# EVERY $m \times 4$ BIQUADRATIC SUM OF SQUARES IS A SUM OF $3m-1$ SQUARES ($5 \le m \le 16$)

CARLES MARÍN
*Independent researcher*

LIQUN QI
*Zhejiang University of Science and Technology; The Hong Kong Polytechnic University*

CHEN OUYANG
*Dongguan University of Technology*

ABSTRACT. Let $\mathrm{BSR}(m,n)$ be the least $r$ such that every real biquadratic sum of squares in $m+n$ variables is a sum of $r$ squares of bilinear forms. We prove $\mathrm{BSR}(m,4) \le 3m-1$ for $5 \le m \le 16$, one square below the previous bound $3m$. In particular $\mathrm{BSR}(5,4) \le 14$, the upper half of the conjecture $\mathrm{BSR}(5,4) = 14$ proposed by two of the present authors, Qi and Ouyang; with the known lower bounds this gives $13 \le \mathrm{BSR}(5,4) \le 14$ and $16 \le \mathrm{BSR}(6,4) \le 17$. As a corollary, $\mathrm{BSR}(m,5) \le 4m-1$ for every $m \ge 2$, one below the Choi–Lam–Reznick bound $4m$ for $5 \le m \le 10$. The proof studies boundary points of the sets of sums of a fixed number of squares. Minimality of a representation bounds the negative index $q$ of its Hankel multiplier through the dimension of its syzygies: $N(q-1) \le \dim R_2 - \binom{N+2}{2}$ for a boundary of the sums of $N$ squares, a boundary inertia inequality (a transport to this setting of the negative-index bound of Boumal, Voroninski and Bandeira for Burer–Monteiro critical points) whose case $q=1$ recovers the Barvinok–Pataki bound. For $m \times 4$ forms it gives $q \le \lfloor (m+13)/6 \rfloor$, which is at most 4 exactly for $m \le 16$. A classification of isotropic restrictions involving at most four rows and a low-negative-index lemma for $q \le 4$ then exclude the remaining boundary configurations; compression to fewer rows, with the base values $\mathrm{BSR}(k,4) \le 3k-1$ for $k = 2,3,4$, handles the smaller supports; and an explicit inductive family of tuples gives the regular points. The argument first fails at $m = 17$, where the inertia bound allows $q = 5$. The arithmetic, the enumeration of cases and several linear-algebra steps are checked in Isabelle/HOL or by exact computation; these checks cover individual steps, not the whole theorem.

## Results at a glance

- $\mathrm{BSR}(m,4) \le 3m-1$ **for** $5 \le m \le 16$ (Theorem 1), by the boundary strategy of Blekherman, Dunbar and Sinn [1]: one square fewer than the previous bound $3m$; in particular $\mathrm{BSR}(5,4) \le 14$, the upper half of the conjecture $\mathrm{BSR}(5,4) = 14$ proposed by two of the present authors in [13], and $13 \le \mathrm{BSR}(5,4) \le 14$, $16 \le \mathrm{BSR}(6,4) \le 17$ (Figure 1). written proof, pieces formalised
- **Five columns:** $\mathrm{BSR}(m,5) \le 4m-1$ for every $m \ge 2$ (Corollary 2), one below the Choi–Lam–Reznick bound for $5 \le m \le 10$. written proof, arithmetic formalised, chain exact computation
- **A boundary inertia inequality** $N(q-1) \le \dim R_2 - \binom{N+2}{2}$ for the negative index $q$ of the Hankel multiplier at a boundary point of the sums of $N$ squares (Lemma 14; in substance the negative-index bound of Boumal, Voroninski and Bandeira [8], with a self-contained proof here); its case $q=1$ is the Barvinok–Pataki bound, and for $m \times 4$ it gives $q \le \lfloor (m+13)/6 \rfloor$ (Figure 3). written proof, arithmetic formalised

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- **The case analysis** over the support triples $(a, b, r)$: compression for small row support, a four-row isotropic lemma and a low-negative-index lemma for $q \leq 4$ for the rest (Lemmas 17, 20 and Theorem 22; Figure 5). written proof, enumeration formalised, Schur coefficients (L2) in the free algebra exact computation
- **Explicit regular tuples for every** $m$ (Lemma 21) formalised, and a self-contained **boundary selection lemma**, the stratification argument of [1, §2] made self-contained, whose codimension-two hypothesis is sharp (Lemma 9, Figure 4) written proof
- **Toward** $\mathrm{BSR}(5, 4) = 13$**:** a negative-syzygy obstruction for boundary points of $\Sigma_s$ (Proposition 23) written proof, trace step formalised; exact certificates at one rational multiplier of each of 11 subtypes (Appendix B) exact computation. It does not decide 13 versus 14.
- **Where the method stops:** at $m = 17$ the inertia bound allows $q = 5$; slope $n - 1$ cannot persist beyond four columns. **Inputs:** $\mathrm{BSR}(2, 4) \leq 5$, $\mathrm{BSR}(3, 4) \leq 8$, $\mathrm{BSR}(4, 4) \leq 11$ from earlier papers.

formalised pieces in Isabelle/HOL exact computation exact computation written proof proof in the text, read in AI-assisted referee-style rounds (not journal peer review).

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**Companion site.** https://karlesmarin.github.io/blueprints/bsr-m-four/: every statement with its dependencies and its checks, its Isabelle/HOL proof or the script that certifies it. This paper: doi:10.5281/zenodo.24522234 (Zenodo), with its supplementary material.

# 1. The results

**Theorem 1.** *For $5 \leq m \leq 16$, every real $m \times 4$ biquadratic sum of squares is a sum of at most $3m - 1$ squares of real bilinear forms:* $\mathrm{BSR}(m, 4) \leq 3m - 1$.

**Corollary 2.** $\mathrm{BSR}(m, 5) \leq 4m - 1$ *for every $m \geq 2$. For $5 \leq m \leq 10$ this is one below the bound $4m$ of Choi, Lam and Reznick [22], also in the strict form of Blekherman, Sinn, Smith and Velasco* [9, Theorem 2.1].

*Proof.* For $5 \leq m \leq 16$, adding one column (see below) gives $\mathrm{BSR}(m, 5) \leq m + \mathrm{BSR}(m, 4) \leq m + 3m - 1$. For $m \geq 12$, the bound $\Lambda$ of [22] (the largest $r$ with $\binom{r+1}{2} \leq \dim R_2 = 15\binom{m+1}{2}$) is at most $4m - 1$, since $r = 4m$ would need $4m(4m+1) \leq 15m(m+1)$, that is, $m \leq 11$. For

$m = 2, 3, 4$, symmetry gives $\mathrm{BSR}(m, 5) = \mathrm{BSR}(5, m)$, which is 6, 10 and at most 14 by [24], [3] and Theorem 1. □

| $m$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{BSR}(m,4) \le$ | **14** | **17** | **20** | **23** | **26** | **29** | **32** | **35** | **38** | **41** | **44** | **47** |
| previous, $3m$ | 15 | 18 | 21 | 24 | 27 | 30 | 33 | 36 | 39 | 42 | 45 | 48 |
| lower bound | 13 | 16 | 19 | 21 | 24 | 27 | 29 | 32 | 35 | 37 | 40 | 43 |

The previous bounds come from $\mathrm{BSR}(m, 3) = 2m$ [3] by adding one column: $\mathrm{BSR}(m, n) \le m + \mathrm{BSR}(m, n-1)$, because an orthogonal change of summand coordinates puts the coefficients of one column in at most $m$ summands, and the others use only the remaining $n-1$ columns. The lower bounds are the recursive-line Zarankiewicz numbers $z_{RL}(m, 4) \le \mathrm{BSR}(m, 4)$, whose known values are collected in [26]:

| $m$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|
| $z_{RL}(m,4)$ | 13 | 16 | 19 | 21 | 24 | 27 | 29 | 32 | 35 | 37 or 38 |

and $z_{RL}(m, 4) = \lfloor (5m+6)/2 \rfloor$ for $m \ge 15$, by Chen and Chen [14]. For $m = 5, 6, 7$ these are the second-order Zarankiewicz numbers $z_2(m, 4)$ (see [11] for $m = 5, 6$ and Xu and Yan [12] for $m = 7$); the extended number $z_{2M}$ introduced in [6] has the same values [5]. The quadratic persistence of $\mathbb{P}^{m-1} \times \mathbb{P}^3$ gives the weaker bound $\lfloor 8m/3 \rfloor$ [15], equal to $z_{RL}(m, 4)$ for $m = 5, 6, 8, 9, 11, 12, 15$ (and for $m = 14$ if $z_{RL}(14, 4) = 37$). The gap between $3m-1$ and the lower bound is $1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4$ for $m = 5, \ldots, 16$ (at $m = 14$ relative to $z_{RL}(14, 4) \ge 37$, whose exact value is open). The notation BSR and the general bound $\mathrm{BSR}(m, n) \le mn - 1$ were introduced and established, respectively, in [23]; the augmented Zarankiewicz numbers studied in [24] give the earlier lower bounds $z_L(4, 4) = 10$ and $z_L(5, 4) \ge 12$ [24], so $\mathrm{BSR}(4, 4) = 11$ [4] shows that $z_L$ and BSR differ at $4 \times 4$. In [13], two of the present authors (L.Q. and C.O.) discuss $5 \times 4$ as the frontier case. Motivated by the progression $\mathrm{BSR}(m, 4) = 3m - 1$ for $m = 2, 3, 4$, they formulate the conjecture $\mathrm{BSR}(5, 4) = 14$ [13, Conjecture 1] and ask whether $\mathrm{BSR}(5, 4)$ is 13 or 14 [13, Question 1]. Theorem 1 proves the upper half of this conjecture, $\mathrm{BSR}(5, 4) \le 14$, and the upper bound $3m-1$ of the progression for $m \le 16$; it does not settle the exact value.

**The frontier.** The known exact values and bounds form the chain

$$\begin{aligned} \mathrm{BSR}(m, 2) &= m + 1 && [2, 24], \\ \mathrm{BSR}(m, 3) &= z_2(m, 3) = z_{RL}(m, 3) = 2m && [3, 11], \\ \mathrm{BSR}(4, 4) &= z_{2A}(4, 4) = z_{2M}(4, 4) = 11 && [4, 25, 6], \\ z_{2A}(5, 4) = z_2(5, 4) = 13 &\le \mathrm{BSR}(5, 4) \le 14 && [13, 11], \text{ this paper}, \end{aligned}$$

so $5 \times 4$ is the first open case. Together with the known lower bound $\mathrm{BSR}(5, 4) \ge z_{2A}(5, 4) = 13$, our main theorem yields $13 \le \mathrm{BSR}(5, 4) \le 14$, and the exact value remains open. Every form produced by the admissible augmented displays counted by $z_{2A}(5, 4) = 13$ [25, 13] has an explicit representation by at most 13 squares of bilinear forms; so a $5 \times 4$ form that needs 14 squares, if one exists, cannot arise from such a construction. Section 4 discusses the slope of $\mathrm{BSR}(m, 4)$ and partial results toward $\mathrm{BSR}(5, 4) = 13$.

**Method.** We follow the boundary strategy of Blekherman, Dunbar and Sinn [1, §§2–3] in the form of [3, Proposition 3], as in the case $\mathrm{BSR}(4, 4) = 11$ [4]; a supporting functional at a boundary point of the sums of $N$ squares is also Scheiderer's tool for quaternary quartics [21,

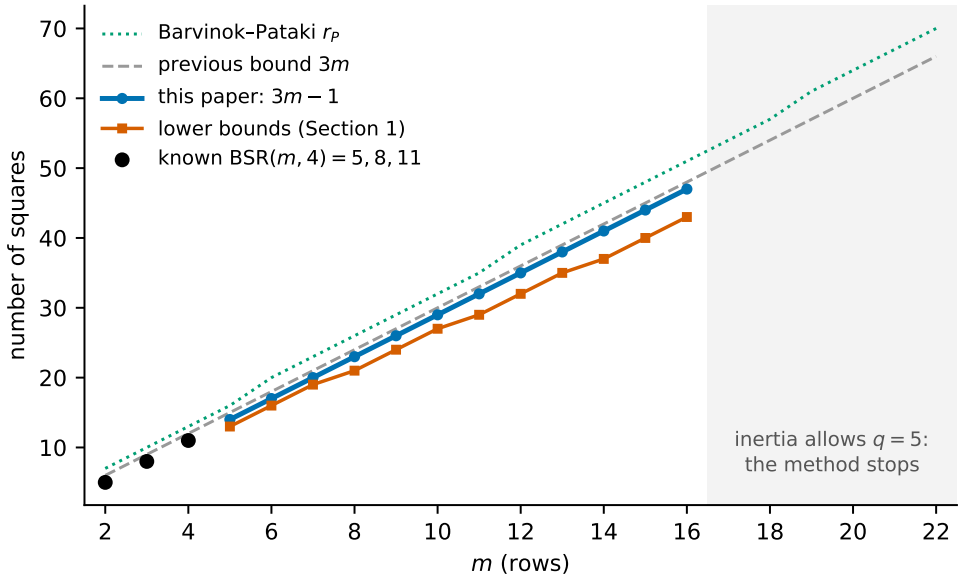



**Figure 1.** Bounds on BSR$(m, 4)$. Blue: Theorem 1; grey dashed: the previous bound $3m$; vermilion: the lower bounds of the table; black: the known values $5, 8, 11$ for $m = 2, 3, 4$, which the induction uses; green dotted: the Barvinok–Pataki rank $r_P$. In the shaded range the inertia bound allows $q = 5$ and the method stops. The gap to the lower bound is 1 for $m = 5, 6, 7$ and grows to 4 at $m = 16$.

