer from 0 to its total as a subset sum: by induction, adding a term d_j to the interval $[0, H]$ already reached gives $[0, H] \cup [d_j, H + d_j]$, and $d_j \leq H + 1$ leaves no gap. The pairs version: pick one element of each pair $(u_j, v_j = u_j + 2d_j)$; the choices realise $U + 2y$ for every $0 \leq y \leq H$, the even offsets from $U = \sum u_j$, and adding an odd q_0 or not supplies the other parity.

Evidence & scope

- **Formalized in Lean 4 + Mathlib:** `SegmentoII.complete_range` (indexed form; also `complete_list`) and `SegmentoII.pairs_subset_sum`. No sorry in the file; the supplement's README states standard axioms only, and the file ends with `#print axioms` for each theorem.
- **What the Lean statement is, exactly:** `pairs_subset_sum` says every x in $[U + q_0 - 1, U + 2H + 1]$ equals $(q_0$ or 0) plus one chosen element of each pair. The step “this is a subset sum of $\{q_0, u_j, v_j\}$ ” uses that these numbers are distinct, which the paper assumes and the Lean file states only in a comment. The Lean versions assume less than the paper: neither $d_j \geq 1$ nor $q_0 \leq 2H + 1$, and of the numbers only that q_0 is odd (not that the u_j, v_j are odd, positive or distinct). In the proof of Theorem A the numbers are distinct primes, which `II_comprobador.py` checks for every block.
- **Checked by program** `II_lemma_pares_directo.py` against the true subset sums of the first 8 blocks (`II_lemma_pares_directo_OUT.txt`).

Proposition 3.1. p. 6

Let F be a finite set of primes and $\varepsilon_0 : F \rightarrow \{\pm 1\}$. There is $M(F)$ such that for every admissible $m \geq M(F)$ some ε with $\varepsilon|_F = \varepsilon_0$ has $S_\varepsilon(m) = 0$. For $F = \{2, 3, 5, 7\}$ one may take $M(F) = 60000$.

↳ Used in: Theorem B (p. 6)

Proof

For $m > 2 \max F$ the set F is disjoint from Q . Conditioning on the event $\varepsilon|_F = \varepsilon_0$, of probability $2^{-|F|}$, multiplies $\mathbb{E} R_Q^2$ by at most $2^{|F|}$, since $R_Q^2 \geq 0$; the proof of Theorem A then goes through with $2^{|F|/2} B_*$ in place of B_* , and both B_*/P_Q and K/P_Q tend to 0. For $F = \{2, 3, 5, 7\}$ the factor is 4: every block with $b \geq 60000$ satisfies $90D^2 > 16b^3(130 + 7e)$, and beyond 10^8 , $4B_*/P_Q + K/P_Q < 0.07$.

In plain words (commentary, not part of the paper)

Fixing the signs of finitely many primes in advance does not prevent zeros, once m is large. The reason is in the second moment of Lemma 2.1: conditioning on an event of probability $2^{-|F|}$ can multiply $\mathbb{E} R_Q^2$ by at most $2^{|F|}$, since $R_Q^2 \geq 0$, so B_* is replaced by $2^{|F|/2} B_*$, and the proof of Theorem A still closes for large m , both B_*/P_Q and K/P_Q tending to 0. For $F = \{2, 3, 5, 7\}$ the factor on B_* is 4, and the paper verifies the strengthened inequalities explicitly, giving $M(F) = 60000$. This proposition is the density half of Theorem B: any finite prefix of signs can be continued to a zero at a large length.

Evidence & scope

- **General F : proof on paper only.**
- $F = \{2, 3, 5, 7\}$, **block range, checked by program:** the inequality $90D^2 > 16b^3(130 + 7e)$ at the 78 blocks with $b \geq 60000$, by `II_certificados_A.py` and again by the independent `II_comprobador.py` (PASS; `II_comprobador_OUT.txt`).
- $F = \{2, 3, 5, 7\}$, **beyond 10^8 :** `II_cola_analitica.py` gives $4B_*/P_Q + K/P_Q = 0.06799$ at $M = 10^8$, and the same maximum on the grid up to

3 The digit recursion

Theorem 4.1. ✓ Lean ✓ p. 6

For $k \geq 0$ and $0 \leq r < q$,

$$S_\chi(qk + r) = s q S_\chi(k) + k(T_q + q C(r)) + T(r).$$

✓ Checked when this page was built: [SegmentoII.lean](#) compiles with Lean 4.30.0-rc2, no errors, no sorry. Axioms — digit_recursion: propext, Classical.choice, Quot.sound.

Reproduce: Lean 4.30.0-rc2, Mathlib 701fb6e9c3b9; in that Lean project, lake env lean

SegmentoII.lean (SHA-256 of the compiled file: 3c5a9eac3055c58e...; a source page on this site may show it with comments edited, under its own hash).

NEW **Classical and new:** for the unweighted sums the recursion is in Borwein–Choi–Coons and in Klurman–Mangerel–Pohoata–Teräväinen; the weight adds a term, and with it the zeros.

↳ Used in: [Corollary 4.3, first statement](#) (p. 13), [Corollary 4.3](#) (p. 14), [Proposition 4.4](#) (p. 16), [Corollary 4.5](#) (p. 17), [Theorem N1 \(new\)](#) (p. 19)

Proof

The $n \leq qk + r$ prime to q are $jq + r'$ with $j < k$, $1 \leq r' < q$, and $kq + r'$ with $r' \leq r$; as $\chi(jq + r') = \chi(r')$ and $C(q - 1) = 0$, they contribute $kT_q + kq C(r) + T(r)$. The multiples jq , $j \leq k$, contribute $\sum_j jq s \chi(j) = sq S_\chi(k)$.

