A divisor-sum criterion for Kassel–Reutenauer polynomial divisibility modulo odd primes

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Abstract

We answer the question posed by J. M. Rodríguez Caballero on when $p \mid \sigma(n)$ fails to imply $[p]_q \mid P_n(q)$ in $\mathbb{F}_p[q]$, where P_n is the Kassel–Reutenauer q-analogue of the divisor sum. Writing $n = 2^e m$ with m odd, we give a finite criterion in terms of the odd power sums of the divisors of m. We also classify the primes for which the implication holds for every n and construct infinitely many counterexamples for every other prime. Although infinite, the exceptional integers have natural density zero. Exact polynomial calculations in Sage and a separate divisor-sum implementation check the formulas; the general statements rest on the proofs below.

1 Statement

Write $\sigma_s(m) = \sum_{d \mid m} d^s$ and $[p]_q = 1 + q + \cdots + q^{p-1}$. We use the polynomial $P_n(q)$ of Kassel and Reutenauer in the form recorded by Rodríguez Caballero [1, 2]. His defining sum over all $d \mid n$ reduces to odd divisors: the even-divisor terms cancel in pairs under $d \mapsto 2n/d$, since $\gamma(2n/d) = 1 - \gamma(d)$.

Theorem 1 (Exact criterion)**.** *Let p be an odd prime and $n = 2^e m$, where m is odd. Put $r = 2^{e+1}$. Then*

$$[p]_q \mid P_n(q) \quad \text{in } \mathbb{F}_p[q] \quad \Longleftrightarrow \quad (r^s - 1)\sigma_s(m) \equiv 0 \pmod{p} \quad \text{for every odd } s \in [1, p-2]. \tag{1}$$

The condition at $s = 1$ is precisely $p \mid \sigma(n)$. If it holds, the first failed condition at odd index s is the first nonzero Hasse derivative of P_n at $q = 1$, of order $s - 1$.

Thus the integers requested in Caballero's question [3] are exactly those satisfying the $s = 1$ congruence but failing at least one of $s = 3, 5, \ldots, p-2$. The test uses only the factorization of n.

Theorem 2 (Primes with no exceptions)**.** *For an odd prime p, the implication*

$$p \mid \sigma(n) \quad \Longrightarrow \quad [p]_q \mid P_n(q) \text{ in } \mathbb{F}_p[q]$$

holds for every positive integer n if and only if

$$2^\ell \equiv 1 \pmod{p} \qquad \text{for every odd prime } \ell \mid p - 1. \tag{2}$$

Equivalently, $p - 1$ is a power of 2, or it has a unique odd prime factor ℓ and $\operatorname{ord}_p(2) = \ell$. If (2) fails for ℓ, there are infinitely many exceptions of the form $n = a^{\ell-1}$ with a prime of order ℓ modulo p.

For example, $p = 5, 7, 17, 23$ have no exceptions. At $p = 11$, the example $n = 81$ from [1] first fails at $s = 5$: $D^4 P_{81}(1) = 2$ in $\mathbb{F}_{11}$. At $p = 13$, already $n = 9$ fails at $s = 3$, with $D^2 P_9(1) = 9$ in $\mathbb{F}_{13}$.