§4.6]. Section 2 sets up the boundary method and its algebraic lemmas at $m = 5$; Section 3 adds the boundary inertia inequality, the four-row isotropic lemma and the low-negative-index lemma, which prove the theorem for every $5 \le m \le 16$ with a uniform case analysis. Appendix A gives an alternative proof for $m = 5$ with a case analysis specific to $5 \times 4$; it shares the preliminaries of Section 2.

**How the proof goes.** Put $N = 3m - 1$ and let $\Sigma_N$ be the set of sums of $N$ squares. If $\Sigma_N$ were not the whole cone $\Sigma$, its boundary would meet the interior of $\Sigma$. At a well-chosen boundary point $F$ the supporting hyperplane gives a linear functional $\ell$, and $H(u, v) = \ell(uv)$ is a symmetric form on bilinear forms (a *Hankel multiplier*). Three facts about $H$ drive everything: it kills every summand of every representation of $F$, so $\operatorname{rank} H \le \dim V - N = m + 1$; its image contains no product $\alpha \otimes \beta$, because $F$ has no complex singular point off the real locus (*product-free*); and it is indefinite, with a second-order sign condition along syzygies. The proof then lists the possible triples $(a, b, r)$ of row support, column support and rank of $H$ and excludes each one: small row support by *compression* (most summands live on fewer rows, where induction applies), large row support by *isotropic-row lemmas* (rows of $H$ cannot map into a totally isotropic space) and by counting the syzygies that the sign condition makes nonnegative. The boundary inertia inequality is the count that keeps the negative index $q$ small enough for the isotropic lemmas we can prove.

**The mechanism in three counts.** Three numbers decide where the method works, and why it stops where it does.

(a) *The rank budget.* The summands of a shortest representation span an $N$-dimensional space of bilinear forms killed by $H$, so $\operatorname{rank} H \le \dim V - N = 4m - (3m - 1) = m + 1$. At $N = 3m - 1$ the multiplier is therefore small: its rank grows like $m$, while $V$ has dimension

$4m$, and its image, being product-free, has dimension at most $(a-1)(b-1)$ by Parthasarathy's bound [7].

(b) *The room for negative directions.* Indefiniteness forces one negative direction of $H$; by Lemma 14 each further one costs $N$ dimensions of syzygies, out of a room of $M - \binom{N+2}{2}$. For $m \times 4$ forms and $N = 3m-1$ this room is exactly $m(m+7)/2$, so

$$q \le q_* = 1 + \frac{m(m+7)}{2(3m-1)},$$

which grows by one every six rows (the table below and Figure 3). The isotropic and low-negative-index lemmas handle $q \le 4$, and $\lfloor q_* \rfloor \le 4$ exactly for $m \le 16$. This is where the theorem stops: the first failure, at $m = 17$, is that $q = 5$ becomes possible (remark (iii) below says what a version for $q = 5$ would need).

(c) *The induction line.* Compression puts all but at most $4a - r$ summands on the $m - a$ rows outside the row support, so $\mathrm{SOS}_{\mathbb{R}}(F) \le 4a - r + \mathrm{BSR}(m-a, 4)$. With $r \ge a+1$ this is at most $3m - 2 < N$ precisely when $\mathrm{BSR}(k,4) \le 3k-1$ for $k = m-a$. The known values $5, 8, 11$ for $k = 2, 3, 4$ lie exactly on that line, so the induction closes; it cannot start from $k = 1$, since $\mathrm{BSR}(1,4) = 4 > 2$. That is why the two largest row supports, $a = m-1$ and $a = m$, are excluded by the isotropic lemmas instead of compression (Figure 5).

| $m$ | $N = 3m-1$ | rank budget $m+1$ | room $m(m+7)/2$ | $q_*$ | $\lfloor q_* \rfloor$ |
|---|---|---|---|---|---|
| 5 | 14 | 6 | 30 | $22/7 \approx 3.14$ | 3 |
| 10 | 29 | 11 | 85 | $114/29 \approx 3.93$ | 3 |
| 16 | 47 | 17 | 184 | $231/47 \approx 4.91$ | 4 |
| 17 | 50 | 18 | 204 | $127/25 = 5.08$ | 5 |
| 22 | 65 | 23 | 319 | $384/65 \approx 5.91$ | 5 |
| 23 | 68 | 24 | 345 | $413/68 \approx 6.07$ | 6 |

**What the proof rests on.** Besides the boundary strategy of [1], the proof uses three earlier values: $\mathrm{BSR}(2,4) \le 5$ [2, Theorem 1.1], $\mathrm{BSR}(3,4) = \mathrm{BSR}(4,3) \le 8$ [3] and $\mathrm{BSR}(4,4) \le 11$ [4]. These lie exactly on the line $3k-1$ ($5, 8, 11$ for $k = 2, 3, 4$), which is why the induction on $m$ closes: compression reduces to $\mathrm{BSR}(k,4)$ with $k \ge 2$, and every such value is at most $3k-1$. These three numbers are the only inputs taken from other papers. The two from our notes [3, 4] are published with complete proofs and permanent identifiers; each note has a companion site that maps every statement to its proof and to the parts checked in Isabelle/HOL. The existence of a suitable boundary point (Proposition 10) follows the stratification argument of [1, §2] and [3, Proposition 3]; here it is proved in full through Lemma 9. The rest is proved here. The inertia inequality is, in substance, known (remark (i) below); the bound $\operatorname{rank} H \le (a-1)(b-1)$ is Parthasarathy's (remark (ii)).

**What is checked by machine.** The proof uses no numerical optimization or finite search. Its displayed computations C1–C12 (Section 2 and Appendix A) and the counts of Section 3 are of three kinds. The integer counts and evaluations (C5, C7, C9–C11, and the general-$m$ counts: incidence $M-2$, compression $3m-2$, the ranks of the alternating maps) are rechecked by an exact script. Exact symbolic computation rechecks, in small dimensions: the step from (4) to (5) in C1 (Jacobi's identity), C2 (the Schur coefficients), C4 (the polar map $X \mapsto \frac{1}{2}(\operatorname{tr} X\, I + X^t)$ and its rank 16), and the scalar case of (L2), including the coefficient $-5p^t A^2 p$. The Schur formula (M1) of Lemma 18 is checked symbolically in dimension 2 and,

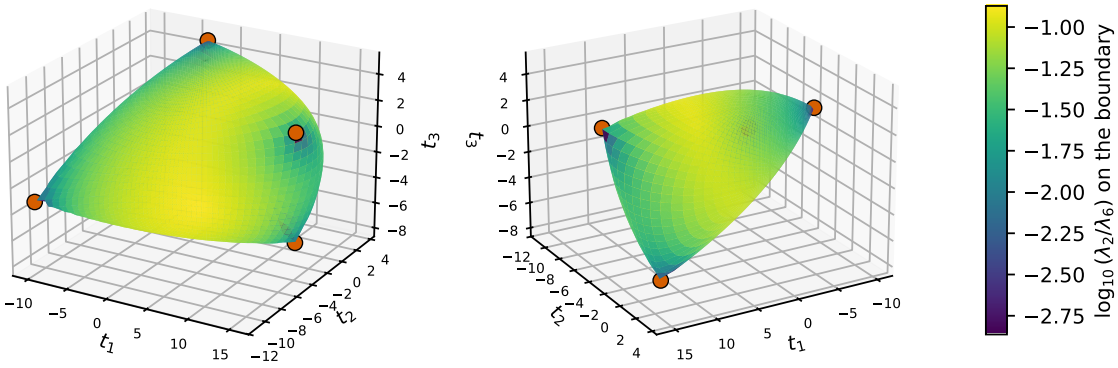



**Figure 2.** The Gram spectrahedron [27] of a $2 \times 3$ biquadratic sum of squares $F = \sum_{k=1}^{6} \ell_k^2$ (integer coefficients, two views): the positive semidefinite $6 \times 6$ matrices $G$ with $F = v^t G v$, an affine three-dimensional family (21 entries, 18 monomials). Representations of $F$ by $r$ squares are the points of rank $r$. On the boundary the colour is $\log_{10}(\lambda_2/\lambda_6)$. This is a numerical illustration in floating point: on all 9800 rays drawn the boundary point has one numerically zero eigenvalue (rank 5), and a local search finds four numerically rank-four points (vermilion), at the corners; four is $\mathrm{BSR}(2,3)$. The form, the Gram directions and the search are in the supplementary script `laboratorio.py`. In this language Theorem 1 says that the Gram spectrahedron of every $m \times 4$ sum of squares, $5 \le m \le 16$, contains a point of rank at most $3m - 1$.

in dimensions 3 and 4, at random rational points in exact arithmetic (a spot check, not a proof). The matrix case of (L2) is computed in the free algebra generated by the blocks, with an indefinite metric on $W_0$, so it holds in every dimension; a wrong coefficient is rejected. It is also checked at exact points over $\mathbb{Q}(i)$ of the relevant variety (dimensions 3 and 4 of $W_0$). The regular point C6, an explicit identity, is also confirmed by an exact rank computation. Formalized in Isabelle/HOL, each session without unproved steps (this says nothing about the links between these components, which are in the written proof), in six sessions of our public repository of formalizations (`BoundaryInertia`, `BSRFourColumnsPieces`, `BSRFourColumnsLinear`, `BSRFourColumnsLinear2`, `BSRFourColumnsLinear3`, `BSRFourColumnsSmallRows`): the three algebraic steps of Lemma 17 (the rank-one row observation, the evaluation step and the final contradiction, with the dimension counts and the common eigenvector of Lemma 15 as hypotheses); the arithmetic of the inertia inequality, its $m \times 4$ form $2(3m-1)(q-1) \le m(m+7)$ and its equivalence with $6q \le m + 13$ for $m \ge 5$; the enumeration of the support triples for every $m \ge 5$; the regular tuples of Lemma 21 for every $m$; the kernel of the polar map (9); the factorization in Lemma 3; the polarization of symmetric trilinear, 4-linear and 5-linear forms; the $2 \times 2$ step of Lemma 16; the key step of Lemma 15; the last step of Lemma 4 (a binary form of positive degree has a zero); the averaging step of Lemma 11; the classification of linear spaces of matrices of rank at most one used in Lemma 7; and the support count. Three further sessions, not yet in the public repository, are in the supplementary material: the algebraic core of Lemma 6 (Jacobi's identity turns (4) into (5), which gives $K_{ij} = t_i u_j - t_j u_i$ for independent $u_i$, and the bordered maps commute), computation C3 (the tensor $p_i^t(A_j - t_j I)p_k$ vanishes and $A_c P \subset P$ when $P^{\perp} = P$), and the two steps of computation C8 (single-wedge

**Table 1.** The formal components in Isabelle/HOL (six sessions, none with unproved steps) and the statements of this paper that use them.

| Session | Main statements | Used in |
|---|---|---|
| `BoundaryInertia` | `inertia_pataki_form`, `inertia_m4`, `inertia_m4_floor`, `q_values`; `incidence`, `compression_count`, `psi_counts` | Lemma 14, remark (i); counts of Section 3 |
| `BSRFourColumnsPieces` | `support_triples_m4`, `regular_point_m4`, `cross_product_identity` | Lemmas 21, 5 |
| `BSRFourColumnsLinear` | `rank5_polar_kernel`, `singular_2x2_space`, `symmetric_trilinear_polarization` | (9), Lemma 16 |
| `BSRFourColumnsLinear2` | `line_decomposable`, `rank_one_sum_minor`, `common_line_step_kernel`, `common_line_step_image`, `pencil_core` | Lemmas 3, 15, 4 |
| `BSRFourColumnsLinear3` | `symmetric_4linear_polarization`, `symmetric_5linear_polarization`; `positive_averaging_nonneg`, `diagonal_terms_vanish`; `rank_le_one_space_full`, `support_count_inequality` | Lemmas 15, 11, 7; supports |
| `BSRFourColumns-SmallRows` | `row_rank_one_simple`, `evaluation_step`, `row_in_line`, `small_rows_contradiction` | Lemma 17 |

pairings give $S \succeq 0$ and the radical statement for $n \geq 3$; inertia subtraction for a diagonal metric $\mathrm{diag}(I_p, -I_q)$); the reduction of $J^{\oplus n}$ to that diagonal form and the remaining steps of Lemma 11 stay in the text. Four more sessions, also in the supplementary material, cover (the last two items share one session):

- the arithmetic of Corollary 2, with the cited inputs as hypotheses;
- the trace inequality behind Proposition 23;
- the algebraic core of Lemma 18: the unperturbed coefficients and the induction over the moments, with the word analysis of the Neumann expansion as a hypothesis;
- every polarization and positivity step of Lemma 19;
- the polarization steps, the rank-two case and the integer counts of Lemma 20. Its Schur coefficients (L2) remain an exact computation.

