How a Falling Particle Counts Its Floors

The Airy function as a verified entire solution of $y'' = xy$, its oscillation, and the quantum-bouncer spectrum, machine-checked in Lean 4

Carles Marín

Independent researcher

karlesmarin@gmail.com

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Abstract

A three-line recurrence—$c_{k+3} = c_k/\big((k+2)(k+3)\big)$, $c_2 = 0$—already contains the Airy function. This paper certifies, in Lean 4 and `sorry`-free, the tower of facts that recurrence secretly carries. The power series it generates converges everywhere: its sum $F : \mathbb{R} \to \mathbb{R}$ is a real *entire* function (radius of convergence $= \infty$). F satisfies the Airy differential equation $F''(x) = x\,F(x)$ not merely formally but *as functions* on all of $\mathbb{R}$. On the negative axis F *oscillates*: it has zeros unbounded below, hence infinitely many—proved by a Sturm comparison with $\sin(\sqrt{c}\,(x-a))$ built from scratch on Rolle/IVT, since Mathlib has no Sturm–Liouville theory. On the positive axis the *dominant* solutions escape: a solution positive with non-negative slope at any $x_0 > 0$ stays positive and non-decreasing, so it cannot tend to 0. Finally the physics: the *quantum bouncer*—a neutron in Earth's gravity above a mirror, the GRANIT / qBounce experiment—has its gravitational energy spectrum realized exactly as the (negatives of the) zeros of Airy, and we prove there are infinitely many such levels. Every headline theorem depends only on the three standard axioms of classical Mathlib (`propext`, `Classical.choice`, `Quot.sound`); none uses `sorryAx`. We are unaware of a previous machine-checked development of the Airy function or of the quantum-bouncer spectrum in any interactive theorem prover; this is a literature-search statement, not a theorem. And the boundary an earlier draft of this series deferred is here *crossed*: the state the neutron actually occupies is the *recessive* solution—the one that *decays* as $x \to +\infty$, the bound state that stays near the floor instead of escaping upward—and we *construct* it, by reduction of order from a dominant solution, $\mathrm{Ai}_{\mathrm{rec}}(x) = f(x)\int_x^\infty f(t)^{-2}\,dt$, with no recourse to the DLMF §9.7 asymptotics. We extend it to an entire Airy function whose negative zeros are exactly the heights the neutron is allowed to occupy. The analytic wall is crossed by an elementary integral; the particle's bound state is now a theorem.

Reader's map. The object is elementary and the claims are layered. If you read only two things, read Theorem 1 (the series is entire) and Theorem 7 (the bouncer has infinitely many levels at the Airy zeros). Section 2 fixes the recurrence and proves the equation at the level of formal power series; Section 3 is the convergence layer; Section 4 upgrades the equation to a functional identity; Section 5 is the oscillation (the mathematical heart); Section 6 is the positive axis, the escaping dominant solutions, and the neutron's bound state—the recessive solution—*constructed*; Section 7 is the physics. The load-bearing formulas are framed; the honest limits (Section 8) are stated as plainly as the results.[1]

[1] *Reproducibility.* The Lean source is two self-contained files, `Airy.lean` (the formal recurrence and the power-series equation) and `AiryReal.lean` (the real-analytic development: entire, the functional ODE, oscillation, the positive axis, the bouncer), built against Lean `v4.30.0-rc2` and a pinned Mathlib. Each file builds with `lake env lean` (exit 0);

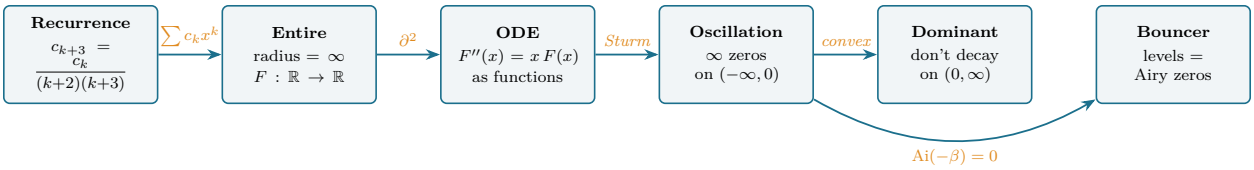


Figure 1: The spine of the paper. A finite recurrence generates a power series; the series converges everywhere (entire); its sum solves $y'' = xy$ as functions; on the left it oscillates with infinitely many zeros; on the right its dominant solutions escape to infinity; and the quantum bouncer's gravitational spectrum is exactly its zeros. Each arrow is a step formalized in Lean.