💡 In plain words (commentary, not part of the paper)

Notation: $\chi = \chi_{q,s}$; for $0 \leq r < q$, $C(r) = \sum_{r' \leq r} \chi(r')$, $T(r) = \sum_{r' \leq r} r' \chi(r')$, and $T_q = T(q - 1)$. Split the integers $n \leq qk + r$ into multiples of q and the rest. The multiples jq , $j \leq k$, give $sq S_\chi(k)$, because $\chi(jq) = s \chi(j)$. The rest come in k complete blocks of residues plus a partial block up to r ; since χ only sees the residue and $C(q - 1) = 0$, they give $k(T_q + q C(r)) + T(r)$. So S_χ at m is determined by S_χ at m with its last base- q digit removed, plus a term that depends on that digit and, linearly, on the remaining number k . The zeros of [Corollary C](#), [Corollary 4.3](#), [Proposition 4.4](#) and [Corollary 4.5](#) are all read off this recursion. The unweighted recursion, $\sum_{n \leq qk+r} \chi(n) = s \sum_{n \leq k} \chi(n) + C(r)$, is in Borwein–Choi–Coons (for $s = 1$) and Klurman–Mangerel–Pohoata–Teräväinen (any s); the weight brings the factor q and the term $k(T_q + q C(r))$.

🔵 Evidence & scope

- **Formalized in Lean 4 + Mathlib:** `SegmentoII.digit_recursion`. Hypotheses in Lean: $q \geq 2$, $\chi : \mathbb{N} \rightarrow \mathbb{Z}$ completely multiplicative, $\chi(jq + r) = \chi(r)$ for $1 \leq r < q$, $\chi(q) = s$, and $C(q - 1) = 0$; neither primality of q nor values ± 1 is assumed. No sorry; standard axioms (supplement README). The file does not prove that $\chi_{q,s}$ satisfies these hypotheses (elementary properties of the Legendre symbol).
- **Checked by program `II_ceros_digitos.py`:** the recursion at every $m \leq 10^7$ for the primes $5 \leq q \leq 73$ and $q = 163$, both signs (`II_ceros_digitos_OUT.txt`).

Theorem 4.2. (Kanemitsu–Kuzumaki, Kučera) p. 7

Let f be odd, let χ be an odd Dirichlet character modulo f , and put

$B_{1,\chi} = f^{-1} \sum_{a=1}^f a \chi(a)$. Then

$$\sum_{c=1}^{(f-1)/2} c \chi(c) = \frac{f}{2} B_{1,\chi} (\chi(2)^{-1} - 1).$$

If moreover χ is primitive, so that $B_{1,\chi} \neq 0$, the sum vanishes if and only if $\chi(2) = 1$.



Classical and new: not new: in Kanemitsu–Kuzumaki and in Kučera.

↳ Used in: [Corollary 4.3](#) (p. 14), [Corollary 4.5](#) (p. 17)

Proof

Write $T = \sum_{c < f/2} c \chi(c)$, $U = \sum_{c < f/2} \chi(c)$, $W = \sum_{c \text{ even}} \chi(c)$ and $X = \sum_{c \text{ even}} c \chi(c)$, all over $1 \leq c < f$. Pairing $b \mapsto f - b$ exchanges odd and even residues, since f is odd, and $\chi(f - b) = -\chi(b)$; hence $\sum_{b \text{ odd}} b \chi(b) = -fW + X$, and adding X gives $fB_{1,\chi} = 2X - fW$, that is $X = \frac{f}{2}(B_{1,\chi} + W)$. Substituting $c = 2c'$ in the even sums, $W = \chi(2)U$ and $X = \chi(2) \cdot 2T$. Finally $\sum_b \chi(b) = 0$ splits as $\sum_{\text{odd}} + \sum_{\text{even}}$, and the first equals $(2\chi(2) - 1)B_{1,\chi}$ by the same substitution applied to $\chi(2)fB_{1,\chi} = \sum_a a \chi(2a)$; so $-\chi(2)U = (2\chi(2) - 1)B_{1,\chi}$, i.e. $U = (\chi(2)^{-1} - 2)B_{1,\chi}$. Putting the three together, $2\chi(2)T = \frac{f}{2}(B_{1,\chi} + \chi(2)U)$ gives the displayed formula. For χ primitive and odd, $B_{1,\chi} = -L(0, \chi) \neq 0$.



In plain words (commentary, not part of the paper)

Here $B_{1,\chi} = f^{-1} \sum_{a=1}^f a \chi(a)$. The weighted sum of an odd character over the first half of the residues equals $B_{1,\chi}$ times $\frac{f}{2}(\chi(2)^{-1} - 1)$, that is, half the full weighted sum $\sum_{a \leq f} a \chi(a) = f B_{1,\chi}$ times $\chi(2)^{-1} - 1$, the Euler factor at 2. The proof pairs b with $f - b$, which swaps odd and even residues because f is odd and flips the sign because χ is odd, substitutes $c = 2c'$ in the even sums, which brings out $\chi(2)$, and uses $\sum_b \chi(b) = 0$. For primitive χ , $B_{1,\chi} = -L(0, \chi) \neq 0$, so the half sum vanishes exactly when $\chi(2) = 1$. This criterion gives $T(\frac{q-1}{2}) = 0$ when $q \equiv 7 \pmod{8}$; with the class number formula it puts the digit $(q-1)/2$ into R_q (Corollary C). The identity is not new: it is in Kanemitsu–Kuzumaki and Kučera; the paper gives a short proof for the half sum. Its unweighted twin is the Jacobi–Dirichlet class number formula, equation (3) of the paper.



Evidence & scope

- **Proof on paper only** (not formalized).
- **Checked by segment0_II.py autotest**, exactly in $\mathbb{Z}[\zeta_N]$: the closed form holds for all 1510 odd characters modulo an odd $f \leq 121$; the criterion holds on the 1236 primitive ones; without primitivity it fails 19 times (imprimitive, $B_{1,\chi} = 0$), as it must; off-diagonal and even-character decoys ($f < 60$) fail (autotest_OUT.txt).

Corollary 4.3, first statement.