Everything else (the identity C12 and the remaining arguments of Lemmas 5, 6, 7, 25, 11, 17 and 20) is proved in the text only. Scripts and theory files are in the supplementary material. These formal components check individual steps; they are not a machine-checked proof of Theorem 1 as a whole. Table 1 lists them, at revision `28dfc47` of the repository.

**Four remarks.** (i) The inertia inequality reads $N(q-1) \leq M - \binom{N+2}{2}$ with $M = \dim R_2$. It is, in substance, Theorem 3.4 of Boumal, Voroninski and Bandeira [8] (a bound on the number of negative eigenvalues of the dual multiplier at second-order critical points of the Burer–Monteiro factorization), applied at a boundary point of $\Sigma_N$, where the boundary multiplier satisfies the second-order conditions described by Blekherman, Sinn, Velasco and Zhang [10]; Lemma 14 gives the self-contained proof we need, without a constant-rank assumption. Since the multiplier is indefinite, $q \geq 1$, and for $q = 1$ it says that a boundary of $\Sigma_N$ inside the cone exists only if $\binom{N+2}{2} \leq M$, that is, only if $N + 1 \leq r_P$ for the Barvinok–Pataki rank $r_P = \max\{r : \binom{r+1}{2} \leq M\}$ of extreme Gram matrices; so it recovers the Barvinok–Pataki bound $\Sigma_N = \Sigma$ for $N \geq r_P$; for the full set of bilinear monomials this is also the bound of Choi, Lam and Reznick for sums of squares with prescribed monomials [22], and for $m \times 4$ it exceeds $3m-1$ by 2 to 4 when $5 \leq m \leq 16$, so it does not give Theorem 1. Each further

negative direction costs $N$ dimensions. For $m \times 4$ forms ($N = 3m-1$, $M = 5m(m+1)$) the bound is $q \le \lfloor (m^2+13m-2)/(6m-2) \rfloor = \lfloor (m+13)/6 \rfloor$ for $m \ge 5$, since the difference of the two fractions is $(m+7)/(6(3m-1)) < 1/6$. Thus the admissible negative index grows by one every six values of $m$; to go further one needs the low-negative-index lemma for larger $q$ (see (iii)). (ii) The image of a Hankel multiplier is product-free exactly when it is a completely entangled subspace of $\mathbb{C}^a \otimes \mathbb{C}^b$ in the sense of quantum information; the bound $\operatorname{rank} H \le (a-1)(b-1)$ used below is Parthasarathy's maximal dimension of such a subspace [7]. (iii) *Where the method stops, and why four columns.* At $m = 17$ the inertia bound allows $q = 5$. The low-negative-index lemma (Lemma 20) needs $q \le 4$: its proof maps several rows into a totally isotropic space of dimension at most four, excludes this with the small-row lemma (Lemma 17), which we prove for at most four rows, and uses Lemma 19, which needs negative index at most three on a hyperplane. A version for $q = 5$ (which needs at least a five-row small-row lemma, an invariant-plane lemma for negative index four, and the new cases of the $(k,h)$ table) would carry the theorem to $m \le 22$, since $\lfloor (m+13)/6 \rfloor \le 5$ exactly for $m \le 22$. More generally, in the $m \times n$ setting the Barvinok–Pataki bound gives $\mathrm{BSR}(m,n) \le r_P \le \sqrt{n(n+1)/2}\, m + O(1)$. The known values $\mathrm{BSR}(m,2) = m+1$ (the upper bound from [2, Theorem 1.1]; see [13] for the known exact values) and $\mathrm{BSR}(m,3) = 2m$ [3], and the bound $3m-1$ here, grow with slope $n-1$ (for $n = 4$ this is the progression of [13], who caution that the linear pattern may hold only for small grids); since $(n-1)^2 > n(n+1)/2$ exactly when $n \ge 5$, slope $n-1$ cannot persist beyond four columns, and four columns is the last width where it fits under the Pataki slope ($3 < \sqrt{10}$). For $N = cm$ the inertia inequality asks for roughly $q \approx \frac{n(n+1)-2c^2}{4c}\, m$ negative directions, which is $m/6$ for $(n,c) = (4,3)$ but $m/2$ for $(n,c) = (3,2)$: this is why the boundary-inertia route suits four columns, while $\mathrm{BSR}(m,3) = 2m$ needed a different argument. We do not know the slope of $\mathrm{BSR}(m,n)$ for $n \ge 5$. (iv) *Two observations for going beyond* $m = 16$. First, the proof of Lemma 14 gives more than (I): it bounds the kernel of multiplication on $W \otimes K$ by $N \dim K - \dim(WK)$ and only then uses $\dim(WK) \ge \binom{N+1}{2}$. Keeping $\dim(WK)$ gives $Nq \le M - 1 - \dim(WK)$, that is, $N(q-1) \le M - \binom{N+2}{2} - \delta$ with $\delta = \dim(WK) - \binom{N+1}{2} \ge 0$. For $m = 17$ ($N = 50$, $M = 1530$) this reads $50q \le 254 - \delta$, so $q \le 4$ as soon as $\delta \ge 5$; the case $q = 5$ needs $\delta \le 4$, and $\delta = 0$ when $K = W$. Second, a five-dimensional version of Lemma 16 cannot hold without using flatness: a five-dimensional subspace $S \subset \Lambda^2 \mathbb{C}^4$ of skew-symmetric matrices contains no matrix of rank one, yet every evaluation $A \mapsto Ab$ has image in $b^\perp$ and so rank at most $3 < \dim S - 1$. Any extension of the method to $q = 5$ must use the Hankel identities, not only the absence of simple tensors.

## 2. The boundary method at $m = 5$

Put $A = \mathbb{R}^5$, $B = \mathbb{R}^4$, $V = A \otimes B$, and

$$R_2 = \operatorname{Sym}^2 A \otimes \operatorname{Sym}^2 B, \quad \Phi_k : V^k \longrightarrow R_2, \quad \Phi_k(h) = \sum_{i=1}^{k} h_i^2.$$

Thus $\dim V = 20$ and $\dim R_2 = 150$. Write $\Sigma_k = \operatorname{im} \Phi_k$, $\Sigma = \bigcup_k \Sigma_k$. For $F \in \Sigma$, the least $k$ with $F \in \Sigma_k$ is $\mathrm{SOS}_{\mathbb{R}}(F)$; $\mathrm{BSR}(m,n)$ is the largest real SOS length in size $m \times n$. We use $\mathrm{BSR}(2,4) \le 5$. By [2, Theorem 1.1], if $X \subset \mathbb{P}^N$ is a nondegenerate irreducible variety of minimal degree with Zariski-dense real points, every quadratic form nonnegative on $X$ is a sum of $\dim X + 1$ squares in $\mathbb{R}[X]$. Biquadratic forms in $2+4$ variables are the quadratic

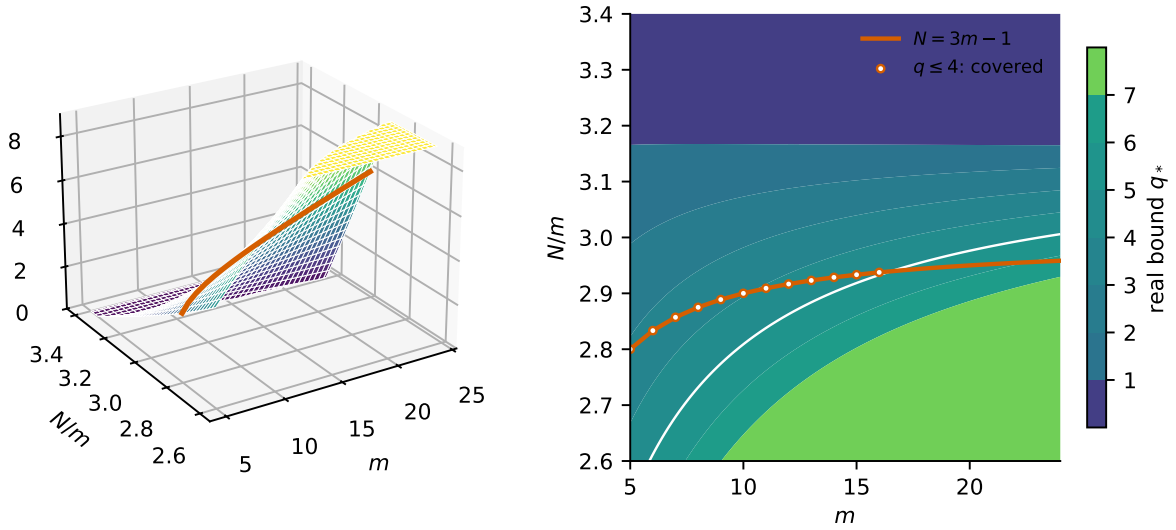



**Figure 3.** The boundary inertia inequality for $m \times 4$ forms: the real bound $q_* = 1 + \big(5m(m+1) - \binom{N+2}{2}\big)/N$, so that the negative index (an integer) satisfies $q \le \lfloor q_* \rfloor$, as a function of $m$ and $N/m$ (left, surface; right, its level sets). Along $N = 3m-1$ (vermilion) $q_* = 22/7 \approx 3.14$ at $m = 5$, $231/47 \approx 4.91$ at $m = 16$ and $127/25 = 5.08$ at $m = 17$, so $\lfloor q_* \rfloor \le 4$ exactly for $5 \le m \le 16$ (white dots), the range of Theorem 1; the white curve is $q_* = 5$.

forms on the Segre variety $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$, which has dimension 4 and degree $4 = \operatorname{codim} + 1$ (minimal), and a biquadratic sum of squares is nonnegative on it; hence it is a sum of 5 squares of linear forms on the Segre variety, i.e. of bilinear forms.

A Hankel multiplier is a symmetric bilinear form $H$ on $V$ with

$$H(x \otimes b, y \otimes c) = H(x \otimes c, y \otimes b). \tag{H}$$

Such forms are precisely the functionals on $R_2$, evaluated on products of bilinear forms. The first radical consists of the $x$ such that $H(x \otimes B, V) = 0$; the first support is its codimension. Define the second radical and second support in the other factor. The image of $H : V \to V^*$ is called product-free if its complexification contains no nonzero decomposable covector.

For the next algebraic lemmas all spaces are over $\mathbb{C}$. Factor a rank-$r$ multiplier as

$$H = T^* J T, \qquad T : A \otimes B \to E \text{ onto}, \quad \dim E = r,$$

where $J$ is nondegenerate symmetric. Set $T_b x = T(x \otimes b)$; write $\langle\,,\,\rangle$ for $J$. Condition (H) becomes

$$\langle T_b x, T_c y \rangle = \langle T_c x, T_b y \rangle. \tag{F}$$

**Lemma 3.** *If $x \neq 0$ and $T_c x = \beta(c) e$ for all $c$, with $\beta \neq 0$, $e \neq 0$, then $\operatorname{im}(T^* J)$ contains a nonzero decomposable covector.*

*Proof.* Choose $d$ with $\beta(d) = 1$. Equation (F) gives $\langle e, T_c y \rangle = \beta(c) \langle e, T_d y \rangle$. This is decomposable and nonzero, because $T$ is onto. □

Choose a generic $b$ where $s = \operatorname{rank} T_b$ is maximal, and put $U = \operatorname{im} T_b$, $Q(b) = T_b^* J T_b$. The linear terms of the bordered $(s+1)$-minors of $T_b + \tau T_c$ give $T_c(\ker T_b) \subset U$. Equation (F) also gives orthogonality to $U$. Thus

$$T_c(\ker T_b) \subset U \cap U^{\perp}. \tag{1}$$

If the first radical is zero and the image is product-free, then, for every nonzero $x \in \ker T_b$,

$$\dim\langle T_c x : c \in B\rangle \geq 2. \tag{2}$$

Indeed, dimension zero is excluded by the first radical and dimension one by Lemma 3. Moreover $Q$ is not identically zero: otherwise separate polarization of $H(x \otimes b, x \otimes b) = 0$, using (H), gives $H = 0$. Therefore at generic $b$,

$$\dim(U \cap U^{\perp}) \leq \min(s-1, r-s). \tag{3}$$

**Lemma 4.** *Let $G_1, G_2 : B \to S^*$ have the property that every nonzero linear combination $G_\lambda$ is onto. If $F : B \to S$ and $F^*G_1, F^*G_2$ are symmetric, then $F = 0$.*

*Proof.* If $k = \operatorname{rank} F > 0$, then $F^*G_\lambda$ has rank $k$. Symmetry gives $F^*G_\lambda = G_\lambda^* F$, so its kernel equals $\ker F$. It induces a nondegenerate symmetric form on $B/\ker F$ for every $[\lambda] \in \mathbb{P}^1$. Its determinant is a homogeneous polynomial of positive degree $k$ having no zero on $\mathbb{P}^1$, impossible over $\mathbb{C}$. □

**Lemma 5** (Isotropic row lemma)**.** *Assume $T$ is onto, the first radical is zero, and its dual image is product-free. There is no nonzero $D \subset A$ for which $S = T(D \otimes B)$ is totally isotropic and $\dim(A/D) \leq 2$.*

*Proof.* The injective map $z \mapsto T^*Jz$, for $z \in S$, takes values in $(A/D)^* \otimes B^*$. If the quotient has dimension zero, then $S = 0$; if it has dimension one, every nonzero image is decomposable. Both possibilities are excluded. For quotient dimension two choose lifts $a_1, a_2$, and define

$$G_i(b)(z) = \langle z, T(a_i \otimes b)\rangle.$$

If $G_\lambda$ were not onto, a nonzero $z \in S$ annihilating its image would make the two rows of $T^*Jz$ dependent. This would give a nonzero decomposable covector. Hence every $G_\lambda$ is onto. For $d \in D$, put $F_d(b) = T(d \otimes b) \in S$. Equation (F) says $F_d^* G_i$ is symmetric. Lemma 4 gives $F_d = 0$ for every $d$, contradicting the zero first radical. □

**Lemma 6** (Nonsingular slice)**.** *Let $\dim A = d \geq 3$, $\dim B = 4$. A rank-$(d+1)$ multiplier with supports $(d, 4)$ and an invertible $Q(b)$ cannot have product-free image.*

*Proof.* Take $E = \mathbb{C}^d \perp \mathbb{C}$, $J = I$, and $T_b x = (x, 0)$. For a complementary basis $c_1, c_2, c_3$ write

$$T_i x = (A_i x, u_i^t x).$$

Flatness gives the following exact identities, denoted computation C1:

$$A_i^t = A_i, \quad [A_i, A_j] = u_j u_i^t - u_i u_j^t. \tag{4}$$

Put $K_{ij} = A_i u_j - A_j u_i$, $v \odot w = vw^t + wv^t$. Substituting (4) into Jacobi's identity gives

$$K_{12} \odot u_3 + K_{23} \odot u_1 + K_{31} \odot u_2 = 0. \tag{5}$$

If the $u_i$ span zero dimensions, $T$ is not onto. If they span a line, the last coordinate of $T$ is a nonzero decomposable covector.