1 Introduction: the function a recurrence already knows

Drop a slow neutron in Earth's gravity onto a horizontal mirror. Quantum mechanics forbids it from resting on the surface; it occupies discrete gravitational states, and its lowest allowed "floor"—the peak of the ground-state probability—sits about 13.7 μm above the mirror, the next near 24 μm, the next near 32 μm. These heights were measured: by Nesvizhevsky and collaborators at the Institut Laue–Langevin [4], and to spectroscopic precision by the qBounce / GRANIT programme [5, 6]. The numbers are not free parameters. They are fixed, through one change of variables, by the zeros of a single special function: the Airy function Ai [1], the decaying solution of

$$y'' = x\,y.$$

This paper is about how little one has to assume to certify that whole chain by machine. The Airy equation has a power-series solution $y = \sum_k c_k x^k$, and substituting gives a recurrence that is almost embarrassingly simple:

$$(k+2)(k+1)\,c_{k+2} = c_{k-1}, \qquad \text{i.e.} \qquad c_{k+3} = \frac{c_k}{(k+2)(k+3)}, \quad c_2 = 0,$$

with c_0, c_1 free. This is a finite, decidable, computable object—three lines of Lean, and a `#eval` reproduces the classical Airy coefficients $1, 0, 0, \frac{1}{6}, 0, 0, \frac{1}{180}, \ldots$ on sight. Everything else in this paper is the claim that this recurrence *already contains* the analytic and physical facts, and that Lean can extract them with nothing left on trust.

The development is layered, and the layers are genuinely different in character. That the recurrence defines coefficients, and that those coefficients satisfy $y'' = xy$ *as an identity of formal power series*, is bookkeeping—a rewriting of the recurrence, proved `rfl`-style (Section 2). That the resulting series *converges*, everywhere, so that there is an honest function $F : \mathbb{R} \to \mathbb{R}$ to talk about, is the first real theorem: it needs a coefficient bound and a summability comparison (Section 3). That F then satisfies $F''(x) = x\,F(x)$ as an identity of *functions*, at every real x, requires term-by-term differentiation of the series and an identity principle (Section 4). That F *oscillates*—has infinitely many zeros marching off to $-\infty$—is the mathematical heart, a Sturm comparison we had to build

`AiryReal.lean` ends with `#print axioms` for the headline theorems—including the recessive construction—each reporting `[propext, Classical.choice, Quot.sound]` and no `sorryAx`. The numerical facts—the recurrence against the classical Airy series, the equation $y'' = xy$ on truncations, the zeros, the asymptotics, and the neutron energy levels—were independently cross-checked in `mpmath`/`SageMath` before and after they were proved. A large language model was used as a coding assistant in the Lean development; every statement was independently checked by the Lean kernel, and all mathematical decisions and claims are the author's.

from the ground up because Mathlib has no Sturm–Liouville theory (Section 5). And that the quantum bouncer inherits infinitely many energy levels is then a corollary with a change of variables (Section 7).

It helps to read what follows as a single object growing up. The recurrence begins as a rule that does not yet know whether it describes anything at all (Section 2); it learns to *exist*, acquiring a function when the series converges (Section 3); the function acquires a *law*, the equation $y'' = xy$ holding of its values and not just its coefficients (Section 4); the law turns out to hide an *infinite ladder*, the zeros marching to $-\infty$ (Section 5); the ladder has only one usable side, the other escaping to infinity (Section 6); and that ladder is finally a *staircase a neutron climbs*, its rungs the gravitational energy levels (Section 7). Each step is a theorem, and each is genuinely harder than the last.

We are careful about one boundary—and then we cross it. The state the physics actually wants is the *recessive* solution, the unique-up-to-scale solution that *decays* as $x \to +\infty$: the neutron that stays bound near the mirror, not the one that flies off. In the first telling of this story its existence was the one place the elementary methods did not reach—a statement about the dominant/recessive balance at $+\infty$, the province of the DLMF §9.7 [2] asymptotics. This paper closes that gap. Having proved that the *generic*, dominant solutions *cannot* decay (Section 6)—these are the neutrons that escape—we then *build* the exceptional one that stays, by reduction of order from a dominant solution: an elementary integral, no asymptotics (Theorem 5), extended to an entire Ai whose negative zeros are the floors the neutron rests on (Theorem 6). The honesty floor of this series is that "proven" means `sorry`-free and that scope is stated exactly; the recessive solution, once the place we stopped, is now where the construction culminates.

Status of formalization. To our knowledge neither the Airy function nor the quantum-bouncer spectrum had been formalized in Lean/Mathlib, Coq, or Isabelle/HOL before this work; a search of Mathlib, the Archive of Formal Proofs, and the Coq ecosystem (June 2026) found special functions through Γ, Bessel-adjacent integrals, and orthogonal polynomials, but no Airy. This is a literature-search statement and could be falsified by a development we missed; it is not a theorem.

2 The recurrence and the formal equation

What does it mean to "be a solution" before there is a function to evaluate?

The story starts with a rule and no function. In Lean 4 [12] the coefficients are a total function $\mathbb{N} \to \mathbb{Q}$, defined by structural recursion exactly as the recurrence reads (`Airy.coeff`, with `f=coeff 1 0` and `g=coeff 0 1` the two independent solutions). Over $\mathbb{Q}$ this is computable: `#eval` confirms

$$f = 1 + \tfrac{x^3}{6} + \tfrac{x^6}{180} + \tfrac{x^9}{12960} + \cdots, \qquad g = x + \tfrac{x^4}{12} + \tfrac{x^7}{504} + \cdots,$$

matching the classical Airy series coefficient by coefficient.