Lean

(Propagation of zeros along digits (first statement✓ of Corollary 4.3))

Let $R_q = \{0 \leq r < q : T_q + q C(r) = 0 = T(r)\}$. Every $m \geq 1$ whose base- q digits all lie in R_q has $S_\chi(m) = 0$, for both values of s .

✓ Checked when this page was built: [SegmentoII.lean](#) compiles with Lean 4.30.0-rc2, no errors, no sorry. Axioms — `zero_step`, `zeros_from_digits`: `propext`, `Classical.choice`, `Quot.sound`.
 Reproduce: Lean 4.30.0-rc2, Mathlib 701fb6e9c3b9; in that Lean project, `lake env lean SegmentoII.lean` (SHA-256 of the compiled file: 3c5a9eac3055c58e...; a source page on this site may show it with comments edited, under its own hash).

↳ Used in: [Corollary 4.3](#) (p. 14)

Proof

Formalized in Lean (see the linked declarations).

💡 In plain words (commentary, not part of the paper)

R_q is the set of digits $0 \leq r < q$ with $T_q + qC(r) = 0$ and $T(r) = 0$: exactly the digits for which both digit-dependent terms of the recursion of Theorem 4.1 vanish. If $S_\chi(k) = 0$ and $r \in R_q$, all three terms on the right are 0, so $S_\chi(qk + r) = 0$. Starting from $S_\chi(0) = 0$ and appending digits one at a time, every m written only with digits of R_q is a zero, whatever the value s at q . With $0, q-1 \in R_q$ this gives at least $2^J - 1$ zeros below q^J (Corollary 4.3(a)); with the single digit $(q-1)/2$, the chain $(q^j - 1)/2$ (Corollary C). Which digits lie in R_q , and why, is the subject of Corollary 4.3, Theorem 4.2 and Proposition 4.6.

🔍 Evidence & scope

- **Formalized in Lean 4 + Mathlib:** `SegmentoII.zero_step` (one digit) and `SegmentoII.zeros_from_digits` (all digits, via Mathlib's `Nat.digits`), under the same hypotheses as `digit_recursion`. No sorry; standard axioms (supplement README).
- **Checked by `segmento_II.py autotest`:** for $q = 5, 13, 29$ the zeros $\leq 10^6$ are exactly the digit tree (255, 31, 255 of them), both s (`autotest_OUT.txt`).

Corollary 4.3. p. 7

Let $R_q = \{0 \leq r < q : T_q + qC(r) = 0 = T(r)\}$. Every $m \geq 1$ whose base- q digits all lie in R_q has $S_\chi(m) = 0$, for both values of s .

1. If $q \equiv 1 \pmod{4}$, then $0, q-1 \in R_q$, and S_χ has at least $2^J - 1$ zeros below q^J , hence at least $\frac{1}{2}X^{\log 2 / \log q} - 1$ below X .
2. If $q \equiv 7 \pmod{8}$, then $(q-1)/2 \in R_q$, and $S_\chi\left(\frac{q^j-1}{2}\right) = 0$ for every $j \geq 1$.
3. If $q \equiv 3 \pmod{4}$, $q > 3$, $h(-q) = 1$ and $s = 1$, then $S_\chi(z) = 0$ along $z_0 = q$, $z_{i+1} = q(z_i + 1)$. For $q = 3$ and $s = 1$ the same holds with 27 in place of q : $z_0 = 27$, $z_{i+1} = 27(z_i + 1)$.
4. For every q and s : if $S_\chi(qk) = 0$ and $C(r) = T(r) = 0$, then $S_\chi(qk + r) = 0$. When $q > 3$ and $\chi(2) = \chi(3) = -1$ this applies with $r = 4$, since $1 - 2 - 3 + 4 = 0$. For $q = 3$ and $s = -1$, where also $\chi(2) = \chi(3) = -1$, the conclusion holds as well: $S_\chi(3k) = 0$ implies $S_\chi(3k + 4) = 0$.
5. If $s = -1$, then $S_\chi(q^2m) = q^2S_\chi(m)$ for every m : a single zero of S_χ gives infinitely many.
6. If $q \equiv 3 \pmod{8}$, then $S_\chi\left(\frac{q^j-1}{2}\right) = h(-q)q^j(1 + s + \dots + s^{j-1})$ for every $j \geq 1$, with $h(-3)$ read as the weighted class number $1/3$. In particular, for

$$s = -1, S_\chi\left(\frac{q^{2i}-1}{2}\right) = 0 \text{ for every } i \geq 1.$$

↳ Used in: [Corollary C](#) (p. 6)