If their span has dimension two, change the complementary basis so that $u_3 = 0$ and $u_1, u_2$ are independent. Equation (4) makes $A_3$ commute with $A_1, A_2$. Comparing coefficients of (5) in a basis beginning with $u_1, u_2$ gives $A_3 u_1 = t u_1$, $A_3 u_2 = t u_2$ for some $t$. Put $P = \operatorname{im}(A_3 - tI)$. This space is invariant under every $A_i$ and is orthogonal to $u_1, u_2$. It is nonzero: otherwise $T_{c_3 - tb} = 0$, contradicting full second support. Equation (4) shows that the restricted operators commute on $P$. A common eigenvector $x \in P \setminus \{0\}$ then satisfies $T_c x \in \mathbb{C}(x, 0)$ for every $c$, contrary to Lemma 3.

If $u_1, u_2, u_3$ are independent, equation (5) implies

$$K_{ij} = t_i u_j - t_j u_i$$

for scalars $t_i$. To see this, complete the three $u_i$ to a basis. Coefficients involving the additional basis vectors first remove those components of $K_{ij}$. Coefficients of $u_i^2$ and $u_i u_j$ give the displayed expression. Consequently

$$\widehat{A}_i = \begin{pmatrix} A_i & u_i \\ u_i^t & t_i \end{pmatrix}$$

are commuting symmetric matrices, by (4). Choose a common eigenvector $z \neq 0$, with eigenvalues $\lambda_i$. Since $T_i = \widehat{A}_i T_b$, $\langle z, T_i x\rangle = \lambda_i \langle z, T_b x\rangle$. Thus $T^* J z$ is decomposable; it is nonzero because $T$ is onto. This is the final contradiction. □

**Lemma 7** (Rank-one factorization)**.** *If the symmetric polynomial matrix $Q(b)$ has homogeneous quadratic entries and generic rank one, then*

$$x^t Q(b) x = g(b) L(x,b)^2.$$

*Here either $L$ is independent of $b$, and the first support is one, or $g$ is constant and $L$ is bilinear.*

*Proof.* Over the fraction field $\mathbb{C}(b)$, a symmetric rank-one matrix gives a scalar times the square of a linear form in $x$. Clear denominators and make that linear form primitive over the UFD $\mathbb{C}[b]$. Gauss's lemma makes the scalar a polynomial. Bihomogeneity permits $L$ to be chosen homogeneous in $b$; if its degree is $e$, then $\deg g + 2e = 2$, so $e = 0$ or 1. If $e = 0$, write $L(x) = \lambda^t x$. Separate polarization of $q(x,b) = x^t Q(b) x = g(b)(\lambda^t x)^2$ in both factors gives

$$H(x \otimes b,\, y \otimes c) = G(b,c)\,(\lambda^t x)(\lambda^t y),$$

with $G$ the polar form of $g$; hence every $x$ with $\lambda^t x = 0$ lies in the first radical, and the first support is at most one. □

We now work over $\mathbb{R}$.

**Lemma 8** (Regular fourteen-tuple)**.** *The differential $d\Phi_{14}$ is onto at*

$$S = \{x_i y_i : 1 \le i \le 4\} \cup \{x_i y_j + x_j y_i : 1 \le i < j \le 4\} \cup \{x_5 y_j : 1 \le j \le 4\}.$$

*Proof.* Suppose a Hankel multiplier kills every element of $S$. The last four elements make $x_5$ a first-radical vector. The first ten span the symmetric tensors in the remaining identified $4 \times 4$ factors. For any $x, y$ in these factors,

$$H(x \otimes y, x \otimes y) = -H(x \otimes y, y \otimes x) = -H(x \otimes x, y \otimes y) = 0.$$

The first equality uses a symmetric tensor in the kernel; the second uses (H); the last uses $x \otimes x$ in the kernel. Separate polarization gives $H = 0$. Thus the annihilator of $S \cdot V$ is zero and $S \cdot V = R_2$, proving surjectivity. This explicit identity, not a numerical rank test, is computation C6. □

**Lemma 9** (Boundary selection)**.** *Let $\Omega \subset \mathbb{R}^d$ be a connected open semialgebraic set, let $K \subset \mathbb{R}^d$ be a closed semialgebraic set with $\Omega \cap \operatorname{int} K \neq \varnothing$ and $\Omega \not\subset K$, and let $Z \subset \mathbb{R}^d$ be semialgebraic with $\dim Z \le d-2$. Then there are a point $F \in \Omega \cap \partial K$ with $F \notin Z$, an open ball $U \ni F$ and a $C^\infty$ function $g$ on $U$ with $g(F) = 0$, $dg_F \neq 0$ and $K \cap U = \{g \ge 0\}$.*

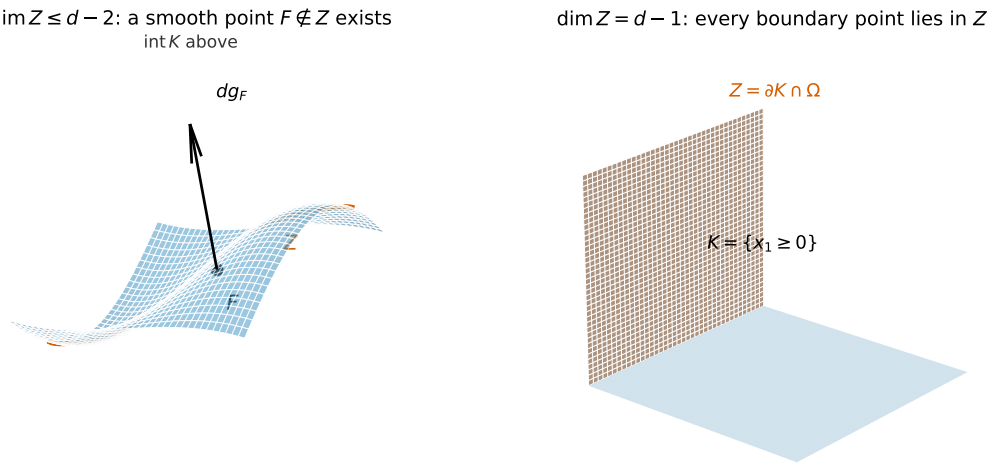



**Figure 4.** Lemma 9 in $d = 3$ (schematic). Left: $\dim Z \le d-2$, a curve on the boundary; off it there is a smooth boundary point $F$ with $dg_F \neq 0$ and $K$ on one side. Right: $\dim Z = d-1$ is not enough: with $K = \{x_1 \ge 0\}$ and $Z = \partial K \cap \Omega$ every boundary point lies in $Z$.

*Proof.* Put $K' = \overline{\operatorname{int} K} \subset K$. Then $\Omega \cap \operatorname{int} K' \neq \varnothing$, and $\Omega \not\subset K'$ because $K' \subset K$. A semialgebraic set $E \subset \Omega$ with $\dim E \le d-2$ does not disconnect $\Omega$: two points of $\Omega \setminus E$ are joined by a polygonal path in $\Omega$, and the pairs of points whose segment meets $E$ form a set of dimension at most $(d-2)+1+d < 2d$, so the vertices can be moved off it. Since $\Omega \cap \operatorname{int} K'$ and $\Omega \setminus K'$ are disjoint, open and nonempty, the frontier $\Omega \cap \partial K'$ therefore has dimension $d-1$. Take a $C^\infty$ semialgebraic Whitney stratification of $\mathbb{R}^d$ compatible with $K$, $K'$ and $Z$; it satisfies the frontier condition. Let $S$ be a stratum of dimension $d-1$ in $\Omega \cap \partial K'$. The closure of another stratum meets $S$ only if it has dimension $d$, so the union $Y$ of the closures of the strata of dimension at most $d-1$ other than $S$, together with the closure $\overline{Z}$ (semialgebraic, with $\dim \overline{Z} = \dim Z \le d-2$), is closed and meets $S$ in a set of dimension at most $d-2$. Choose $F \in S \setminus Y$ and a ball $U \ni F$ disjoint from $Y$ in which $S$ is a connected hypersurface $\{g = 0\}$ with $dg \neq 0$. Each of the two components of $U \setminus S$ is covered by open strata of dimension $d$, hence lies in one of them, and so lies either in $\operatorname{int} K$ or outside $K$. If both lay in $K$, $F$ would be interior to $K'$; if both lay outside $K$, then $\operatorname{int} K' \cap U \subset S$ would be empty, contradicting $F \in \overline{\operatorname{int} K'}$. So one side lies in $\operatorname{int} K$ and the other outside $K$, and after changing the sign of $g$, $K \cap U = \{g \ge 0\}$ ($S \subset K$ since $K$ is closed). Finally $F \in \partial K$, and $F \notin Z$. □

**Proposition 10** (Boundary package)**.** *If $\Sigma_{14} \neq \Sigma$, there exist $F \in \operatorname{int}\Sigma \cap \partial\Sigma_{14}$ and a nonzero indefinite real Hankel multiplier $H$ with the following properties. The real SOS length of $F$ is fourteen. For every representation $F = \sum_{i=1}^{14} h_i^2$, the forms $h_i$ are independent, their span $W$ lies in $\ker H$, and the complex annihilator of $W$ contains no nonzero simple tensor. In particular $r = \operatorname{rank} H \le 6$ and its image is product-free. For every syzygy $\sum_i h_i D_i = 0$,*

$$\sum_i H(D_i, D_i) \ge 0. \tag{11}$$

*Proof.* Each $\Sigma_k$ is semialgebraic, as a polynomial image, and is closed. To prove closedness, let $\tau(F)$ be the sum of the coefficients of all $x_i^2 y_j^2$. If $h_\nu = \sum_{ij} c_{\nu ij} x_i y_j$, then

$$\tau\left(\sum_{\nu=1}^{k} h_\nu^2\right) = \sum_{\nu,i,j} c_{\nu ij}^2.$$

For a convergent sequence of outputs the representing tuples are bounded, and a convergent subsequence proves closedness. A PSD Gram on $V$ has rank at most twenty, so $\Sigma = \Sigma_{20}$. Its interior is nonempty, since the positive definite Grams form an open set and the linear multiplication map onto $R_2$ is surjective. Lemma 8 gives $\Sigma_{14}$ nonempty interior.

Put $\Omega = \operatorname{int}\Sigma$. It is open and convex. If $\Sigma_{14}$ is proper in $\Sigma$, its complement in $\Omega$ is nonempty; otherwise closedness and $\overline{\Omega} = \Sigma$ would give equality. Lemma 9, applied to $\Omega$, $K = \Sigma_{14}$ and the singular incidence below as $Z$, gives a point of $\partial\Sigma_{14} \cap \Omega$ off $Z$ with a local defining function $g$, $\Sigma_{14} = \{g \geq 0\}$ near it.

Consider a complex point of $\mathbb{P}^4 \times \mathbb{P}^3$ with neither projective factor real. It and its conjugate are distinct in both factors. Complex changes of coordinates put them at $(e_1, e_1)$, $(e_2, e_2)$. Each first jet of a $(2,2)$-form has $1+4+3 = 8$ coefficients. At the first point they are coefficients of

$$x_1^2 y_1^2, \quad x_1 x_i y_1^2 \ (i \neq 1), \quad x_1^2 y_1 y_j \ (j \neq 1).$$

The eight coefficients at the second point are disjoint from these. Therefore the joint complex jet map has rank sixteen. Conjugation exchanges the two jets, so imposing them on real forms has real codimension sixteen. The point moves in real dimension $2(4+3) = 14$. Written in real and imaginary coordinates, the singular incidence, including the conditions that neither factor is real, is semialgebraic, so its projection to forms is semialgebraic by Tarski–Seidenberg. The singular incidence, and hence its projection to forms, has real dimension at most

$$150 - 16 + 14 = 148. \tag{12}$$

These jet coordinates and the count are computation C7. A singular point with a real factor is impossible for an interior SOS form: fixing that nonzero real factor gives a positive definite real quadratic form in the other factor, whose complex gradient has no nonzero kernel. This incidence is the set $Z$ used above, and $F$ is the point given by Lemma 9.