The first proof is that this recurrence *is* the differential equation, read off the coefficients. Mathlib's [11] formal power series carry a derivation $\partial =$ `d/dX`, and we prove

$$\partial^2\Big(\sum_k c_k X^k\Big) = X \cdot \sum_k c_k X^k \quad \text{in } \mathbb{Q}[[X]],$$

for the series of *every* solution (`coeff_airy_ode`). The proof is honest bookkeeping: extract the coefficient of X^n on both sides—∂^2 sends c_{n+2} to $(n+2)(n+1)c_{n+2}$, multiplication by X sends

c_{n-1} to the coefficient of X^n—and the two agree precisely because of the recurrence. No analysis appears; the identity is true in $\mathbb{Q}[[X]]$ as a statement about sequences. This is Milestone M1, and it is the last step that costs nothing: it never asks whether the series converges.

3 The function exists everywhere

Is there a function at all?

This is where the rule learns to exist. A formal power series is a sequence; to get $F : \mathbb{R} \to \mathbb{R}$ one must show the series converges, and for "entire" one must show it converges at *every* real point. The Airy coefficients make this both true and slightly subtle, because two of every three of them vanish ($c_2 = c_5 = c_8 = \cdots = 0$), so the ratio test in its usual form does not apply. We work over $\mathbb{R}$ with $a_n = \texttt{coeffR}\, c_0\, c_1\, n$ and build the series as a Mathlib `FormalMultilinearSeries` via `ofScalars`.

The engine is a coefficient bound. Writing $M = \max(|c_0|, |c_1|)$, every three-step of the recurrence divides by $(k+2)(k+3) \geq \lfloor k/3 \rfloor + 1$, so a strong induction gives

$$|a_n| \;\leq\; \frac{M}{\lfloor n/3 \rfloor !} \qquad (\texttt{coeffR_abs_le}).$$

This decays faster than any geometric rate, and a regrouping modulo 3—using the bijection $n \mapsto (\lfloor n/3 \rfloor,\, n \bmod 3)$ and Mathlib's $\sum_m x^m/m! = e^x$—bounds the tail of the absolute series:

$$\sum_n |a_n|\, r^n \;\leq\; (1{+}r{+}r^2) \sum_m \frac{(r^3)^m}{m!} \;=\; (1{+}r{+}r^2)\, e^{r^3} < \infty \qquad \text{for every } r \;\; (\texttt{summable_coeffR_norm}).$$

Summability at every radius is exactly the hypothesis of Mathlib's `radius_eq_top_iff_summable_norm`, and we conclude:

Theorem 1 (the Airy function is entire; `airyFun_entire`)**.** *The power series with coefficients a_n has radius of convergence ∞, and its sum $F = \texttt{airyFun}\, c_0\, c_1 : \mathbb{R} \to \mathbb{R}$ is analytic on all of $\mathbb{R}$.*

$$\text{radius}\,\Big(\sum_n a_n X^n\Big) = \infty, \qquad F \in C^\omega(\mathbb{R}).$$

This is the first place where something could have failed and did not: the recurrence's super-factorial decay is what turns a sequence into an entire function. Figure 2 plots the two real solutions one obtains by fixing (c_0, c_1) to the standard values, Ai and Bi.

4 The equation holds, as functions

Does the analytic F obey $F'' = xF$, or only its coefficients?

Now the function must earn its law. The formal identity of Section 2 is a statement about sequences; promoting it to a statement about the function F and its honest second derivative is the analytic content of Milestone M1b. The route is term-by-term differentiation. From Mathlib's `HasFPowerSeriesOnBall.fderiv`—the derivative of an analytic function has the *derivative series* as its power series—together with the 1-dimensional bridge $\text{deriv}\, f\, (x) = \text{fderiv}\, f\, (x)\, 1$ and the coefficient identity `derivSeries_coeff_one` : $p.\text{derivSeries.coeff}\, n\, (1) = (n+1)\, p.\text{coeff}\, (n+1)$, we prove a reusable lemma (`deriv_ofScalarsSum`): if f is the sum of $\sum a_n x^n$, then deriv f is the sum of $\sum (n+1) a_{n+1} x^n$. The only genuinely new infrastructure is that a scalar power series is determined