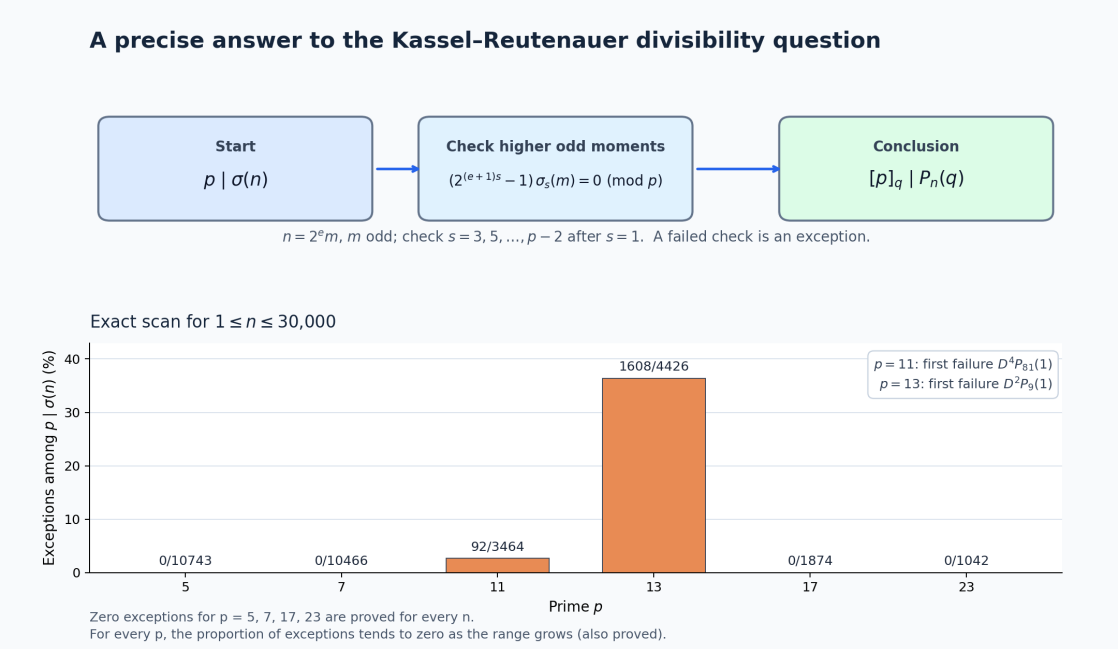


Figure 1: The criterion and exact counts up to $n = 30{,}000$. Zero exceptions for $p = 5, 7, 17, 23$ follow from Theorem 2 for all n, not from the finite scan.

2 Proof of the exact criterion

In characteristic p, $q^p - 1 = (q-1)^p$, so $[p]_q = (q-1)^{p-1}$. For each odd $d \mid m$ set $M_d = 2n/d - d = rm/d - d$ and $\gamma(d) = (M_d+1)/2$. In the truncated power-series ring $\mathbb{F}_p[[u]]/(u^{p-1})$ we can substitute $q = \exp(u)$: all factorials needed have index below p. The odd-divisor formula becomes

$$e^{(1-n)u} P_n(e^u) = \sum_{d \mid m} \frac{\sinh(M_d u/2)}{\sinh(u/2)}. \tag{3}$$

The quotient on the right is even in u. For odd $s \le p-2$, the coefficient of u^{s-1} is a triangular linear combination of $S_1, S_3, \ldots, S_s$, where $S_t = \sum_{d \mid m} M_d^t$. Its coefficient at S_s is $1/(2^{s-1}s!)$, a unit in $\mathbb{F}_p$. Since $e^u - 1$ is u times a unit, $[p]_q \mid P_n(q)$ is equivalent to $S_s = 0$ for all these odd indices.

Expanding the odd power of M_d and pairing terms with complementary binomial indices gives

$$S_s = \sum_{j=0}^{(s-1)/2} (-1)^j \binom{s}{j} r^j m^j (r^{s-2j} - 1)\sigma_{s-2j}(m). \tag{4}$$

Here we used $\sum_{d \mid m} d^{-t} = m^{-t}\sigma_t(m)$ for $t > 0$. The $j = 0$ term is $(r^s - 1)\sigma_s(m)$, while all other terms involve smaller odd indices. Triangular induction proves (1). Finally $\sigma(n) = (2^{e+1} - 1)\sigma_1(m) = (r-1)\sigma_1(m)$. The same triangular argument, together with the invertible changes of variable and prefactor in (3), identifies the first nonzero Hasse derivative. □