Let $g$ be the function of Lemma 9, nonnegative on $\Sigma_{14}$ near $F$, and $\ell = dg_F \neq 0$. For every tuple $h$ representing $F$, the function $g \circ \Phi_{14}$ has a local minimum at $h$. Its first derivative gives $H(h_i, v) = 0$ for every $i, v$, where $H(u, v) = \ell(uv)$. For a syzygy $D$, $d\Phi_{14}(D) = 0$; hence its second derivative gives (11). This uses the same $g, \ell, H$ at every representing tuple.

We have $\ell(F) = 0$. Since $F \in \operatorname{int}\Sigma$, the nonzero functional $\ell$ can be neither nonnegative nor nonpositive on $\Sigma$. Equivalently, $H$ is indefinite. If $F$ had at most thirteen squares, pad that representation with a zero slot. Applying (11) to a variation only in that slot would give $H(v, v) \geq 0$ for all $v$, a contradiction. Thus its length is fourteen. A linear dependence among fourteen summands could be removed by an orthogonal change of summand coordinates, producing a zero slot. The summands are therefore independent, and $W \subset \ker H$ gives $r \leq 20 - 14 = 6$. Finally, a simple covector in $W^\perp$ gives a Segre point where every $h_i$ vanishes; $F = \sum h_i^2$ and all its first derivatives vanish there. This contradicts the choice of $F$. Since $\operatorname{im} H \subset W^\perp$, its image is product-free. □

From here on the boundary multiplier $H$ is real. Over $\mathbb{R}$ it factors as $H = T^{\mathsf{T}} J T$ with $T$ onto a real space $E$ of dimension $r = \operatorname{rank} H$ and $J$ a real nondegenerate symmetric form of signature $(p, q)$, $p + q = r$; $H$ indefinite means $p, q \geq 1$.

**Lemma 11** (Positive annihilator; $n \geq 3$)**.** *Suppose a radical slice $x_0 \otimes L$, with $x_0$ in the first radical, $L \subseteq B$ and $\dim L = n \geq 3$, lies in a boundary span $W$. Let $A'$ be a complement of the first radical, of dimension $a$, and restrict $H$ to $A' \otimes L$. Assume that $T : A' \otimes L \to E$ is onto, that the restriction has zero first radical in $A'$, and that it has full second support on $L$. Choose orthogonal summand coordinates so that $h_i = x_0 \otimes y_i$, $1 \leq i \leq n$, where $y_i$ is a basis of $L$. Put $T_i a = T(a \otimes y_i)$, $H_{ij} = T_i^* J T_j$, and $S = \sum_i H_{ii}$. For the alternating map*

$$\Psi : A' \otimes \Lambda^2 L \longrightarrow E^n, \qquad \Psi(z)_i = \sum_j T_j z_{ij}, \quad z_{ij} = -z_{ji},$$

*if $p$ is the positive index of $J$, then*

$$S \succ 0, \qquad \operatorname{rank} \Psi \leq np - a. \tag{13}$$

*Proof.* The change of summand coordinates exists because the map $\mathbb{R}^{14} \to W$, sending coordinate vectors to the independent summands, is an isomorphism. Choose an orthonormal basis of the preimage of the fiber, then extend it orthogonally. The variations

$$D_i = \sum_j z_{ij} \otimes y_j \ (i \leq n), \qquad D_i = 0 \ (i > n)$$

are syzygies by antisymmetry. Thus $J^{\oplus n}$ is nonnegative on $L_\Psi = \operatorname{im} \Psi$.

A single $(i,j)$-wedge with coefficient $a \in A'$ gives $H_{ii}(a,a) + H_{jj}(a,a) \geq 0$. Summing all pairs gives $(n-1)S(a,a) \geq 0$. If $S(a,a) = 0$, every one of these wedge images is null in $L_\Psi$, hence orthogonal to all of $L_\Psi$. Pairing its $(i,j)$-wedge with the $(i,k)$-wedge of any $b$, for three distinct indices, gives $H_{jk}(a,b) = 0$. Pairing equal wedges gives $H_{ii}(a,b) + H_{jj}(a,b) = 0$. As $n \geq 3$, all diagonal terms also vanish. Therefore $a$ belongs to the first radical of the restriction. Effective support gives $a = 0$, proving $S \succ 0$.

For $a \in A'$, set $C_a = (T_i a)_i$. Flatness makes $C_a$ orthogonal to $L_\Psi$, and

$$J^{\oplus n}(C_a, C_b) = S(a,b).$$

These natural annihilator vectors span an $a$-dimensional positive definite space $M$. The nondegenerate space $M^\perp$ has positive index $np - a$. It contains the nonnegative space $L_\Psi$, whose dimension cannot exceed that positive index. This proves (13). The single-wedge pairings and inertia subtraction are computation C8. □

**Lemma 12** (Annihilator and extension)**.** *The annihilator of $\operatorname{im} \Psi$, with respect to $J^{\oplus n}$, consists exactly of the maps $C : L \to E$ satisfying*

$$\langle C y_i, T(a \otimes y_j) \rangle = \langle C y_j, T(a \otimes y_i) \rangle \quad \text{for every } a, i, j. \tag{14}$$

*It contains the $a$-dimensional space $C_a = T(a \otimes \cdot)$. Assume that $L = B$, or that $B = L \oplus L'$ with $L'$ contained in the second radical of $H$ (one of the two holds in every application below). Any additional independent map extends $T$ by one first-factor row, preserving rank, flatness, product-freeness, and full second support, and increasing first support by one.*

*Proof.* Pair an arbitrary tuple $(Cy_i)_i$ with each single-wedge image. Its vanishing is exactly (14). The natural maps satisfy (14) by flatness, and they are independent by zero first radical. Define the new row of $T'$ to be $C$. Equation (14) is the cross-flatness condition with each old row; the self-pairing condition holds because $J$ is symmetric. The map remains onto the same $E$. A first-radical relation involving the new row would express $C$ as a natural map, and an old second-radical relation remains excluded by the old full second support.

If $T'^* J z$ were a nonzero simple covector, its restriction to the old tensor space $A' \otimes L$ would be either a simple covector or zero. A simple covector $\alpha \otimes \lambda$ on $A' \otimes L$ pulls back along the

projection $A \otimes B \to A' \otimes L$ (kernel $x_0 \otimes B + A \otimes L'$, for a complement $B = L \oplus L'$ inside the second radical when $L \neq B$) to a simple covector of $V^*$ lying in $\operatorname{im} H$; this is excluded because $\operatorname{im} H$ is product-free. In the latter case $z$ is orthogonal to the old surjective image $E$, hence $z = 0$, also impossible. This argument uses one additional map; it requires no mutual compatibility between different additional maps. □

**Lemma 13** (Compression)**.** *For the boundary form $F$, if $H$ has first support $a$ and rank $r$, then*

$$\mathrm{SOS}_{\mathbb{R}}(F) \leq (4a - r) + \mathrm{BSR}(5 - a, 4). \tag{15}$$

*Proof.* Choose $A = A' \oplus A_0$, where $A_0$ is the first radical. The projections to $A' \otimes B$ of the fourteen summands lie in the kernel of the restricted multiplier, of dimension $4a - r$. An orthogonal change of summand coordinates therefore makes at most $4a - r$ summands have nonzero $A'$-projection. All remaining summands lie in $A_0 \otimes B$. Replace their sum of squares by a representation using at most $\mathrm{BSR}(5 - a, 4)$ squares. □

## 3. Larger $m$

**From five rows to $m$ rows.** Proposition 10, Lemmas 11, 12 and 13, and the algebraic Lemmas 3–7 extend to $m \times 4$ biquadratic forms after the following modifications, with 14 replaced by $N = 3m - 1$, $\dim V = 20$ by $4m$, $\dim R_2 = 150$ by $M = 5m(m+1)$, the bound $r \leq 6$ by $r \leq 4m - N = m + 1$, and $\mathrm{BSR}(5-a, 4)$ by $\mathrm{BSR}(m-a, 4)$. The algebraic lemmas are stated for tensor spaces $A \otimes B$ of any dimension, under structural hypotheses (flatness, surjectivity of $T$, zero radicals, product-freeness) that do not involve $m$. The proofs of Proposition 10 and of Lemmas 11–13 use the numbers above only through the counts displayed in Section 2, which are recomputed for general $m$ below. The regular point is supplied by Lemma 21, and the singular-point incidence count by the proof of Theorem 22. Lemmas 26, 27 and Theorem 25, which are specific to $5 \times 4$ (Appendix A), are not transported; for larger $m$ their role is played by the support bound below and by the low-negative-index lemma. Throughout, *effective supports* $(a, b)$ means the first and second supports of $H$ (codimensions of the two radicals). Theorem 22 below is Theorem 1, now proved for all $5 \leq m \leq 16$.

### 3.1. The boundary inertia inequality.

**Lemma 14** (Boundary inertia inequality)**.** *Let $V$ be a finite-dimensional real vector space of forms, and let $R_2$ be the span of their products. Put*

$$v = \dim V, \qquad M = \dim R_2, \qquad \Phi_N(h) = \sum_{i=1}^{N} h_i^2, \qquad \Sigma_N = \operatorname{im} \Phi_N.$$

*Let $\Sigma$ be the full sum-of-squares cone.*

*Suppose $F \in \Sigma_N \cap \operatorname{int} \Sigma$ admits a $C^2$ local defining function $g$ with*

$$g(F) = 0, \qquad dg_F \neq 0, \qquad g \geq 0 \quad \text{on } \Sigma_N \text{ near } F.$$

*Define*

$$H(u, w) = dg_F(uw), \qquad H = T^* J T, \qquad T : V \to E \text{ onto}, \qquad \mathrm{signature}(J) = (p, q).$$

*Then*

$$\text{(I)} \qquad \boxed{Nq \leq M - 1 - \binom{N+1}{2}}.$$

*Proof.* For every representing tuple $F = \Phi_N(h)$, the function $g \circ \Phi_N$ has a local minimum at $h$. Its first derivative gives

$$H(h_i, u) = 0 \qquad (i = 1, \dots, N,\ u \in V).$$

Its second derivative along a syzygy $\sum_i h_i D_i = 0$ gives

$$\sum_i H(D_i, D_i) \geq 0. \tag{I1}$$

The Hessian term of $g$ vanishes because $d\Phi_N(D) = 0$. The same $g$ and multiplier apply at every representing tuple.

The multiplier is nonzero because products span $R_2$. Also $dg_F(F) = \sum_i H(h_i, h_i) = 0$. Since $F$ is interior to $\Sigma$, a nonzero linear functional vanishing at $F$ cannot be nonnegative or nonpositive on all of $\Sigma$. Thus $H$ is indefinite.

If $F$ had length below $N$, pad a shorter representation with a zero slot. In (I1), vary only that slot. This would force $H$ to be positive semidefinite, a contradiction. Therefore $F$ has minimum length $N$. Every representing $N$-tuple is linearly independent, since an orthogonal change of summands would otherwise produce a zero slot.

Put

$$W = \operatorname{span}\{h_1, \dots, h_N\}, \qquad K = \ker H.$$

Then $\dim W = N$ and $W \subset K$. Multiplication $\operatorname{Sym}^2 W \to R_2$ is injective: a nonzero symmetric relation matrix $R$ would make $I + tR$ a Gram matrix for the same form. Moving to a singular positive semidefinite endpoint would express $F$ with at most $N - 1$ squares.

Consider the entire syzygy space

$$\mathcal{Z} = \ker(W \otimes V \longrightarrow R_2).$$

The nonzero functional $dg_F$ annihilates the multiplication image, so

$$\dim \mathcal{Z} \geq Nv - M + 1.$$

Project $\mathcal{Z}$ to $W \otimes E$ using $\mathrm{id}_W \otimes T$. The kernel is precisely the kernel of multiplication on $W \otimes K$. The multiplication image of $W \otimes K$ contains the independent products $\operatorname{Sym}^2 W$. Hence this projection kernel has dimension at most

$$N \dim K - \binom{N+1}{2}.$$

Writing $r = p + q$ and $\dim K = v - r$, the projected syzygy space consequently has dimension at least

$$Nv - M + 1 - N(v - r) + \binom{N+1}{2} = Nr + \binom{N+1}{2} - M + 1.$$

By (I1), this projected space is nonnegative for $J^{\oplus N}$. Since that form is nondegenerate and has positive index $Np$, the space has dimension at most $Np$. Therefore

$$Nr + \binom{N+1}{2} - M + 1 \leq Np.$$

Substituting $r = p + q$ gives (I). □

**Orientation and inertia.** At the chosen one-sided boundary point $F$, let $g$ be a $C^2$ local defining function satisfying

$$g(F) = 0, \qquad dg_F \neq 0, \qquad g \geq 0 \quad \text{on } \Sigma_N \text{ near } F.$$

Set

$$\ell = dg_F, \qquad H(u,v) = \ell(uv), \qquad H = T^*JT, \qquad J \cong \operatorname{diag}(I_p, -I_q).$$

Throughout the argument, $q$ denotes the negative index of this oriented multiplier. The orientation is fixed by

$$\sum_i H(D_i, D_i) \geq 0 \quad \text{for every syzygy } \sum_i h_i D_i = 0.$$

Replacing $H$ by $-H$ reverses this inequality. In particular, $q$ is not defined as the smaller of the two inertia indices.

