

 Map of *Exact zeros of weighted sums of completely multiplicative functions*

 Number theory · arithmetic functions11A2511K6511L4011P7011R29

17 results

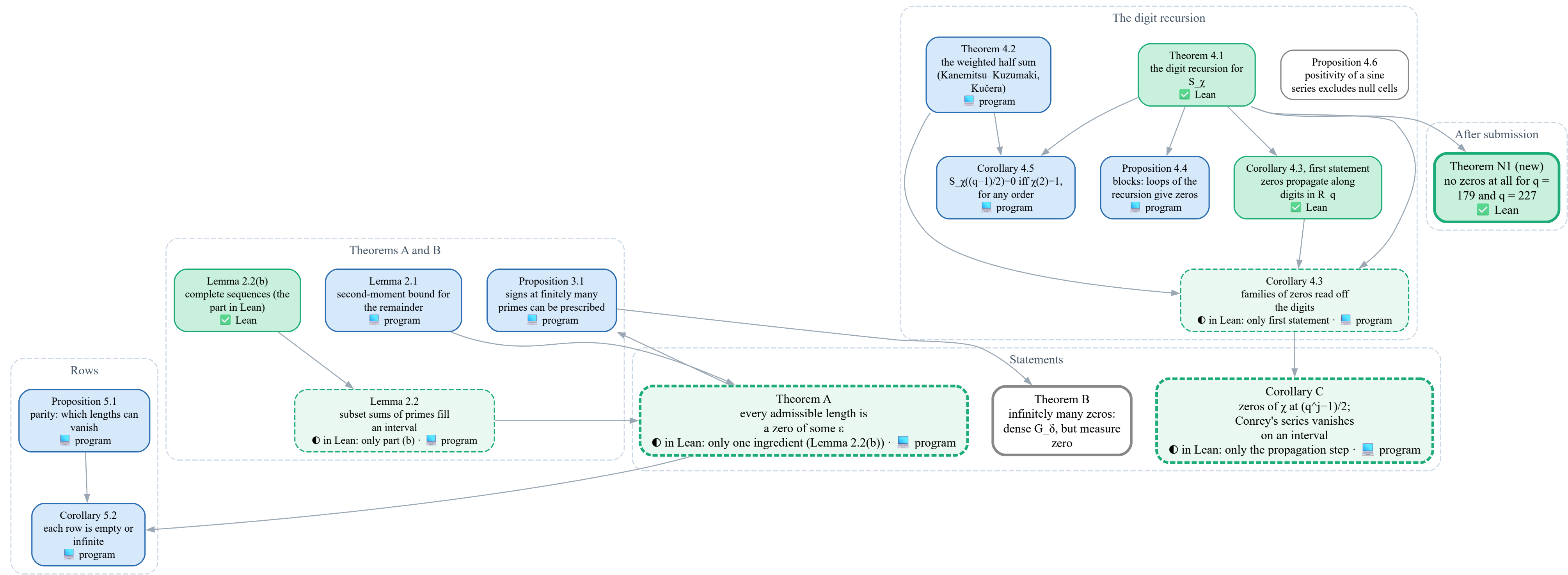
✔ 4 formalized in Lean

🔍 4 with a part in Lean

💻 11 checked by a program

written proof only (a result can carry several tags)

Dependencies in the written proof: an arrow $A \rightarrow B$ means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's “Evidence & scope” block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



What the paper contributes

For a completely multiplicative ε with values ± 1 , the weighted sum $S_\varepsilon(m) = \sum_{n \leq m} n \varepsilon(n)$ can vanish only when $m \equiv 0$ or $3 \pmod{4}$.

- **Every such length is a zero** of some ε , depending on m (Theorem A), even with the signs at finitely many primes prescribed in advance, once m is large (Proposition 3.1).
- **Zeros of one fixed function are rare in measure and generic in category:** the ε with infinitely many zeros form a dense G_δ of probability zero (Theorem B). For the unweighted sums the picture is the reverse: they vanish infinitely often almost surely, by a theorem of Aymone.
- **Quadratic characters give explicit zeros.** For primes $q \equiv 7 \pmod{8}$, $S_\chi((q^j-1)/2) = 0$ for every j , and Conrey's sine series of the Legendre symbol vanishes identically on an interval (Corollary C). The case $j = 1$ is classical (Lebesgue; Aebi–Cairns); the iteration in j and the null interval are the corollary's content. They come from a digit recursion in base q (Theorem 4.1) and the weighted half sum (Theorem 4.2, due to Kanemitsu–Kuzumaki and Kučera).
- **Added after submission:** for $q = 179$ and $q = 227$ the function $\chi_{q,1}$ has no zero at all (Theorem N1), by a finite certificate checked in Lean.

Main tools: a second-moment bound and Sárközy's progressions in subset sums of primes (Theorem A); the anticoncentration theorem of Sárközy and Szemerédi (Theorem B).

 **Start here:** [Theorem A](#): every admissible length is a zero of some ε .

Main results:

- [Theorem A](#): every admissible length is a zero of some ε
- [Theorem B](#): infinitely many zeros: dense G_δ , but measure zero
- [Corollary C](#): zeros of χ at $(q^j-1)/2$; Conrey's series vanishes on an interval
- [Theorem N1 \(new\)](#): no zeros at all for $q = 179$ and $q = 227$



Fields: where the paper lives and what it connects

Tools taken from other branches

- **Additive combinatorics** → arithmetic progressions in subset sums (Sárközy, with Lev's constants in the form of Bringmann–Wellnitz); complete sequences — [Theorem A](#), [Lemma 2.2](#)
- **Probability** → a second-moment bound; the anticoncentration theorem of Sárközy and Szemerédi — [Lemma 2.1](#), [Theorem B](#)
- **Topology (Baire category)** → dense G_δ sets in the product space of sign patterns — [Theorem B](#)
- **Algebraic number theory** → the class number formula, through the weighted half sum (Kanemitsu–Kuzumaki, Kučera) — [Theorem 4.2](#), [Corollary C](#)
- **Formal verification** → Lean 4 + Mathlib: the digit recursion, its propagation step, complete sequences, and the finite certificate of Theorem N1 — [Theorem 4.1](#), [Corollary 4.3](#), [first statement](#), [Theorem N1 \(new\)](#)

What the results serve

- **Probabilistic number theory** ← random multiplicative functions: with the weight n the picture of the unweighted sums (Aymone) is reversed — [Theorem A](#), [Theorem B](#)

- **Analytic number theory (character sums)** ← Conrey's sine series of the Legendre symbol, from his approach to the Riemann hypothesis: it vanishes on an interval for $q \equiv 7 \pmod{8}$; for $q \equiv 3 \pmod{8}$ the null cells are excluded under a positivity hypothesis — [Corollary C](#), [Proposition 4.6](#)
- **Partial sums of Legendre symbols** ← the digit recursion of Borwein–Choi–Coons and of Klurman et al., with the weight added — [Theorem 4.1](#), [Corollary 4.3](#)