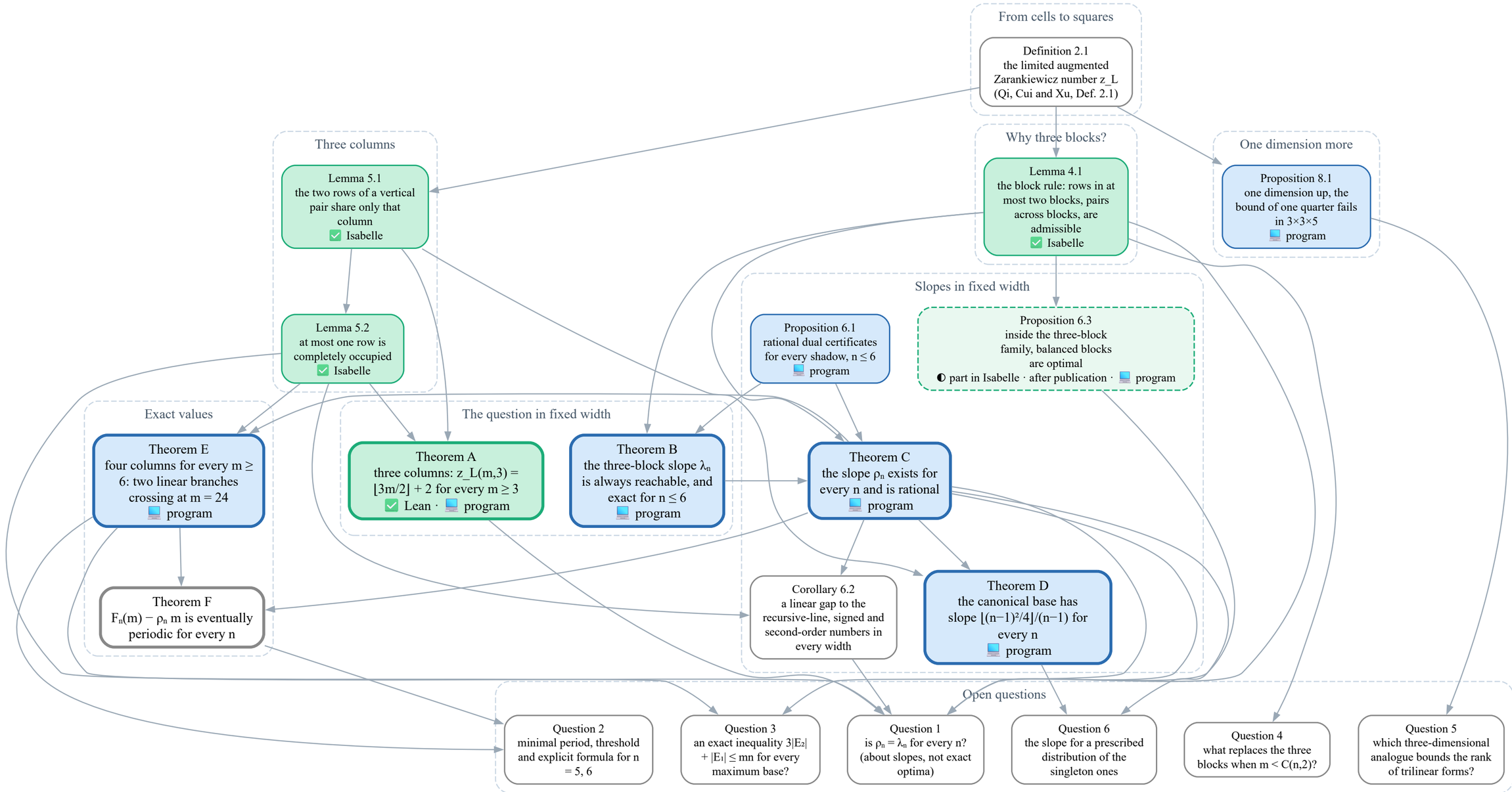
Map of *Why three blocks? The limited augmented Zarankiewicz number in fixed width*

Extremal combinatorics · Zarankiewicz problems · sums of squares of biquadratic forms

20 results | **1** entirely in Lean | **1** partly in Lean | **3** in Isabelle | **8** checked by a program | written proof only | (a result can carry several tags)

Dependencies in the written proof: an arrow A → B means that this proof of B uses A. Every result rests on its written proof; the tags record what was checked besides, and each statement's "Evidence & scope" block says how far. The colour of a box is its strongest tag, in the order above; the tags in the box give the rest. Boxes with a thick border are the main results; click a box to read the statement.