### 3.2. The four-row isotropic lemma.

**Lemma 15** (Common-line commutators)**.** *Let $\mathcal{C}$ be a family of endomorphisms of a finite-dimensional complex vector space $S$. Suppose that a fixed subspace $K \subset S$, of dimension at most one, satisfies*

$$[C, D](S) \subset K \qquad (C, D \in \mathcal{C}).$$

*Then $\mathcal{C}$ is simultaneously triangularizable.*

Lemma 15 is a special case of a theorem of Omladič, Radjavi and Šivic [16, Theorem 6]: a linear space of complex matrices whose commutators all have rank at most one is simultaneously triangularizable. Here the linear span of $\mathcal{C}$ has all its commutators with image in $K$, hence of rank at most one. Their theorem extends Laffey's on semigroups [17]. We keep the short proof to make the paper self-contained.

*Proof.* Induct on $\dim S$. If every member is scalar, there is nothing to prove. Otherwise subtract an eigenvalue from a nonscalar member to obtain a nonzero singular operator $X$.

If $\ker X$ is invariant under the entire family, it is a nonzero proper common invariant subspace. Otherwise choose $Y \in \mathcal{C}$ and $v \in \ker X$ with $XYv \neq 0$. Then

$$0 \neq XYv = [X, Y]v \in K \cap \operatorname{im} X.$$

Thus $K \subset \operatorname{im} X$, and for every $Z \in \mathcal{C}$,

$$ZXw = XZw - [X, Z]w \in \operatorname{im} X.$$

Hence $\operatorname{im} X$ is a common proper invariant subspace.

On a resulting common invariant subspace $U$, restricted commutators have image in $K \cap U$; on the quotient their images lie in $(K + U)/U$. Both have dimension at most one. Apply induction to $U$ and $S/U$. □

**Lemma 16** (Small tableaux)**.** *Let*

$$\varphi : S \hookrightarrow W \otimes B^*$$

*be injective, with no nonzero rank-one tensor in its image. Write $G_b(s) = \varphi(s)(b)$ and $s = \dim S$. If $2 \leq s \leq 3$, some $G_b$ has rank at least $s - 1$, without a restriction on $\dim B$. The same conclusion holds for $s = 4$ when $\dim B \leq 4$.*

*Proof.* For $s = 2$, the assertion follows from injectivity. For $s = 3$, suppose all $G_b$ have rank at most one. A linear space of rank-at-most-one maps has a common image line or a common kernel hyperplane: two rank-one maps whose image lines and covector lines are both independent have a rank-two sum (this is the classical description of spaces of matrices of rank at most one [18, 19]; the last step is [20, Lemma 1.8]). A common kernel contradicts injectivity; a common image line makes each nonzero $\varphi(s)$ rank one.

Now let $s = 4$. Padding $B$ with zero coordinates permits $\dim B = 4$. Maximum rank at most one is excluded as above. Suppose the maximum is two. Choose bases with

$$G_{b_0} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad G_b = \begin{pmatrix} A_b & B_b \\ C_b & D_b \end{pmatrix}.$$

The Schur complement of $G_{b_0} + tG_b$ vanishes. Its first three coefficients give

$$D_b = 0, \qquad C_b B_b = 0, \qquad C_b A_b B_b = 0.$$

If some $B_b$ is invertible, polarization gives $C_c = 0$ for every $c$. All evaluation images then lie in a fixed two-plane. A four-dimensional subspace of $\mathbb{C}^2 \otimes (\mathbb{C}^4)^*$ contains a rank-one tensor: write its two rows as maps $R_1, R_2 : S \to (\mathbb{C}^4)^*$. If $R_1$ is singular, use a nonzero vector in its kernel; otherwise use an eigenvector of $R_1^{-1} R_2$.

Thus every $B_b$ is singular. A linear space of singular $2 \times 2$ matrices has a common kernel or a common image. To see this, normalize one rank-one member to $E_{11}$. Every other member has the form $\left(\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right)$, and $bc = 0$ identically on the linear space.

A common kernel of the $B_b$ gives a common kernel of all $G_b$, impossible. Hence $B_b = u\beta_b$ for a fixed $u \neq 0$. The identity $C_b B_b = 0$ gives $C_b u = 0$ for all $b$. If all $C_b$ vanish, use the fixed-two-plane argument. Otherwise $\ker C_b = \mathbb{C}u$ generically, and $C_b A_b B_b = 0$ gives $A_b u \in \mathbb{C}u$, first generically and then identically. Thus $G_b(u, 0)$ always lies in a fixed line. The nonzero tensor $\varphi(u, 0)$ has rank one, a final contradiction. □

**Lemma 17** (Small row images)**.** *Let $T : A \otimes B \to E$ be a surjective complex linear map to a nondegenerate symmetric space. Assume flatness, zero first radical, and that $T^*J(E)$ contains no nonzero simple tensor.*

*There is no $d$-dimensional subspace $D \subset A$ with*

$$\dim T(D \otimes B) \leq d$$

*when $d = 2$ or $3$. There is also no such subspace for $d = 4$ if $\dim B \leq 4$.*

*In particular, the three-row and four-row isotropic lemmas hold.*

*Proof.* Every nonzero row map $T_a : B \to E$ has rank at least two. Indeed, rank zero contradicts zero first radical. If $T_a(b) = \lambda(b)e$ has rank one, flatness gives

$$\lambda(b)\langle e, T(c, d)\rangle = \lambda(d)\langle e, T(c, b)\rangle.$$

Consequently $T^*Je$ is a simple tensor. It is nonzero by surjectivity and nondegeneracy, a contradiction.

Suppose $S = T(D \otimes B)$ has dimension $s \leq d$. The preceding observation excludes $s \leq 1$. Define

$$F_a(b) = T(a, b) \in S, \qquad G_b(z)(c) = \langle z, T(c, b)\rangle.$$

The tableau $z \mapsto G_\bullet(z)$ is injective and contains no nonzero rank-one tensor. Lemma 16 supplies $b_0$ with $\dim \ker G_{b_0} \leq 1$.

Mixed flatness says

$$G_b F_a(c) = G_c F_a(b).$$

If $F_a(b_0) = 0$, then $F_a(B) \subset \ker G_{b_0}$. The row-rank observation forces $a = 0$. Thus evaluation at $b_0$ is injective on $D$, so $d \le s$. Hence $s = d$, and evaluation at $b_0$ identifies $D$ with $S$.

Write, under this identification,

$$F_a(b) = C_b a, \qquad C_{b_0} = I.$$

Then

$$G_b = G_{b_0} C_b, \qquad G_{b_0}[C_b, C_c] = 0.$$

All commutators have image in the fixed space $\ker G_{b_0}$, of dimension at most one. Lemma 15 supplies a common eigenvector $z \neq 0$, with $C_b z = \lambda(b) z$. Therefore

$$G_b z = \lambda(b) G_{b_0} z.$$

If $G_{b_0} z = 0$, tableau injectivity fails. Otherwise this is a nonzero rank-one tableau member. Both alternatives are impossible. □

### 3.3. Schur information at negative index four.

**Lemma 18** (All moments)**.** *Let $W$ be a real nondegenerate symmetric space. In Witt coordinates $(e, W, f)$ suppose*

$$T_0(z, w) = (z, w, 0), \qquad T_c(z, w) = (\alpha z + \langle \ell, w \rangle,\ pz + Aw,\ \langle p, w \rangle),$$

*where $A$ is self-adjoint. If $(T_0 + tT_c)^* J (T_0 + tT_c)$ is singular identically in $t$, then*

$$\langle p, A^j p \rangle = 0 \qquad (j \ge 0).$$

*Proof.* Set $B = A - \alpha I$, $\tau = t/(1 + t\alpha)$ and $\mu_j = \langle p, B^j p \rangle$. After division of $T_0 + tT_c$ by $1 + t\alpha$, the scalar Schur complement, divided by $\tau^2$, is

$$\mu_0 - \left\langle (2I + \tau B)p,\ \left((I + \tau B)^2 + \tau^2(\ell p^* + p\ell^*)\right)^{-1} (2I + \tau B)p \right\rangle. \tag{M1}$$

The $W$-block is invertible as a formal power series, so (M1) vanishes identically.

Without the rank-two perturbation, (M1) equals

$$\mu_0 - \left\langle p, (I + 2(I + \tau B)^{-1} + (I + \tau B)^{-2})p \right\rangle.$$

Its coefficient of $\tau^j$ is $(-1)^{j+1}(j+3)\mu_j$, including $-3\mu_0$ for $j = 0$. In one variable these are the coefficients of $1 - \left(1 + \frac{1}{1+x}\right)^2 = -\frac{2x+3}{(1+x)^2}$, and none of them vanishes.

Expand the perturbed inverse by its formal Neumann series. Put $X = I + \tau B$, $u = (2I + \tau B)p$ and $E = \tau^2(\ell p^* + p\ell^*)$. Then

$$\left(X^2 + E\right)^{-1} = \sum_{n \ge 0} (-1)^n X^{-2} (E X^{-2})^n,$$

and the term with $n \ge 1$ contributes $\tau^{2n}$ times a sum of words

$$(u^* X^{-2} a_1)(b_1^* X^{-2} a_2) \cdots (b_{n-1}^* X^{-2} a_n)(b_n^* X^{-2} u), \qquad (a_k, b_k) \in \{(\ell, p), (p, \ell)\}.$$

*Each word contains a factor $\langle p, g(\tau B)p \rangle$ for a power series $g$.* Indeed, if $a_1 = p$ the first factor is such, because $u = (2I + \tau B)p$. Otherwise $(a_1, b_1) = (\ell, p)$ and the second factor begins with $p^*$; it is of this form if $a_2 = p$, and otherwise $b_2 = p$. Continuing, either some $a_k = p$ closes a factor $p^* X^{-2} p$, or all $a_k = \ell$, all $b_k = p$, and the last factor is $p^* X^{-2}(2I + \tau B)p$. All operators in these factors are functions of $\tau B$, so the factor expands as $\sum_i c_i \tau^i \mu_i$. Since the word carries the prefactor $\tau^{2n}$ with $n \ge 1$, its coefficient of $\tau^j$ is a sum of products, each containing some $\mu_i$ with $i \le j - 2$. Now argue by induction. If $\mu_0 = \cdots = \mu_{j-1} = 0$, the coefficient of $\tau^j$ in (M1) reduces to $(-1)^{j+1}(j+3)\mu_j$, which must vanish; hence $\mu_j = 0$. Finally $A = B + \alpha I$, so the moments with $A$ also vanish. □

**Lemma 19** (Invariant extension of an isotropic plane)**.** *Let $W$ be a real symmetric space of negative index at most three. Let $p_c$ and self-adjoint $B_c$ depend linearly on $c$. Suppose $P = \operatorname{span}\{p_c\}$ is an isotropic two-plane,*

$$B_c p_d = B_d p_c, \qquad [B_c, B_d]P \subset P, \qquad [B_c, B_d]P^\perp \subset P,$$

*and $\langle p_c, B_c^j p_c\rangle = 0$ for $j = 1, 2, 3, 4$. Then $P$ is contained in a common invariant totally isotropic space $P'$ of dimension two or three.*

*Proof.* The trilinear form $\langle p_c, B_d p_e\rangle$ is fully symmetric. Its diagonal vanishes, so $B_c P \subset P^\perp$.

The four-linear form

$$R(c, d, e, f) = \langle B_c p_d, B_e p_f\rangle$$

is fully symmetric. Interchange within either pair uses $B_c p_d = B_d p_c$; interchange of the operator indices changes its value by $\langle p_d, [B_c, B_e]p_f\rangle = 0$. Its diagonal is the second moment, hence $R = 0$. Thus

$$P' = P + \sum_c B_c P$$

is totally isotropic and has dimension at most three. If $P' = P$, it is already invariant.

Suppose $\dim P' = 3$. The quotient $P'^\perp/P'$ is positive definite. The five-linear form

$$U(i, j, k, l, m) = \langle B_i B_j p_k, B_l p_m\rangle$$

is fully symmetric: swapping $i, j$ introduces $\langle [B_i, B_j]p_k, B_l p_m\rangle = 0$; swapping the outer operator indices introduces $\langle B_j p_k, [B_i, B_l]p_m\rangle = 0$; the other swaps follow from $B_j p_k = B_k p_j$. Its diagonal is the third moment, so $U = 0$. Together with $R = 0$, this gives $B_i P' \subset P'^\perp$.