Proof

By induction on the number of digits, from $S_\chi(0) = 0$ and Theorem 4.1. (a) χ is even, so $r \leftrightarrow q - r$ gives $T_q = 0$; with $C(q - 1) = 0$, both 0 and $q - 1$ lie in R_q . (b) χ is odd and $\chi(2) = 1$. Theorem 4.2 gives $T((q - 1)/2) = 0$, the Euler factor $\chi(2)^{-1} - 1$ vanishing; the half sum (3) gives $C((q - 1)/2) = (2 - \chi(2))h(-q) = h(-q)$; and $T_q = qB_{1,\chi} = -qh(-q)$. So $T_q + qC = 0$, and $(q^j - 1)/2$ is the number whose j digits are all $(q - 1)/2$. (c) With $r = 0$ and $k = z + 1$, Theorem 4.1 gives $S_\chi(q(z + 1)) = q(z + 1)(s\chi(z + 1) - h(-q))$ when $S_\chi(z) = 0$; $S_\chi(q) = T_q + qs = q(s - h) = 0$, and $z_i \equiv 0 \pmod{q}$ gives $\chi(z_i + 1) = 1$. At $q = 3$, $T_3 = -1$, and three steps with $r = 0$ give $S_\chi(27k) = 27S_\chi(k) - 27k$; so $S_\chi(27) = 0$, and $S_\chi(z) = 0$ with $27 \mid z$ gives $S_\chi(27(z + 1)) = 27(z + 1)\chi(z + 1) - 27(z + 1) = 0$. (d) Subtracting the cases r and 0 of Theorem 4.1, $S_\chi(qk + r) = S_\chi(qk) + kqC(r) + T(r)$; for $q > 3$ the digit $r = 4$ exists. For $q = 3$, $s = -1$ it does not, and directly: $T_3 = 1 - 2 = -1$, so $S_\chi(3k) = -3S_\chi(k) - k$ and $S_\chi(3k) = 0$ forces $3 \mid k$; and, as $\chi(3k + 1) = \chi(3k + 4) = 1$, $\chi(3k + 2) = -1$ and $\chi(3k + 3) = -\chi(k + 1)$, $S_\chi(3k + 4) - S_\chi(3k) = (3k + 3)(1 - \chi(k + 1))$, which vanishes because $k + 1 \equiv 1 \pmod{3}$ gives $\chi(k + 1) = 1$. (e) With $r = 0$ and $s = -1$, $S_\chi(qm) = -qS_\chi(m) + mT_q$; applied twice, the terms in T_q cancel. (f) Put $c = (q - 1)/2$ and $m_j = (q^j - 1)/2$, so that $m_{j+1} = qm_j + c$. Now $\chi(2) = -1$, so Theorem 4.2 gives $T(c) = qh(-q)$, (3) gives $C(c) = 3h(-q)$, and $T_q = -qh(-q)$; at $q = 3$, where $c = 1$, directly $T(1) = C(1) = 1$ and $T_3 = -1$, the same values with $h(-3) = 1/3$. Theorem 4.1 with $k = m_j$ and $r = c$ gives $S_\chi(m_{j+1}) = sqS_\chi(m_j) + q^{j+1}h(-q)$, and $S_\chi(m_1) = T(c) = qh(-q)$.

💡 In plain words (commentary, not part of the paper)

$h(-q)$ is the class number, given for prime $q \equiv 3 \pmod{4}$, $q > 3$, by the Jacobi–Dirichlet formula $h(-q) = \frac{1}{2-(2|q)} \sum_{0 < n < q/2} (n \mid q)$ (equation (3) of the paper). The first statement is Lemma 4.3, first statement. The items say where the recursion produces zeros: (a)–(c) and (e) give infinitely many, (d) adds one digit to a zero, and (f) gives the exact value along $(q^j - 1)/2$. (a) For $q \equiv 1 \pmod{4}$ the character is even, and 0 and $q - 1$ are always in R_q . (b) For $q \equiv 7 \pmod{8}$ the Euler factor at 2 makes the weighted half sum vanish, and with the class number formula this puts $(q - 1)/2$ in R_q . (c) For class number one and $s = 1$ a chain $z_{i+1} = q(z_i + 1)$ of zeros, which works also where R_q is empty ($q = 11, 19, 43, 67, 163$, Table 2 of the paper). (d) A digit r with $C(r) = T(r) = 0$ can be appended to a zero of the form qk . (e) For $s = -1$, $S_\chi(q^2m) = q^2S_\chi(m)$: one zero gives infinitely many. (f) For $q \equiv 3 \pmod{8}$ the value at $(q^j - 1)/2$ is explicit, $h(-q)q^j(1 + s + \dots + s^{j-1})$, and vanishes for $s = -1$ and even j .

🔵 Evidence & scope

- **First statement formalized in Lean** (node Lemma 4.3, first statement). Items (a)–(f): proof on paper only.
- `II_ceros_R.py`: for all primes $5 \leq q \leq 2000$, $0, q - 1 \in R_q$ at the 147 primes $q \equiv 1 \pmod{4}$, $(q - 1)/2 \in R_q$ at the 78 primes $q \equiv 7 \pmod{8}$, $R_q = \emptyset$ at the 76

primes $q \equiv 3 \pmod{8}$; 0 failures.

- `II_ceros_heegner.py` (item (c)): the chains for $q = 7, 11, 19, 43, 67, 163$, $s = 1$, all members listed are zeros by direct summation; $s = -1$ control nonzero.
- `II_s_menos_y_131.py` (item (e)): 246 of 246 cases at $q = 11, 19, 23$.
- `II_ceros_menos1.py` (item (f)): primes $q \equiv 3 \pmod{8}$, $3 < q < 400$: zero at $j = 2$ for $s = -1$ (and $j = 4$ for $q = 11, 19, 43$); $S\left(\frac{q^2-1}{2}\right) = 2q^2h(-q)$ for $s = 1$; 0 failures.
- `segmento_II.py autotest`: items (b) and (c) hold where they apply for $q = 3, 7, 11, 19, 43, 67, 163$, $s = 1$, with the recursion checked at every $m \leq 3 \cdot 10^5$ (`autotest_OUT.txt`). Item (f) is checked by `II_ceros_menos1.py` (above); item (d) has no dedicated check in the supplement.

Proposition 4.4. p. 9

Let $q \equiv 3 \pmod{4}$, $q > 3$, $s = 1$, $h = h(-q)$ and $D(r) = C(r) - h$. Suppose that B has base- q digits $d_{L-1}, \dots, d_0$ with

$$\sum_{i < L} D(d_i) = 0 \quad \text{and} \quad S_\chi(B) = hB.$$

Then, with $Q = q^L$, every $m_t = qB(1 + Q + \dots + Q^{t-1})$, $t \geq 1$, is a zero of S_χ .

↳ Not used by another result of the paper.

Proof

With $s = 1$ and $T_q = -qh$, Theorem 4.1 reads $S_\chi(qk + r) = qS_\chi(k) + qkD(r) + T(r)$. Appending the digits of B one at a time, by induction, $S_\chi(Qk + B) = QS_\chi(k) + Qk \sum_i D(d_i) + S_\chi(B)$, so $S_\chi(k) = hk$ implies $S_\chi(Qk + B) = h(Qk + B)$; starting from $k = 0$, $k_t = B(1 + Q + \dots + Q^{t-1})$ has $S_\chi(k_t) = hk_t$. Appending the digit 0, $S_\chi(qk_t) = q(S_\chi(k_t) + k_t D(0)) = q(hk_t - hk_t) = 0$.