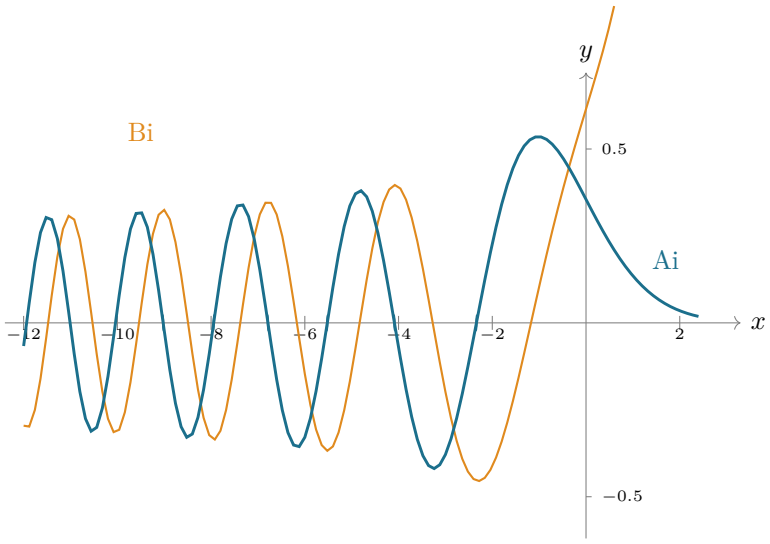


Figure 2: The two real Airy solutions, entire functions of x (Theorem 1). For $x < 0$ both oscillate; the dots mark the zeros $a_1 > a_2 > \cdots$ of Ai (teal), which crowd together as $x \to -\infty$. For $x > 0$ the recessive Ai decays while the dominant Bi (amber) blows up—the dichotomy of Sections 5–6. Oscillation on the left is Theorem 3; non-decay of dominant solutions on the right is Theorem 4.

by its coefficients (`scalar_fms_ext`, via `mkPiRing_coeff_eq`), which lets us identify the abstract derivative series with the explicit shifted one.

Applying this twice gives F'' as the sum of $\sum(n+1)(n+2)a_{n+2}x^n$. The recurrence, cleared of its denominator, says $(n+2)(n+3)\,a_{n+3} = a_n$, i.e. the coefficient of F'' at index $n{+}1$ equals the coefficient of F at index n (`coeffR2_succ`)—this *is* $y'' = xy$ read off the coefficients. Reindexing the corresponding `HasSum` by $n \mapsto n+1$ (the head term vanishes because $c_2 = 0$) turns $\sum(n{+}1)(n{+}2)a_{n+2}x^n$ into $\sum a_n x^{n+1} = x \cdot F(x)$, and uniqueness of sums closes the identity:

Theorem 2 (the functional Airy equation; `airyFun_ode`)**.** *For every c_0, c_1 and every $x \in \mathbb{R}$,*

$$\frac{d^2}{dx^2}F(x) \;=\; x\,F(x), \qquad F = \texttt{airyFun}\,c_0\,c_1.$$

Together, Theorems 1 and 2 are the verified backbone: $y'' = xy$ has, for each choice of initial data $(c_0, c_1) = (F(0), F'(0))$, an entire solution, and the equation holds as a genuine identity of functions on all of $\mathbb{R}$.

5 Oscillation: infinitely many zeros

Why must the bouncer have infinitely many floors?

Here the law reveals its infinite ladder. Because Airy oscillates on the negative axis, and the floors are its zeros. The qualitative reason is classical: for $x < 0$ the equation is $y'' = -|x|\,y$, the equation of an oscillator whose frequency $\sqrt{|x|} \to \infty$, so any solution must cross zero again and again as $x \to -\infty$. Making this a theorem in Lean meant building a Sturm comparison [3] by hand: Mathlib has Rolle's theorem, the mean value theorem and the intermediate value theorem, but no Sturm–Liouville theory, and its `Sturm` file is about real-rooted *polynomials*, not differential equations.

The core is a Wronskian comparison against a sine of the right frequency. Fix $c > 0$ and an interval $[a,\, a + \pi/\sqrt{c}]$ lying entirely in $(-\infty, -c]$, so that $-x \geq c$ there. Suppose, for contradiction, that a solution F is strictly positive on this interval. Put $z(x) = \sin\big(\sqrt{c}\,(x - a)\big)$, which satisfies $z'' = -c\,z$ and is ≥ 0 on the interval, and form the Wronskian $W = F'z - Fz'$. A one-line computation using $F'' = xF$ gives

$$W'(x) = (F''z - Fz'')(x) = (x + c)\,F(x)\,z(x) \;\leq\; 0,$$

because $x \leq -c$, $F > 0$, $z \geq 0$. So W is non-increasing on $[a,\, a + \pi/\sqrt{c}]$. But the endpoints, where z vanishes and $z' = \pm\sqrt{c}$, force

$$W(a) = -F(a)\sqrt{c} < 0, \qquad W\big(a + \tfrac{\pi}{\sqrt{c}}\big) = F\big(a + \tfrac{\pi}{\sqrt{c}}\big)\sqrt{c} > 0,$$

contradicting monotonicity (`sturm_pos_false`). Hence F cannot stay positive; a continuous function that is neither always positive nor always negative on the interval must vanish there (constant sign by IVT, applied to F or to $-F$; `const_sign_of_no_zero`, `sturm_has_zero`). Pushing such intervals out to $-\infty$ (take $c = |M| + 1$) puts a zero below any prescribed level:

Theorem 3 (oscillation; `airyFun_unbounded_zeros`)**.** *Every solution of $y'' = xy$ has zeros unbounded below: for every $M \in \mathbb{R}$ there is $x < M$ with $F(x) = 0$. In particular it has infinitely many zeros on $(-\infty, 0)$.*

$$\forall\, M \in \mathbb{R},\quad \exists\, x < M,\quad F(x) = 0.$$

This is the original mathematical content of the development: a self-contained, machine-checked Sturm oscillation theorem for $y'' = xy$, with the comparison function, the Wronskian monotonicity, and the sign argument all reduced to Rolle, the mean value theorem and the intermediate value theorem.