What the paper contributes

The limited augmented Zarankiewicz number $z_L(m,n)$ of Qi, Cui and Xu is a lower bound for the largest sum-of-squares rank of $m\times n$ biquadratic forms. The paper follows one question: with the number n of columns fixed, how many pairs does each new row buy?

- Three columns, exactly: $z_L(m,3)=\lfloor 3m/2\rfloor+2$ for every $m\ge3$ (Theorem A), answering the question of Qi, Löfberg and Chen; proof formalized in Lean. The gap to their weak variant grows like $m/6$.
- Every width: $F_n(m)=z_L(m,n)-z(m,n)=\rho_n m+O_n(1)$ with ρ_n rational, the optimum of finitely many linear programs (Theorem C), and the error is eventually periodic (Theorem F).
- Pairs around a hollow triangle of three column blocks give $\rho_n\ge\lambda_n=n/3-1/2+O(1/n)$; balanced blocks are optimal inside that family (Proposition 6.3), and $\rho_n=\lambda_n$ for $n\le6$ by rational dual certificates checked in exact arithmetic (Theorem B). Equality for every n is a conjecture.
- Consequences and companions: a linear gap to the recursive-line, signed and second-order numbers in every width (Corollary 6.2); the slope $\lfloor(n-1)^2/4\rfloor/(n-1)$ of Lebedev's canonical base for every n (Theorem D); $z_L(m,4)$ for every $m\ge6$ by certified computation (Theorem E); fourteen new exact values; and a three-dimensional analogue where the naive bound of one quarter already fails in $3\times3\times5$ (Proposition 8.1).

The admissibility lemmas behind the constructions are formalized in Isabelle/HOL (submitted to the Archive of Formal Proofs). Version 1 on Zenodo: doi:10.5281/zenodo.23778502.

🚀 **Start here: Theorem A**: three columns: $z_L(m,3) = \lfloor 3m/2 \rfloor + 2$ for every $m \geq 3$.

Main results:

- **Theorem A**: three columns: $z_L(m,3) = \lfloor 3m/2 \rfloor + 2$ for every $m \geq 3$
- **Theorem B**: the three-block slope λ_n is always reachable, and exact for $n \leq 6$
- **Theorem C**: the slope ρ_n exists for every n and is rational
- **Theorem D**: the canonical base has slope $\lfloor (n-1)^2/4 \rfloor/(n-1)$ for every n
- **Theorem E**: four columns for every $m \geq 6$: two linear branches crossing at $m = 24$
- **Theorem F**: $F_n(m) - \rho_n m$ is eventually periodic for every n

🧭 Fields: where the paper lives and what it connects

Tools taken from other branches

- **Extremal graph theory** → Turán's complete tripartite graph as the model of the three blocks; Lebedev's three-block construction — Lemma 4.1, Theorem B, Proposition 6.3
- **Linear programming and duality** → one linear program per shadow of triples, rational dual certificates checked in exact arithmetic — Proposition 6.1, Theorem C, Theorem E
- **Simple games** → the critical threshold of graphical simple games: its sharp bound $\lfloor k^2/4 \rfloor / k$ is known (Hof 2016; Hof, Kern, Kurz and Paulusma 2018); the paper reproves it with the minimal-norm argument of Hof, Kern, Kurz, Pashkovich and Paulusma — Theorem D
- **Presburger arithmetic** → semilinear sets (Ginsburg–Spanier) applied to the exact profile system — Theorem F
- **Formal verification** → Lean 4 + Mathlib for the three-column theorem; Isabelle/HOL for the admissibility lemmas — Theorem A, Lemma 4.1, Lemma 5.1, Lemma 5.2

What the results serve

- **Sum-of-squares rank of biquadratic forms** ← lower bounds $\mathrm{BSR} \geq z_A \geq z_L$ in fixed width; the gap to the relaxed criteria — Theorem A, Corollary 6.2
- **Higher-dimensional Zarankiewicz problems** ← the box-graph analogue and its first counterexample — Proposition 8.1