In plain words (commentary, not part of the paper)

Notation: $h = h(-q)$ and $D(r) = C(r) - h$. With $s = 1$ the recursion reads $S_\chi(qk + r) = qS_\chi(k) + qkD(r) + T(r)$. When R_q is empty no single digit carries every zero to a zero, but a word of digits can preserve the weaker relation $S_\chi(k) = hk$: appending the digits of a block B with $\sum D(d_i) = 0$ and $S_\chi(B) = hB$ sends $S_\chi(k) = hk$ to the same relation at $q^Lk + B$. Repeating the block keeps the relation; a final digit 0, with $D(0) = -h$, turns it into a zero. At $q = 59$ ($h = 3$) the block $(1, 22, 21, 0)_{59} = 283200$ works, giving zeros from 16 708 800 on; at $q = 347$ a thirteen-digit block. So class number greater than one does not block infinitely many zeros. The first zero at $q = 59$, 4 591 439, is not of this form.

Evidence & scope

- **Proof on paper only** (not formalized).
- `II_bloques_ceros.py`: at $q = 59$, $B = 283200$, $\sum D = 0$, $S(B) = 849600 = 3B$, $S(16708800) = 0$, and $t = 2, 3$ by the recursion; at $q = 347$, $\sum D = 0$, $S(B) = 5B$, $S(347B) = 0$ (`II_bloques_ceros_OUT.txt`).

- `segmento_II.py autotest`: zeros 16708800, 202466578185600, 2453360618309656910400 from the $q = 59$ block; the $q = 347$ block gives zeros.

Corollary 4.5. p. 9

Let q be an odd prime, χ an odd Dirichlet character modulo q of any order, extended completely multiplicatively by any root of unity $\chi(q)$, and $S_\chi(m) = \sum_{n \leq m} n\chi(n)$. Then $S_\chi\left(\frac{q-1}{2}\right) = 0$ if and only if $\chi(2) = 1$, and in that case $S_\chi\left(\frac{q^j-1}{2}\right) = 0$ for every $j \geq 1$. The characters for which this happens are the odd characters trivial on $\langle 2 \rangle$.

↳ Not used by another result of the paper.

Proof

Theorem 4.1 and its proof hold verbatim for χ , with $s = \chi(q)$. By Theorem 4.2, $T((q-1)/2) = \frac{q}{2} B_{1,\chi}(\chi(2)^{-1} - 1)$ with $B_{1,\chi} \neq 0$, which gives the equivalence at $j = 1$. If $\chi(2) = 1$, the proof of Theorem 4.2 gives $C((q-1)/2) = (\chi(2)^{-1} - 2)B_{1,\chi} = -B_{1,\chi}$, and $T_q = qB_{1,\chi}$, so $(q-1)/2 \in R_q$.

💡 In plain words (commentary, not part of the paper)

The mechanism behind Corollary C does not use that the character is quadratic. For any odd Dirichlet character modulo an odd prime q , of any order, extended completely multiplicatively with any root of unity as its value at q , the recursion of Theorem 4.1 holds verbatim, and Theorem 4.2 says the weighted half sum vanishes exactly when $\chi(2) = 1$. When it does, $(q-1)/2$ lies in R_q , and every $(q^j-1)/2$ is a zero. The characters with this property are the odd characters trivial on the subgroup generated by 2.

🔍 Evidence & scope

- **Proof on paper only** (not formalized; the Lean file is over $\mathbb{Z}$ -valued functions).
- `II_ceros_orden_r.py`, exactly in $\mathbb{Z}[\zeta_r]$ (README: orders 2, 6, 8, 18): primes $q \leq 130$, 210 checks with χ odd and $\chi(2) = 1$, 0 failures; control: of 817 odd characters with $\chi(2) \neq 1$, none vanishes at $j = 1$ (`II_ceros_orden_r_OUT.txt`).

Proposition 4.6. p. 10

Let $q \equiv 3 \pmod{8}$, $q > 3$, and let $\chi_{q,-1}$ be the extension with value -1 at q . If $\sum_{n \geq 1} \chi_{q,-1}(n) \sin(2\pi nx)/n^2 \geq 0$ for $0 \leq x \leq 1/2$, then $R_q = \emptyset$.

↳ Not used by another result of the paper.

Proof

By the symmetry $C(q-1-r) = C(r)$, $T(q-1-r) = q(C(r) - h) - T(r)$ it suffices to exclude $r \leq (q-1)/2$. Suppose such an r lies in R_q and put $u = (rq+1)/q^2 \in [0, 1/2]$. Splitting n by its q -adic valuation, the series at u is $\sum_{j \geq 0} (-1)^j q^{-2j} f_q(q^j u)$. The term $j = 0$ vanishes, qu lying inside the cell of r ; the terms $j \geq 2$ vanish, their arguments being integers; and the term $j = 1$ is $-q^{-2} f_q(1/q) = -2\pi^2 h/q^{7/2}$, since $F_q(1) = h$. The series is negative at u .

💡 In plain words (commentary, not part of the paper)

Conrey's sine series is $f_q(x) = \sum_{n \geq 1} (n | q) \sin(2\pi nx)/n^2$; for prime $q \equiv 3 \pmod{4}$, $q > 3$, through his Theorem 4 and the class number formula it equals $\frac{2\pi^2}{q^{3/2}} F_q(qx)$ with

$F_q(y) = T(r) + y(h - C(r))$, $r = \lfloor y \rfloor$, which is linear on each cell and vanishes on the cell of r exactly when $r \in R_q$. For $q \equiv 3 \pmod{8}$, Conrey's case, a null cell is the first alternative of his Theorem 4, which he remarked would make f_q vanish on an interval; the paper does not decide whether it occurs. The proposition excludes it under a positivity hypothesis on the series of $\chi_{q,-1}$ (value -1 at q), not of f_q . Splitting n by its power of q , that series at $u = (rq + 1)/q^2$, for r a putative null cell, is $\sum_j (-1)^j q^{-2j} f_q(q^j u)$: the term $j = 0$ vanishes, the terms $j \geq 2$ have integer arguments, and the term $j = 1$ is $-2\pi^2 h/q^{7/2} < 0$.