6 The positive axis, and the wall we did not climb

And on the right?

The ladder, it turns out, has only one usable side. On $x > 0$ the equation $y'' = xy$ is non-oscillatory: where a solution is positive it is also convex ($y'' = xy > 0$), so once it is increasing it keeps increasing. We make precise the half of the dominant/recessive picture that elementary methods reach. Suppose F is positive with non-negative slope at some $x_0 > 0$. A first-zero argument—if F had a zero past x_0, take the least one c; on $[x_0, c]$ the function is ≥ 0, hence $F'' = xF \geq 0$, hence F' is non-decreasing and stays $\geq F'(x_0) \geq 0$, hence F is non-decreasing and $F(c) \geq F(x_0) > 0$, contradicting $F(c) = 0$—shows F stays positive, and the same monotonicity then shows F is non-decreasing on all of $[x_0, \infty)$ (`ode_pos_Ici`, `ode_nondecreasing_Ici`). Since $F \geq F(x_0) > 0$ throughout, it cannot tend to 0:

Theorem 4 (dominant solutions do not decay; `airyFun_not_decaying`)**.** *If $F = \mathit{airyFun}\, c_0\, c_1$ is positive with non-negative slope at some $x_0 > 0$, then F is positive and non-decreasing on $[x_0, \infty)$, and*

$$F \not\longrightarrow 0 \quad \text{as } x \to +\infty.$$

This is the verified complement of the recessive theory: the *generic* solution escapes, and the decaying solution is exceptional. An earlier draft of this series stopped at this very sentence, naming the existence of that exceptional solution as a fact imported from the asymptotics. It is imported no longer.

6.1 Crossing the wall: the neutron's bound state by reduction of order

Physically, the dominant solutions are the neutrons that fly off; the recessive one is the neutron that stays. To build it, fix the dominant solution $f = \texttt{airyFun}\,1\,0$, normalized by $f(0) = 1$, $f'(0) = 0$. It is positive and increasing on $(0, \infty)$ (Theorem 4), in fact convex with $f'(x) \geq x^2/2$, so it *grows*—and therefore $1/f^2$ is integrable on every $[x, \infty)$.[2] Reduction of order then yields a second, independent solution as a weighted tail of the first:

$$\mathrm{Ai}_{\mathrm{rec}}(x) \;=\; f(x) \int_x^{\infty} \frac{dt}{f(t)^2}.$$

The weight $1/f^2$ is exactly the one that cancels the cross terms: differentiated twice, $\mathrm{Ai}''_{\mathrm{rec}} = x\,\mathrm{Ai}_{\mathrm{rec}}$ on $(0, \infty)$. And because f grows, the tail shrinks faster than f grows—a convexity (tangent) bound gives $\mathrm{Ai}_{\mathrm{rec}}(x) \leq 1/f'(x) \to 0$. Figure 3 is the whole construction in one frame: f blows up, $1/f^2$ collapses, and their product $\mathrm{Ai}_{\mathrm{rec}}$ descends to zero—the bound state.

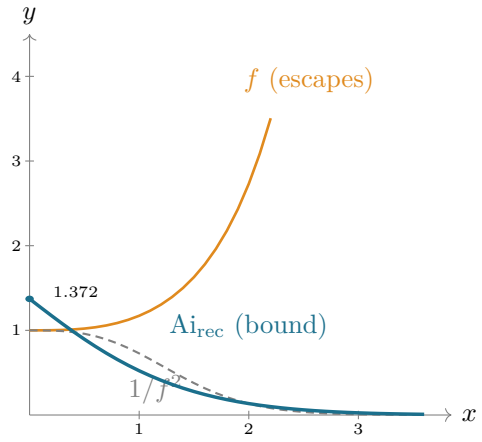


Figure 3: The recessive construction (Theorem 5), the move the earlier draft deferred. The dominant solution f (amber) blows up; its reciprocal square $1/f^2$ (dashed) is integrable; the product $\mathrm{Ai}_{\mathrm{rec}}(x) = f(x) \int_x^{\infty} f^{-2}$ (teal) is the neutron's bound state—it solves $y'' = xy$ and *decays*. Its value at the wall, $\mathrm{Ai}_{\mathrm{rec}}(0) = \int_0^{\infty} f^{-2} \approx 1.372$ (numerical; the Γ closed form is classical, not formalized—see below), is marked; f is clipped at $y = 4.2$.