3 Proof of the classification

Suppose (2) holds and $p \mid \sigma(n)$. For each odd $s \le p-2$, either $\gcd(s, p-1) = 1$ or an odd prime ℓ divides both s and $p-1$. In the second case $r^s = (2^\ell)^{(e+1)s/\ell} = 1$ modulo p. In the first case, $\sigma_1(m) = 0$ implies $\sigma_s(m) = 0$ unless $r = 1$, when the criterion holds already. To see the implication, some prime-power factor $1 + a + \cdots + a^b$ of $\sigma_1(m)$ vanishes in $\mathbb{F}_p$. If $a = 1$, then $p \mid b+1$ and the corresponding factor of $\sigma_s(m)$ also vanishes. If $a \ne 1$, then $a^{b+1} = 1$; because $\gcd(s, p-1) = 1$, we have $a^s \ne 1$ and $(a^s)^{b+1} = 1$, so that factor again vanishes. Theorem 1 now applies.

Conversely, choose an odd prime $\ell \mid p-1$ such that $2^\ell \neq 1$ modulo p. Dirichlet's theorem supplies infinitely many primes a in a residue class of multiplicative order ℓ modulo p. Put $n = a^{\ell-1}$; then $e = 0$ and $r = 2$. For every odd $s < \ell$, $\sigma_s(n) = 0$ modulo p, since a^s has order ℓ. But $\sigma_\ell(n) = \ell \neq 0$ modulo p, and $r^\ell - 1 \neq 0$. Therefore n is an exception and its first failed moment is $s = \ell$. □

4 Sparsity, computation and a further question

Corollary 1 (Sparsity)**.** *For every odd prime p, the integers n with $p \mid \sigma(n)$ but $[p]_q \nmid P_n(q)$ have natural density zero. In fact, $p \mid \sigma(n)$ itself holds for a density-one set of integers.*

Proof. If an odd prime $a \equiv -1 \pmod{p}$ divides n to exponent exactly one, then $1 + a^s = 0$ modulo p for every odd s. Consequently $\sigma_s(m) = 0$ for all odd s, and Theorem 1 applies, while $p \mid \sigma(n)$. For a finite set A of primes $a \equiv -1 \pmod{p}$, the density of integers with no $a \in A$ occurring to exponent exactly one is $\prod_{a \in A}(1 - a^{-1} + a^{-2})$. This product tends to zero as A grows because $\sum_{a \equiv -1 \,(\mathrm{mod}\ p)} a^{-1}$ diverges. The complement of the good set therefore has density zero. □

The source package constructs P_n independently in $\mathbb{Z}[q]$, checks $P_n(1) = \sigma(n)$, symmetry and nonnegative coefficients, and compares polynomial divisibility in Sage with Theorem 1 in 1055 exact cases. A second program checks all $n \leq 30{,}000$ for $p = 3, 5, 7, 11, 13, 17, 23$; the counts in Figure 1 are finite observations, whereas the two theorems are general proofs.

The criterion decides each n and the corollary gives $E_p(X) = o(X)$, but a quantitative problem remains. For a fixed prime p that admits exceptions, determine the asymptotic behaviour of

$$E_p(X) = \#\{n \leq X : p \mid \sigma(n),\ [p]_q \nmid P_n(q)\},$$

including the contribution of each first failed Hasse derivative. This asks how often each arithmetic obstruction actually occurs.

Disclosure. AI assistance was used in developing the proof, code and figure. The argument was checked against independent exact implementations; no external human referee report is claimed.

References

[1] J. M. Rodríguez Caballero, "On Kassel–Reutenauer q-analog of the sum of divisors and the ring $\mathbb{F}_3[X]/X^2\mathbb{F}_3[X]$," *Finite Fields Appl.* **51** (2018), 183–190. doi:10.1016/j.ffa.2018.01.010.

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[3] J. M. Rodríguez Caballero, "Post Riemann Hypothesis Number Theory," Mathstodon, 26 November 2026. https://mathstodon.xyz/@josephcmac/117683681364815836.