The squared norm of $B_c^2 p_c$ is the fourth moment. Hence $B_c^2 p_c \in P'$ by positivity of $P'^\perp/P'$. Modulo $P'$, the trilinear map $(i, j, k) \mapsto B_i B_j p_k$ is symmetric, using $B_j p_k = B_k p_j$ and $[B_i, B_j]p_k \in P$. Its diagonal is zero; polarization gives $B_i B_j p_k \in P'$ for all indices. Therefore $B_i P' \subset P'$. □

### 3.4. The low-negative-index lemma.

**Lemma 20** (Low negative index)**.** *Let $H$ be a nonzero real Hankel multiplier with effective supports $(a, b)$, where $b \le 4$. Suppose its complex image contains no nonzero simple tensor and its negative index satisfies $q \le 4$. Put $r = \operatorname{rank} H$. Then*

$$r \ge a + 1.$$

*If $b = 4$ and $a \ge 3$, then*

$$r \ge a + 2.$$

*Proof.* Remove the factor radicals and write $H = T^* J T$, with $T : A \otimes B \to E$ onto and $J$ of signature $(p, q)$. Choose a generic real $b_0$ maximizing both $s = \operatorname{rank} T_{b_0}$ and $\operatorname{rank}(T_{b_0}^* J T_{b_0})$. Set

$$U = \operatorname{im} T_{b_0}, \quad k = a - s, \quad h = \dim(U \cap U^\perp).$$

The linear terms of bordered $(s + 1)$-minors give $T_c(\ker T_{b_0}) \subset U$. Flatness also gives orthogonality to $U$, hence

$$T_c(\ker T_{b_0}) \subset U \cap U^\perp. \tag{L1}$$

Every nonzero complex row map has rank at least two, by the row-rank argument of Lemma 17. Thus $k > 0$ implies

$$2 \le h \le q \le 4.$$

Suppose $r \le a$. If $k = 0$, then $r = a$ and $T_{b_0}$ is an isomorphism. Under this identification all other slices are self-adjoint and pairwise commuting. A common complex eigenvector gives a rank-one row, impossible. If $k > 0$, then

$$k = a - s \ge r - s \ge h.$$

An $h$-plane in $\ker T_{b_0}$ maps into the isotropic $h$-plane $U \cap U^{\perp}$, contradicting Lemma 17. Hence $r \ge a + 1$.

Now let $b = 4$ and $r = a + 1$. Lemma 6 excludes an invertible $Q(b)$, so every $Q(b)$ is singular.

First suppose $k > 0$. By (L1) and the small-row lemma, $k < h$. Since $h \le r - s = k + 1$, necessarily

$$(k, h) = (1, 2), (2, 3), (3, 4).$$

Choose real Witt coordinates

$$E = W_0 \perp \mathbb{H}^h, \qquad A = W_0 \oplus U_0^h \oplus Z^k, \qquad T_{b_0}(w, u, z) = (w, u, 0).$$

Flatness and (L1) give

$$T_c = \begin{pmatrix} A_c & P_c & 0 \\ L_c & B_c & R_c \\ P_c^* & Q_c & 0 \end{pmatrix}, \qquad A_c^* = A_c, \quad Q_c^t = Q_c.$$

The Schur complement along $W_0$ vanishes identically: $b_0$ maximizes $\operatorname{rank} Q(b)$, so $\operatorname{rank} Q(b_0 + \tau c) \le \operatorname{rank} Q(b_0) = \dim W_0$ for every $c$ and $\tau$, while the $W_0$-block of $Q(b_0 + \tau c)$ is invertible near $\tau = 0$; rank additivity under block elimination forces the complement to vanish. Its successive $U_0$-block coefficients give

$$Q_c = 0, \qquad P_c^* P_c = 0, \qquad P_c^* A_c P_c = 0, \qquad P_c^* A_c^2 P_c = 0. \tag{L2}$$

The coefficients after preceding identities are imposed are, respectively,

$$2Q_c, \quad -3P_c^* P_c, \quad 4P_c^* A_c P_c, \quad -5P_c^* A_c^2 P_c.$$

(In the raw expansion the blocks $L_c$ and $B_c$ do appear; with $G_j = P_c^* A_c^j P_c$ the $\tau^3$ coefficient is $4G_1 - 2(G_0 B_c + B_c^t G_0)$ and the $\tau^4$ coefficient is $-5G_2 + 3(G_1 B_c + B_c^t G_1) - B_c^t G_0 B_c + 4(G_0 L_c P_c + (L_c P_c)^t G_0)$, so they drop out exactly once the preceding identities $G_0 = 0$, $G_1 = 0$ are imposed; $R_c$ never enters.) These coefficients do not depend on the dimensions. Write $J_0$ for the metric on $W_0$, so that $J_0^2 = 1$ in a Witt basis, $A_c^* = J_0 A_c^t J_0$ and $P_c^* = P_c^t J_0$. The $W_0$-block of $Q(b_0 + \tau c)$ is $J_0 + N(\tau)$ with $N(\tau) = O(\tau)$, and its inverse is the finite recurrence $J_0 \sum_{j \le 4} (-N(\tau) J_0)^j$ modulo $\tau^5$. Expanding the Schur complement with this inverse is a computation in the free algebra generated by the blocks $A_c, P_c, P_c^t, L_c, L_c^t, B_c, B_c^t, Q_c$ and $J_0$, with the relation $J_0^2 = 1$. Since the preceding identities are monomials ($Q_c$, $P_c^* P_c$, $P_c^* A_c P_c$), imposing them deletes the words that contain them. The coefficients above are what remains, for every $\dim W_0$ and every $h$. Polarization and cross-flatness imply $P_c^* P_d = 0$ for all $c, d$. Therefore

$$P = \sum_c \operatorname{im} P_c$$

is totally isotropic in $W_0$, and $\dim P \le q - h$.

For $(k, h) = (3, 4)$, $P = 0$ and the seven rows $U_0 \oplus Z$ map into an isotropic four-space. For $(k, h) = (2, 3)$, $\dim P \le 1$ and the five rows $U_0 \oplus Z$ map into an isotropic space of dimension at most four. Both contradict Lemma 17.

For $(k,h) = (1,2)$, the same argument works if $\dim P \le 1$. Suppose $\dim P = 2$. Then $P^\perp/P$ is positive definite. If some $P_c$ has rank two, it is generically onto $P$; (L2) gives $A_cP \subset P^\perp$ and makes every vector of $A_cP$ null. Thus $A_cP \subset P$, first generically and then for every $c$.

If all $P_c$ have rank at most one, the rank-one matrix-space classification gives either a common image line or $P_c = p_c\phi$ with a fixed nonzero covector $\phi$. The first option contradicts $\dim P = 2$. Mixed flatness in the second option gives

$$A_cp_d - A_dp_c \in P.$$

The first moment in (L2) polarizes to $A_cP \subset P^\perp$. The second moment and the positive quotient give $A_cp_c \in P$. Polarizing this relation and combining it with the displayed difference yields $A_cp_d \in P$ for all $c,d$.

Thus $P$ is invariant in either case. The five rows $P \oplus U_0^2 \oplus Z$ map into the isotropic four-space $P \oplus \mathbb{R}^2 e$, again impossible.

It remains that $k = 0$. The generic slice image is a degenerate hyperplane, with one-dimensional radical. In Witt coordinates it has the form

$$T_{b_0}(z,w) = (z,w,0),$$

and flatness gives

$$T_c(z,w) = (\alpha_c z + \langle \ell_c, w\rangle,\ p_c z + A_c w,\ \eta_c z + \langle p_c, w\rangle), \qquad A_c^* = A_c.$$

The linear Schur coefficient is $2\eta_c$, so $\eta_c = 0$. Lemma 18 applies.

Let $P = \operatorname{span}_{\mathbb{R}}\{p_c\}$. The zeroth moment makes $P$ totally isotropic. The covector $T^*Je$ is

$$(z,w;c) \longmapsto \langle p_c, w\rangle.$$

If $P = 0$, surjectivity fails; if $\dim P = 1$, this is a nonzero simple covector. Since $W_0$ has negative index $q-1 \le 3$, we obtain

$$2 \le \dim P \le 3.$$

Put $B_c = A_c - \alpha_c I$. Flatness gives

$$B_cp_d = B_dp_c$$

and

$$[B_c, B_d]x = \ell_d\langle p_c, x\rangle - \ell_c\langle p_d, x\rangle + p_d\langle \ell_c, x\rangle - p_c\langle \ell_d, x\rangle. \tag{L3}$$

In particular, these commutators send both $P$ and $P^\perp$ into $P$.

If $\dim P = 2$, Lemma 19 supplies a common invariant isotropic space $P' \supset P$ of dimension two or three.

If $\dim P = 3$, it is maximal isotropic. Polarizing $\langle p_c, B_cp_c\rangle = 0$ gives $B_cP \subset P^\perp$. Since $\langle B_cp_c, B_cp_c\rangle = 0$ and $P^\perp/P$ is positive definite, $B_cp_c \in P$. Polarization and $B_cp_d = B_dp_c$ give $B_cP \subset P$. Set $P' = P$.

In either case $P'$ is invariant under every $A_c$, and

$$T\big((\mathbb{R}z \oplus P') \otimes B\big) \subset \mathbb{R}e \oplus P'.$$

The source row space and the totally isotropic target both have dimension three or four. After complexification, Lemma 17 gives a contradiction. Thus $r = a+1$ is impossible, proving $r \ge a+2$. □

**Small effective supports.** For effective supports $(a, b)$, product-freeness over $\mathbb{C}$ implies

$$r \le (a-1)(b-1).$$

Indeed, the projective image of $H$, of dimension $r-1$, is disjoint from the Segre variety, of dimension $a+b-2$, inside $\mathbb{P}^{ab-1}$; the projective dimension theorem gives the displayed inequality.

If $b = 1$, every nonzero image vector is simple. If $b = 2$, the upper bound gives $r \le a-1$, whereas Lemma 20 gives $r \ge a+1$. Thus $b \le 2$ is impossible.

Likewise $a \le 2$ is excluded by interchanging the factors and applying the weak lower bound; the second factor after interchange then has dimension at most two.

### 3.5. Regular points for every m.

**Lemma 21** (Uniform regular tuples)**.** *For every $m \ge 5$, $\Phi_{3m-1}$ on $m \times 4$ bilinear forms has a real point with surjective differential.*

*Proof.* For $m = 5$, take

$$\mathcal{S}_5 = \{x_i y_i : 0 \le i \le 3\} \cup \{x_i y_j + x_j y_i : 0 \le i < j \le 3\} \cup \{x_4 y_j : 0 \le j \le 3\}, \qquad h_5^* = x_0 y_0.$$

A Hankel multiplier annihilating this tuple has $x_4$ as a first radical vector. On the remaining block its kernel contains every symmetric tensor. Therefore

$$H(x \otimes y, x \otimes y) = -H(x \otimes y, y \otimes x) = -H(x \otimes x, y \otimes y) = 0.$$

Separate polarization gives $H = 0$, proving regularity.

Maintain two additional properties: $h_m^*$ has zero $y_3$ coefficient, and the $y_3$ coefficients of $\mathcal{S}_m \setminus \{h_m^*\}$ span all old rows. They hold for the base tuple.

Introduce $z = x_m$ and define

$$\mathcal{S}_{m+1} = (\mathcal{S}_m \setminus \{h_m^*\}) \cup \{h_m^* + z y_3,\ z y_0,\ z y_1,\ z y_2\}, \qquad h_{m+1}^* = z y_0.$$

The cardinality increases by three, and the two additional properties persist.

Suppose $H$ annihilates the new tuple. Since $z y_0, z y_1, z y_2$ lie in its kernel, Hankel symmetry leaves only the following possible entries involving $z$:

$$k(a) = H(z y_3, a y_3), \qquad \gamma = H(z y_3, z y_3).$$

Pairing $z y_3$ with the unchanged old forms forces $k = 0$ by their column-three spanning property. The old restriction of $H$ now annihilates the whole old tuple, including $h_m^*$, so it is zero by induction. Finally $H(z y_3, h_m^* + z y_3) = 0$ gives $\gamma = 0$. Thus $H = 0$, proving regularity. □

### 3.6. Proof for larger m.

**Theorem 22.** *For $5 \le m \le 16$,*

$$\boxed{\mathrm{BSR}(m, 4) \le 3m - 1}.$$

*Proof.* Induct on $m$, using

$$\mathrm{BSR}(2, 4) \le 5, \quad \mathrm{BSR}(3, 4) \le 8, \quad \mathrm{BSR}(4, 4) \le 11.$$

Put $N = 3m - 1$ and $M = 5m(m+1)$.

Suppose $\Sigma_N \ne \Sigma$. The regular tuple of Lemma 21 permits Proposition 10 (in its $m \times 4$ form) to be applied. Its singular-incidence exception has codimension at least two: a first

jet has $m+3$ coefficients, while a conjugate pair with neither factor real moves in dimension $2m+4$. The incidence dimension is at most

$$M - 2(m+3) + (2m+4) = M - 2.$$

Thus the boundary proposition supplies a form $F$ of minimum length $N$, a representing span $W$ of dimension $N$, and an indefinite real multiplier $H$ with product-free complex image, such that

$$W \subset K = \ker H, \qquad r = \operatorname{rank} H \le m+1,$$

with nonnegative normal second variation on every syzygy.