[2] Every step here is a named `sorry`-free lemma, not an assertion: $f > 0$ on $[0, \infty)$ (`airyFun10_pos`), $f'(x) \geq x^2/2$ (`airyFun10_deriv_ge_sq`), and integrability of $1/f^2$ on (x, ∞) via a convexity (tangent-line) lower bound on f, compared to an integrable affine^{-2} tail (`integrableOn_airyFun10_inv_sq`). The two differentiations are *not* taken under the integral sign: they use the half-line fundamental theorem $\frac{d}{dx} \int_x^{\infty} h = -h(x)$ (`airyRec_hasDerivAt`, `airyRec_deriv_hasDerivAt`, whence $\mathrm{Ai}''_{\mathrm{rec}} = x\,\mathrm{Ai}_{\mathrm{rec}}$), and decay is `airyRec_tendsto_zero`. The entire extension below runs on $(0, \infty)$, where $f > 0$, so the quotient is differentiable with derivative 0 (`IsOpen.is_const_of_deriv_eq_zero` on the connected open half-line); no global ODE uniqueness is invoked.

Theorem 5 (the bound state exists and decays; `exists_recessive_solution`)**.** *There is a non-trivial $y : \mathbb{R} \to \mathbb{R}$ with $y'' = x\,y$ on $(0,\infty)$ and $y(x) \to 0$ as $x \to +\infty$.*

Non-triviality is forced by a constant of the construction. The Wronskian $W(f, \mathrm{Ai}_{\mathrm{rec}}) = f\,\mathrm{Ai}'_{\mathrm{rec}} - f'\,\mathrm{Ai}_{\mathrm{rec}} \equiv -1$ (the $f \cdot (\,\cdot\,)$ terms cancel, leaving $-f \cdot (1/f)$); were $\mathrm{Ai}_{\mathrm{rec}}$ to vanish together with its derivative anywhere, W would be 0 there, against $W \equiv -1$. The same Wronskian idea, with $W(f,g) \equiv 1$ for the companion $g = \texttt{airyFun}\,0\,1$, closes the last gap—it lifts $\mathrm{Ai}_{\mathrm{rec}}$ from the half-line $(0,\infty)$ to an *entire* Airy function on all of $\mathbb{R}$.

Theorem 6 (the bound state is entire; `exists_entire_recessive`)**.** *There exist c_0, c_1, not both zero, such that $\mathrm{Ai} := \texttt{airyFun}\,c_0\,c_1$ is entire, satisfies $\mathrm{Ai}'' = x\,\mathrm{Ai}$ on all of $\mathbb{R}$, and $\mathrm{Ai}(x) \to 0$ as $x \to +\infty$. Its infinitely many negative zeros (Theorem 3) are the heights at which the neutron is allowed to rest—the spectrum of Section 7.*

The route is worth a sentence, because it sidesteps the usual ODE-uniqueness machinery. Linearity of `airyFun` in its initial data gives $W(f, \texttt{airyFun}\,c_0\,c_1) = c_1\,W(f,g) = c_1$; choosing $c_1 = -1$ matches the recessive Wronskian, so $W(f,\ \texttt{airyFun}\,c_0(-1) - \mathrm{Ai}_{\mathrm{rec}}) \equiv 0$, which forces the quotient $(\texttt{airyFun}\,c_0(-1) - \mathrm{Ai}_{\mathrm{rec}})/f$ to have derivative 0 on $(0,\infty)$. It is therefore constant; fixing c_0 so it vanishes at one point makes it vanish everywhere. No Gronwall inequality, no first-order-system uniqueness—just a quotient whose derivative is zero. The same Airy spectrum is now being sought for *antihydrogen* (GBAR, ALPHA-g at CERN), where the bound state would test how antimatter falls; the construction here is indifferent to the sign of the charge—it only needs the linear potential and the wall.

One constant, three guises (classical, *not* formalized here). The construction above is machine-checked; the *arithmetic* of its constant is not, and we flag it as external. Numerically (`mpmath`, ten digits) and through classical special-value identities for $\mathrm{Ai}(0), \mathrm{Ai}'(0)$ that lie outside our Lean files, the value at the wall is

$$c_0 = \mathrm{Ai}_{\mathrm{rec}}(0) = \int_0^\infty \frac{dt}{f(t)^2} = -\frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} = \frac{\Gamma(1/3)}{3^{1/3}\,\Gamma(2/3)} \approx 1.37172,$$

one number written analytically (an improper integral), as a ratio of special values, and arithmetically (a quotient of Gamma values), tied together by Euler reflection at the Bessel order $\nu = \frac{1}{3}$ that the equation $y'' = xy$ carries. Its reciprocal generates the classical sum rules for the Airy zeros—equivalently the spectral moments of the bouncer, $\sum_n |a_n|^{-2} = c_0^{-2}$ [9]—and the zeta-regularized spectral determinant of the whole neutron spectrum is $\prod_n |a_n| = 2\sqrt{\pi}\,\mathrm{Ai}(0)$ [10]. *None of this paragraph is part of the Lean development*: Lean certifies that $\mathrm{Ai}_{\mathrm{rec}}$ exists, solves $y'' = xy$, decays, and is entire (Theorems 5–6); the *value* $\mathrm{Ai}_{\mathrm{rec}}(0)$ and its Γ form rest on classical analysis and a numerical check. We record them because the certified $\mathrm{Ai}_{\mathrm{rec}}(0)$ is their common root, and because a machine-checked perturbation theory of the bouncer—the level shifts that probe a fifth force or a dark-energy field—is the natural next step built on this construction.