Evidence & scope

- **Proof on paper only.** The positivity hypothesis is not verified for any q in the paper or the supplement.
- **Independent of the hypothesis:** `II_R_vacio.py` finds $R_q = \emptyset$ for all 2408 primes $q \equiv 3 \pmod{8}$, $5 \leq q \leq 10^5$ (`II_R_vacio_OUT.txt`). A finite computation; whether R_q is always empty is Question 6.1 of the paper, open.
- `II_automata_y_conrey.py` checks the identity $f_q = \frac{2\pi^2}{q^{3/2}} F_q(qx)$ numerically at three points for $q = 59$ and $q = 179$ (agreement to the 8 digits printed).

4 Rows

Proposition 5.1. p. 11

Let $n = 2m + 1$ and let ε be any function on $1, \dots, m$ with values ± 1 . Then

$$\sum_{c=1}^m c \varepsilon(c) \equiv \frac{n^2 - 1}{8} \pmod{2}, \quad \text{so it can vanish only if } \left(\frac{2}{n}\right) = 1.$$

In particular, if χ is a primitive quadratic character of conductor f with $\gcd(f, m!) = 1$, then $T(m, \chi)$ does not vanish unless $\left(\frac{2}{n}\right) = 1$.

↳ Used in: [Corollary 5.2](#) (p. 19)

Proof

Changing one sign changes the sum by $2c$, so its parity is that of

$\sum_{c \leq m} c = m(m+1)/2$, and $m(m+1)/2 = (n^2 - 1)/8$. The second supplementary law of quadratic reciprocity is $\left(\frac{2}{n}\right) = (-1)^{(n^2-1)/8}$, so the sum is even exactly when that symbol is 1.

In plain words (commentary, not part of the paper)

Here $n = 2m + 1$ and $\left(\frac{2}{n}\right)$ is the Jacobi symbol. Any ± 1 signs on $1, \dots, m$ give a sum with the parity of $1 + 2 + \dots + m = m(m+1)/2 = (n^2 - 1)/8$, since flipping the sign of c changes the sum by $2c$. By the second supplementary law, that parity is even exactly when $\left(\frac{2}{n}\right) = 1$, that is $n \equiv \pm 1 \pmod{8}$, that is $m \equiv 0, 3 \pmod{4}$: the admissible lengths again. For a primitive quadratic character of conductor f with $\gcd(f, m!) = 1$, the values on $1, \dots, m$ are all ± 1 , so $T(m, \chi) = \sum_{c \leq m} c \chi(c)$ inherits the obstruction. The coprimality hypothesis is needed: at $f = 35$, $m = 6$, $\chi(5) = 0$ and $T = 0$ although $\left(\frac{2}{13}\right) = -1$.

Evidence & scope

- **Proof on paper only** (two lines; not formalized).
- `segmento_II.py autotest`: no zero at the 12 inadmissible $m \leq 22$, by exhaustive search; the $f = 35, m = 6$ example ($T = 0$ with $(2 \mid 13) = -1$, because $\chi(5) = 0$) (`autotest_OUT.txt`).

Corollary 5.2. p. 11

Let $n = 2m + 1$. The row m is non-empty if and only if $\left(\frac{2}{n}\right) = 1$, and then it is infinite.

↳ Not used by another result of the paper.

Proof

Necessity is Proposition 5.1. Conversely, $\left(\frac{2}{n}\right) = 1$ is $m \equiv 0, 3 \pmod{4}$; take ε from Theorem A. For a prime $f \equiv 3 \pmod{4}$, $f > n$, the value $(2 \mid f)$ is fixed by $f \pmod{8}$, and for an odd prime $p \leq m$ reciprocity gives $(p \mid f) = (-1)^{(p-1)/2}(f \mid p)$, fixed by $f \pmod{p}$ and taking both signs. By the Chinese remainder theorem and Dirichlet's theorem there are infinitely many primes $f \equiv 3 \pmod{4}$ with $(p \mid f) = \varepsilon(p)$ for every prime $p \leq m$; for them $\chi = (\cdot \mid f)$ agrees with ε on $1, \dots, m$, and $T(m, \chi) = S_\varepsilon(m) = 0$.

 In plain words (commentary, not part of the paper)

Row m is the set of odd primitive quadratic characters χ of prime conductor $f > 2m + 1$ with $T(m, \chi) = \sum_{c \leq m} c \chi(c) = 0$. Parity (Proposition 5.1) empties the rows with $\left(\frac{2}{n}\right) = -1$. For the others, Theorem A gives a sign pattern ε with $S_\varepsilon(m) = 0$, and the question is whether a Legendre symbol can copy it on $1, \dots, m$. It can: for a prime $f \equiv 3 \pmod{4}$, $(2 \mid f)$ is fixed by $f \pmod{8}$ and, by reciprocity, $(p \mid f)$ by $f \pmod{p}$, taking both signs. The Chinese remainder theorem and Dirichlet's theorem then give infinitely many primes f whose symbol agrees with ε at every prime $p \leq m$, hence on $1, \dots, m$. Every admissible length is the length of a vanishing weighted Legendre sum, for infinitely many primes.

 Evidence & scope

- **Proof on paper only** (not formalized); it inherits the guarantees of Theorem A.
- `segmento_II.py autotest`: rows 2 to 40 occupied iff $(2 \mid n) = 1$, 39 of 39; decoy rule $(3 \mid n)$ matches only 19 of 39 (`autotest_OUT.txt`). A finite check; “infinite” is on paper.

5 After submission

Theorem N1 (new). ✔ Lean (No zeros at $q = 179$ and $q = 227$)✔

Let $q \in \{179, 227\}$ and $\chi = \chi_{q,1}$. Then $S_\chi(m) \neq 0$ for every $m \geq 1$. This answers the second part of the question on $q \equiv 3 \pmod{8}$ in the paper: there are primes $q \equiv 3 \pmod{8}$ for which $\chi_{q,1}$ has no zero at all.

✔ Checked when this page was built: [SegmentoII_SinCeros.lean](#) compiles with Lean 4.30.0-rc2, no errors, no sorry. Axioms — `S_ne_zero_179`, `S_ne_zero_227`: `propext`, `Classical.choice`, `Quot.sound`.
Reproduce: Lean 4.30.0-rc2, Mathlib 701fb6e9c3b9; in that Lean project, `lake env lean SegmentoII_SinCeros.lean` (SHA-256 of the compiled file: `c72f623ab86fb232...`; a source page on this site may show it with comments edited, under its own hash).

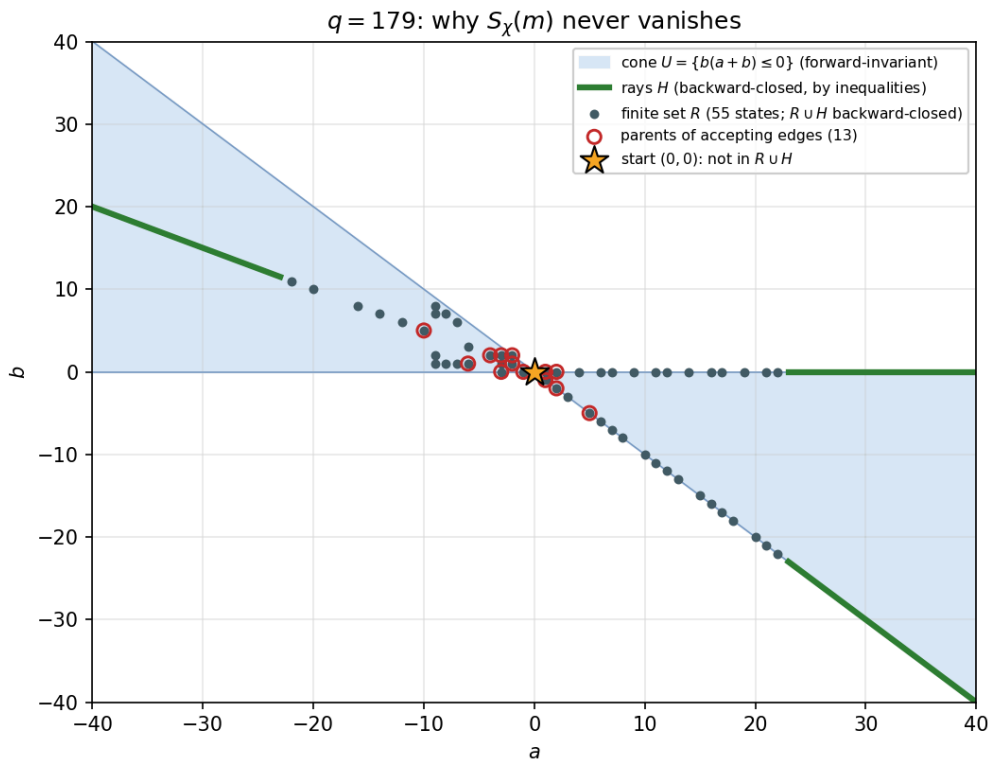
↳ Not used by another result of the paper.

Proof

A finite certificate: a forward-invariant cone $\{b(a + b) \leq 0\}$ of the digit automaton, three rays, and a finite set of 55 (resp. 162) states; inside the cone their union is backward-closed, contains every parent of an accepting edge, and does not contain the origin.

In plain words (commentary, not part of the paper)

Reading the base- q digits of m from the least significant one, the relation $S_\chi(k) + ak = b$ is carried from one state (a, b) to the next by an exact integer transition (Theorem 4.1). A positive zero of S_χ is a path from $(0, 0)$ that ends, after its leading digit, on a state with $b = 0$. The figure shows why no such path exists for $q = 179$: the path can never leave the blue cone, and every state from which the last step could land on $b = 0$ (red circles) lies in a set—the 55 grey states and the three green rays—that, inside the cone, can only be entered from a state already in it. The start $(0, 0)$ is not in that set, so no path from it ever enters the set, and none reaches a red circle.



Evidence & scope

- **Formalized in Lean 4 + Mathlib, checked by the kernel:** `S_ne_zero_179` and `S_ne_zero_227` (links above). The finite part is checked by `decide` inside the kernel; no `native_decide`, no `sorry`. Axioms: `propext`, `Classical.choice`, `Quot.sound`.
- **The checker can fail:** the same Lean file proves that it rejects the backward set computed for $q = 59$ (which has zeros; the set contains the origin), the same set with the origin deleted, and two corrupted certificates for $q = 179$ (one state removed; one table entry changed).

- **The hypothesis is the real Legendre symbol:** the function `leg` of the checker is proved equal to Mathlib’s `legendreSym` (`leg_eq_legendreSym`), the theorems are restated with `legendreSym` (`S_ne_zero_179_legendreSym`, `S_ne_zero_227_legendreSym`), and a function satisfying all hypotheses is constructed (`chi0_spec`, `S_chi0_ne_zero_179`, `S_chi0_ne_zero_227`), so the theorem is not vacuous.