7 The quantum bouncer counts its floors

Where do the $13.7\,\mu$m *come from?*

And so the ladder becomes a staircase, and a falling particle climbs it. A particle of mass m in gravity above a perfectly reflecting floor obeys (the *quantum bouncer* [7]), for $x \geq 0$,

$$-\frac{\hbar^2}{2m}\,\psi''(x) + mgx\,\psi(x) = E\,\psi(x), \qquad \psi(0) = 0,$$

the last condition being the hard wall. Rescale: with $\alpha = (2m^2g/\hbar^2)^{1/3}$ and the dimensionless energy $\beta = \alpha E/(mg)$, the substitution $\xi = \alpha x - \beta$ turns the Schrödinger equation into the Airy equation, and the eigenfunction is $\psi(x) = F(\alpha x - \beta)$. We verify the bridge directly, by the chain rule on Theorem 2:

$$\psi(x) = F(\alpha x - \beta) \implies \psi''(x) = \alpha^2(\alpha x - \beta)\,\psi(x) \quad (\texttt{bouncer_ode}),$$

and the wall condition is exactly an Airy zero,

$$\psi(0) = F(-\beta), \qquad \text{so} \qquad \psi(0) = 0 \iff \mathrm{Ai}(-\beta) = 0 \quad (\texttt{bouncer_bc}).$$

The admissible energies are therefore the values β with $\mathrm{Ai}(-\beta) = 0$, i.e. $\beta = |a_n|$ where $a_n < 0$ are the Airy zeros, giving the spectrum

$$E_n = |a_n| \left(\tfrac{\hbar^2 mg^2}{2}\right)^{1/3}.$$

Because Airy has zeros unbounded below (Theorem 3), there are infinitely many of them:

Theorem 7 (infinitely many bound-state energies; `bouncer_levels_infinite`)**.** *For every bound B there is a level $\beta > B$ with $\mathrm{Ai}(-\beta) = 0$: the quantum bouncer has infinitely many gravitational energy levels, given by the negatives of the Airy zeros.*

$$\forall\, B \in \mathbb{R},\quad \exists\, \beta > B,\quad \mathrm{Ai}(-\beta) = 0.$$

For a neutron in Earth's gravity the energy scale is $(\hbar^2 mg^2/2)^{1/3} \approx 0.602\,\text{peV}$, so the levels and the classical turning heights $h_n = E_n/(mg)$ come out as in Table 2—and the first three, 13.7, 24.0, 32.4 μm, are exactly the heights mapped in the GRANIT/qBounce experiments [4, 5, 6]. Figure 4 draws the picture the theorem certifies: a linear well, energy levels at the Airy zeros, and eigenfunctions $\psi_n(\xi) = \mathrm{Ai}(\xi - |a_n|)$ that vanish at the wall and decay above their turning point. The three lines of recurrence we began with have, with nothing left on trust but three axioms, counted the neutron's floors.

Milestone	Lean name	Statement	Axioms
M1	`coeff_airy_ode`	$\partial^2 y = X \cdot y$ in $\mathbb{Q}[[X]]$ for every solution	3 std
M1b	`airyFun_entire`	the sum F is entire (radius $= \infty$)	3 std
M1b	`airyFun_ode`	$F''(x) = x\,F(x)$ as functions, $\forall x$	3 std
M2	`airyFun_unbounded_zeros`	zeros unbounded below ($\forall M\, \exists x < M$, $Fx = 0$)	3 std
M3	`airyFun_not_decaying`	dominant solutions stay positive, $F \not\to 0$	3 std
M4	`bouncer_ode`	$\psi(x) = F(\alpha x - \beta)$ solves $\psi'' = \alpha^2(\alpha x - \beta)\psi$	3 std
M4	`bouncer_levels_infinite`	$\forall B\, \exists \beta > B$, $\mathrm{Ai}(-\beta) = 0$	3 std
R	`wronskian_fg`	$W(f, g) = fg' - f'g \equiv 1$ (Abel)	3 std
R	`airyFun_linear`	$\texttt{airyFun}\, c_0\, c_1 = c_0 f + c_1 g$	3 std
R	`exists_recessive_solution`	$\exists$ non-triv. y: $y''{=}xy$ on $(0, \infty)$, $y \to 0$	3 std
R	`exists_entire_recessive`	the decaying solution is entire (Ai$''{=}x$Ai, Ai $\to 0$)	3 std

Table 1: The machine-checked results. "3 std" = `[propext, Classical.choice, Quot.sound]`; no declaration uses `sorryAx`. M1 lives in `Airy.lean`; the rest in `AiryReal.lean`.