Tables cited

range of m	method	certificate	margin
$3 \leq m \leq 20000$	explicit ε	10000 lengths	exact
$5000 \leq m < 10^8$	Lem. 2.1, 2.2(a,b)	104 blocks	$B_*^2/D^2 \leq 0.445$
$m \geq 10^8$	Lem. 2.1, 2.2(c)	none	$(B_* + K)/P_Q < 0.027$

Table 1: The proof of Theorem A, range by range. The ranges overlap. The 104 certificates (2128 pairs) and the 10000 explicit functions are generated by programs in the supplementary material and stored there, and a separate program that only reads the stored files checks them in integer arithmetic.

References

AC17

C. Aebi and G. Cairns, *Sums of quadratic residues and nonresidues*, Amer. Math. Monthly **124** (2017), 166–169, [doi:10.4169/amer.math.monthly.124.2.166](https://doi.org/10.4169/amer.math.monthly.124.2.166).

Aym24

M. Aymone, *Sign changes of the partial sums of a random multiplicative function II*, C. R. Math. Acad. Sci. Paris **362** (2024), 895–901, [doi:10.5802/crmath.615](https://doi.org/10.5802/crmath.615).

Ber76

B. C. Berndt, *Classical theorems on quadratic residues*, Enseign. Math. (2) **22** (1976), 261–304.  [find it](#)

BCC10

P. Borwein, S. K. K. Choi and M. Coons, *Completely multiplicative functions taking values in $\{-1, 1\}$* , Trans. Amer. Math. Soc. **362** (2010), 6279–6291, [doi:10.1090/S0002-9947-2010-05235-3](https://doi.org/10.1090/S0002-9947-2010-05235-3).

BW21

K. Bringmann and P. Wellnitz, *On near-linear-time algorithms for dense subset sum*, Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA), 2021, 1777–1796; full version [arXiv:2010.09096](https://arxiv.org/abs/2010.09096), where the result cited is Theorem 4.23, [doi:10.1137/1.9781611976465.107](https://doi.org/10.1137/1.9781611976465.107).

Con24

B. Conrey, *Character sums and the Riemann hypothesis*, Acta Arith. **214** (2024), 327–342, [doi:10.4064/aa230530-13-11](https://doi.org/10.4064/aa230530-13-11).

Dir38

P. G. Lejeune Dirichlet, *Sur l'usage des séries infinies dans la théorie des nombres*, J. Reine Angew. Math. **18** (1838), 259–274; Werke I (1889), 357–374, [doi:10.1515/crll.1838.18.259](https://doi.org/10.1515/crll.1838.18.259).

Dus10

P. Dusart, *Estimates of some functions over primes without R.H.*, preprint, 2010, [arXiv:1002.0442](https://arxiv.org/abs/1002.0442).

HL24

A. Hussain and Y. Lamzouri, *The limiting distribution of Legendre paths*, J. Éc. polytech. Math. **11** (2024), 589–611, [doi:10.5802/jep.260](https://doi.org/10.5802/jep.260).

Jac32

C. G. J. Jacobi, *Observatio arithmetica de numero classium divisorum quadraticorum formae $yy + Azz$, designante A numerum primum formae $4n + 3$* , J. Reine Angew. Math. **9** (1832), 189–192, [doi:10.1515/crll.1832.9.189](https://doi.org/10.1515/crll.1832.9.189).

Kal19

A. Kalmynin, *Long nonnegative sums of Legendre symbols*, preprint, 2019, [arXiv:1911.10634](https://arxiv.org/abs/1911.10634).

KK01

S. Kanemitsu and T. Kuzumaki, *On a generalization of the Maillet determinant II*, Acta Arith. **99** (2001), 343–361, [doi:10.4064/aa99-4-4](https://doi.org/10.4064/aa99-4-4).

KMPT21

O. Klurman, A. P. Mangerel, C. Pohoata and J. Teräväinen, *Multiplicative functions that are close to their mean*, Trans. Amer. Math. Soc. **374** (2021), 7967–7990, [doi:10.1090/tran/8427](https://doi.org/10.1090/tran/8427).

KMS26

O. Klurman, M. Munsch and Y.-C. Sun, *Small values of signed harmonic sums and logarithmic means of multiplicative functions*, preprint, 2026, [arXiv:2605.04694](https://arxiv.org/abs/2605.04694).

Kuč01

R. Kučera, *Formulae for the relative class number of an imaginary abelian field in the form of a determinant*, Nagoya Math. J. **163** (2001), 167–191, [doi:10.1017/S0027763000007959](https://doi.org/10.1017/S0027763000007959).

Leb42

V.-A. Lebesgue, *Démonstration de quelques théorèmes relatifs aux résidus et aux non-résidus quadratiques*, J. Math. Pures Appl. **7** (1842), 137–159.  [find it](#)

Lev03

V. F. Lev, *Blocks and progressions in subset sum sets*, Acta Arith. **106** (2003), 123–142, [doi:10.4064/aa106-2-3](https://doi.org/10.4064/aa106-2-3).

Mar26

C. Marín, *The ring of character values depends only on the segment: $SU(n)$, $Sp(2m)$ and $GL(n)$ at a torsion element*, preprint, 2026, [doi:10.5281/zenodo.22847476](https://doi.org/10.5281/zenodo.22847476).

Ngu12

H. H. Nguyen, *A new approach to an old problem of Erdős and Moser*, J. Combin. Theory Ser. A **119** (2012), 977–993, [doi:10.1016/j.jcta.2012.01.003](https://doi.org/10.1016/j.jcta.2012.01.003).

OEIS

OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, sequence [A360659](https://oeis.org/A360659).

Sár94

A. Sárközy, *Finite addition theorems II*, J. Number Theory **48** (1994), 197–218, [doi:10.1006/jnth.1994.1062](https://doi.org/10.1006/jnth.1994.1062).

SS65

A. Sárközy and E. Szemerédi, *Über ein Problem von Erdős und Moser*, Acta Arith. **11** (1965), 205–208, [doi:10.4064/aa-11-2-205-208](https://doi.org/10.4064/aa-11-2-205-208).