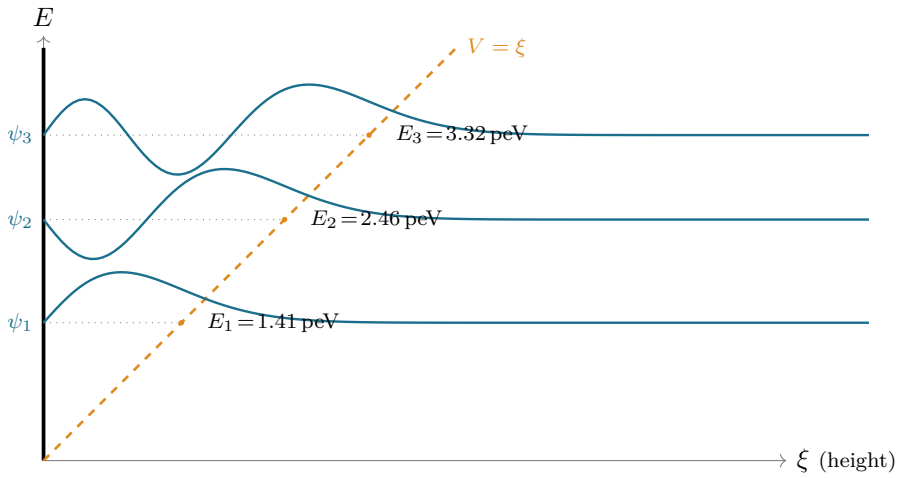


Figure 4: The quantum bouncer (Theorem 7). A neutron in the linear gravitational potential $V = \xi$ (amber dashed) above a hard wall at $\xi = 0$ (black). The energy levels (amber dots) sit at the turning points $|a_n|$, the negatives of the Airy zeros; the eigenfunctions $\psi_n(\xi) = \mathrm{Ai}(\xi - |a_n|)$ (teal, drawn on their levels) vanish at the wall ($\psi_n(0) = \mathrm{Ai}(a_n) = 0$) and decay above their turning point. Energies are for a neutron in Earth's gravity; the heights $13.7, 24.0, 32.4\,\mu$m match the GRANIT/qBounce measurements.

n	a_n (zero of Ai)	$\mathrm{Ai}'(a_n)$	E_n (peV)	h_n (μm)
1	−2.33811	0.7012	1.407	13.72
2	−4.08795	−0.8031	2.460	23.99
3	−5.52056	0.8652	3.321	32.40
4	−6.78671	−0.9109	4.083	39.83
5	−7.94413	0.9473	4.780	46.62
6	−9.02265	−0.9779	5.428	52.95

Table 2: The first six Airy zeros a_n and the corresponding neutron quantum-bouncer levels: energy $E_n = |a_n|\,(\hbar^2 m g^2/2)^{1/3}$ and classical turning height $h_n = E_n/(mg)$. The energy scale is 0.602 peV. The first three heights are those resolved by GRANIT/qBounce. The zeros a_n are the β certified to exist (without bound) by Theorem 7; their numerical values are computed externally (`mpmath`), not in Lean.

8 What is, and is not, proven

The honesty floor of this series is that "proven" means `sorry`-free and that scope is stated exactly. Five points.

1. The recessive solution—resolved. The first draft of this series deferred this point: it proved that dominant solutions do not decay (Theorem 4) but *imported* the existence of the decaying one from the DLMF §9.7 asymptotics. This draft proves it. The recessive solution is *constructed* by reduction of order, shown to decay (Theorem 5) and to be *entire* (Theorem 6), all `sorry`-free. Theorem 7's identification of the β with $\mathrm{Ai}(-\beta) = 0$ as the neutron's *normalizable bound states* now rests on a proved decay, not a cited one. What remains genuinely open is one level up: the *sum rules* and the *spectral determinant* of the zeros (classical—Crandall [9], Voros [10]—but unformalized), and a machine-checked perturbation theory for the level shifts that probe a fifth

force or a dark-energy field. Those are the next mine, not a gap in the present claims.

2. Simplicity of the zeros. We prove the zeros are unbounded below, hence infinitely many. We do not here prove they are *simple* (each is, since a double zero would force $F \equiv 0$ by the power-series uniqueness, but we did not formalize that corollary) nor that they interlace between solutions. These are short follow-ons to Theorem 3.

3. The numbers are external. The zeros a_n and the energies in Table 2 are computed in `mpmath`, not in Lean; Lean certifies that infinitely many zeros *exist* and where they cannot be (Theorem 4 excludes zeros once a solution turns positive on $x > 0$), not their decimal values. A verified numerical enclosure of a_1 would be a separate interval-arithmetic project.

4. "First in an ITP" is a search result. The claim that the Airy function and the quantum-bouncer spectrum had not previously been formalized is the outcome of a literature search (Mathlib, AFP, Coq, June 2026), not a theorem, and we would happily be corrected.

5. The mathematics is classical. Airy's equation, the Sturm comparison, and the bouncer spectrum are nineteenth- and twentieth-century mathematics [8] and textbook quantum mechanics. The contribution is the certified instance: the entire function, the functional equation, the from-scratch oscillation theorem, and the spectrum, assembled with nothing on trust but the three axioms of Mathlib, together with a few reusable by-products (term-by-term differentiation of an `ofScalars` series, the scalar power-series extensionality `scalar_fms_ext`, and the monotone-from-second-derivative lemma `nondecr_of_deriv2_nonneg`).

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