The boundary inertia inequality (Lemma 14, with $v = 4m$) gives

$$\boxed{q \le \left\lfloor \frac{m^2 + 13m - 2}{6m - 2} \right\rfloor \le 4}.$$

Let $(a, b)$ be the effective supports. The small-support exclusions leave $a \ge 3$, $b \in \{3, 4\}$.

If $a \le m-2$, compression and induction give

$$\mathrm{SOS}_{\mathbb{R}}(F) \le 4a - r + \mathrm{BSR}(m-a, 4) \le 4a - (a+1) + 3(m-a) - 1 = 3m - 2 < N,$$

a contradiction. Here $r \ge a+1$ is the weak inequality of Lemma 20, and $\mathrm{BSR}(k, 4) \le 3k-1$ holds for every $k = m - a \ge 2$: by induction for $5 \le k < m$, and by the base values $5, 8, 11$ for $k = 2, 3, 4$, which lie exactly on the line $3k-1$.

Compression stops at $a = m-2$: for $a = m-1$ it would need $\mathrm{BSR}(1, 4) = 4 > 3 \cdot 1 - 1$. For $a = m$ or $m-1$, the constraints $b \in \{3, 4\}$, $a+1 \le r \le m+1$, and $r \ge a+2$ when $b = 4$ (Lemma 20) leave exactly the following triples; in particular the low-index lemma excludes $(m, 4, m+1)$ and $(m-1, 4, m)$. The remaining triples are

$$(m-1, 4, m+1), \quad (m, 3, m+1), \quad (m-1, 3, m+1), \quad (m-1, 3, m).$$

For $(m-1, 4, m+1)$, $W = K$. The positive-annihilator lemma has fiber dimension four. Its annihilator has exactly $m-1$ dimensions: an additional map would extend $H$ to supports $(m, 4)$ and rank $m+1$, contradicting the strong low-index inequality. The rank of $\Psi$ is the dimension of its target, $4(m+1)$, minus that of its annihilator (the target carries the nondegenerate form $J^{\oplus 4}$), that is $4(m+1) - (m-1) = 3m+5$; and $p = r - q \le m$ since $q \ge 1$. Hence

$$3m + 5 = \operatorname{rank} \Psi \le 4p - (m-1) \le 3m + 1,$$

impossible.

For $(m, 3, m+1)$, exchange the factors. The fiber dimension is $m$, the coefficient support has dimension three, and the annihilator has exactly three dimensions by the same extension argument. The target has dimension $m(m+1)$, so $\operatorname{rank} \Psi = m(m+1) - 3$, and again $p \le m$. Hence

$$m^2 + m - 3 = \operatorname{rank} \Psi \le mp - 3 \le m^2 - 3,$$

again impossible.

For the remaining triples choose radical generators $x_0, y_0$, put $u = x_0 \otimes y_0$, and set

$$U = x_0 \otimes B + A \otimes y_0.$$

For active bases $a_i, b_j$ and $v = \sum v_{ij} a_i \otimes b_j$, the rectangle identities give

$$uv = \sum_{ij} v_{ij} (a_i \otimes y_0)(x_0 \otimes b_j). \tag{R}$$

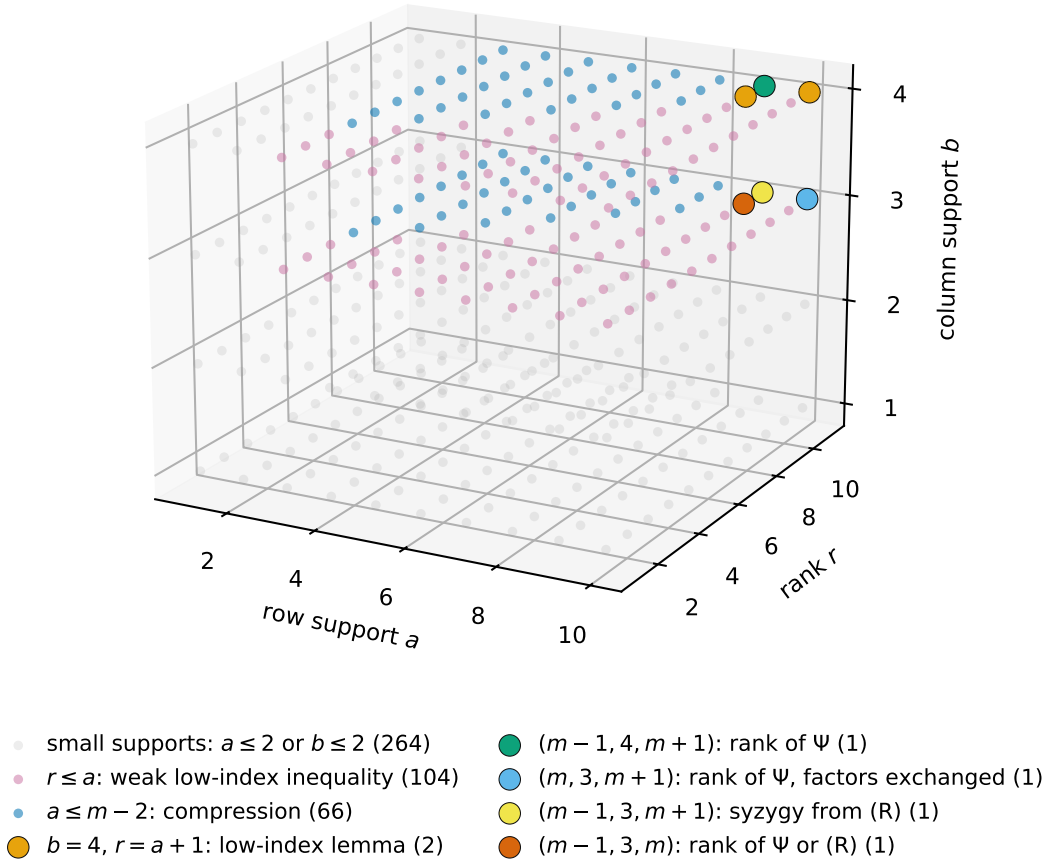



**Figure 5.** Every candidate integer triple $(a, b, r)$ for $m = 10$ (row support, column support, rank of $H \le m+1$; not all are realized by multipliers), coloured by the step of the proof of Theorem 22 that excludes it, with the number of triples in each class. Small supports, the weak inequality $r \ge a+1$ and compression remove all but six triples; the low-index lemma removes two more, and the last four (large markers) each need their own argument. The same partition is checked for every $5 \le m \le 16$ by the script that draws the figure.

For $(m-1, 3, m+1)$, $W = K$. Choose summand coordinates with $h_1 = cu$, $c \neq 0$. Take an active $v$ with $H(v, v) < 0$. Equation (R) implies $h_1 v \in WK$, so write

$$-h_1 v = \sum_i h_i k_i, \qquad k_i \in K.$$

Then $D_i = \delta_{i1} v + k_i$ is a syzygy with $\sum_i H(D_i, D_i) = H(v, v) < 0$, a contradiction.

For $(m-1, 3, m)$, $\dim K = N + 1$, so $W = \ker \lambda$ for a nonzero functional $\lambda$ on $K$. Write $K = U \oplus K_0$, with $K_0$ the active kernel.

If $u \notin W$, then

$$W \cap (x_0 \otimes B) = x_0 \otimes L,$$

where $\dim L = 3$ and $L$ projects isomorphically onto $B/\mathbb{R}y_0$. The restricted multiplier retains rank $m$, first support $m-1$, and signature $(p, q)$. The positive-annihilator lemma has fiber

dimension three. An extra annihilator map would extend first support to $m$ without increasing rank, contradicting $r \geq m+1$. The target has dimension $3m$, so $\operatorname{rank}\Psi = 3m-(m-1) = 2m+1$, while $p \leq m-1$ since $q \geq 1$. Therefore

$$2m+1 = \operatorname{rank}\Psi \leq 3p-(m-1) \leq 2m-2,$$

impossible.

If $u \in W$, arrange $h_1 = cu$. When $\lambda|_U = 0$, equation (R) gives $uv \in WK$ for every active $v$. Otherwise

$$\operatorname{Sym}^2 U/(U \cap W)U$$

is one-dimensional, measured by $(\lambda|_U)^2$. Thus $uv \in WK$ whenever

$$\varphi(v) = \sum_{ij} v_{ij}\lambda(a_i \otimes y_0)\lambda(x_0 \otimes b_j) = 0.$$

The active covector $\varphi$ is zero or decomposable. If nonzero, it cannot annihilate $K_0$, since that would put it in the product-free image of the active multiplier.

Start with a negative active $v$ and, when necessary, add an element of $K_0$ to impose $\varphi(v) = 0$. Its value under $H$ remains negative. The syzygy construction above again contradicts nonnegative second variation.

All support cases are excluded. Hence $\Sigma_N = \Sigma$. □

## 4. Perspectives

**The slope.** Theorem 1 gives slope 3 up to $m = 16$, against $8/3$ from quadratic persistence and $\sqrt{10} \approx 3.16$ from the Barvinok–Pataki bound. We expect the true slope of $\mathrm{BSR}(m,4)$ to be 3: the progression $3m-1$ holds at $m = 2,3,4$, the method reaches it up to $m = 16$, and the stop at $m = 17$ comes from the inertia inequality allowing $q = 5$ (Figure 1), not from a construction. It is open whether $\mathrm{BSR}(m,4) - 3m$ stays bounded, and whether the lower bounds of slope $5/2$ given by $z_{RL}$ can be raised.

**Toward** $\mathrm{BSR}(5,4) = 13$**.** Deciding [13, Question 1] means running the boundary method with $N = 13$. Let $I_2$ be the kernel of $\operatorname{Sym}^2 V \to R_2$ and, for a subspace $E \subseteq V$, put $i(E) = \dim(I_2 \cap \operatorname{Sym}^2 E)$.

**Proposition 23** (Negative-syzygy obstruction)**.** *Let $F$ be a smooth boundary point of $\Sigma_s$ with multiplier $H$ as in Proposition 10, and let $W \subseteq K = \ker H$ be the $s$-dimensional span of the summands of a representation of $F$. Let $N$ be a $q$-dimensional subspace on which $H$ is negative definite, $D = K \oplus N$ and $k = \dim K$. Then*

$$i(D) - i(K) \;\leq\; \operatorname{rk}(I_2 \cap \operatorname{Sym}^2 D \to \operatorname{Sym}^2(D/W)) \;\leq\; \binom{k-s+q+1}{2}.$$

*Proof.* Choose a complement $C$ of $W$ in $D$, so that $\operatorname{Sym}^2 D = (W \odot D) \oplus \operatorname{Sym}^2 C$ and $\operatorname{Sym}^2 C \cong \operatorname{Sym}^2(D/W)$. The kernel of the projection of $I_2 \cap \operatorname{Sym}^2 D$ onto $\operatorname{Sym}^2 C$ is $I_2 \cap (W \odot D)$. Its subspace $I_2 \cap (W \odot K)$ has dimension at most $i(K)$. If the displayed inequality failed, some element of $I_2 \cap (W \odot D)$ would not lie in $W \odot K$. Writing it as $\sum_i h_i \odot g_i$, with $h_1, \dots, h_s$ orthonormal summands of the representation, gives a syzygy $\sum_i h_i g_i = 0$ in $R_2$ with $g_i = k_i + n_i \in K \oplus N$ and not all $n_i$ zero. Since $K$ is the radical of $H$, its second

variation is $\sum_i H(g_i, g_i) = \sum_i H(n_i, n_i) < 0$, against the nonnegative second variation of Proposition 10. □

For $s = 13$ at $5 \times 4$ the multiplier has rank at most 7. Appendix B lists, for one rational multiplier of each of the 11 subtypes of rank at most 6 and negative index at least 2, a certified upper bound for the middle rank that is smaller than $i(D) - i(K)$ exact computation. By the proposition, none of these multipliers supports a smooth boundary point of $\Sigma_{13}$, whatever $W$ is. In 4 of them the bound uses that the $\mathrm{Sym}^2 N$-block of $I_2 \cap \mathrm{Sym}^2 D$ vanishes, so that the rank in the middle is at most $\binom{k-s+q+1}{2} - \binom{q+1}{2}$. Three steps remain open:

- to pass from one rational multiplier to the whole stratum, since $i(K)$ can jump under specialization;
- to treat negative index 1;
- to treat rank 7, where no such syzygy exists and a dual certificate is needed.

The type that admits local candidates under every condition we know is rank 7 with inertia $(6, 1)$. There, the candidates we measured are interior points of $\Sigma_{13}$.

## Appendix A. A second proof for $m = 5$

Theorem 22 covers $m = 5$. This appendix gives an alternative proof for $m = 5$, with a case analysis specific to $5 \times 4$. It uses the notation and Lemmas 3–13 of Section 2, and none of Section 3.

**Lemma 24** (Degenerate hyperplane calculation)**.** *Suppose* $\dim A = d$, $\dim E = d+1$, *a generic* $T_b$ *is injective, and every* $Q(b)$ *is singular. Choose a Witt decomposition*

$$E = W \perp \langle e, f \rangle, \quad \dim W = d-1, \quad e^2 = f^2 = 0, \quad \langle e, f \rangle = 1,$$

*and coordinates with* $T_b(w, z) = (w, z, 0)$. *Then, for every* $c$,

$$T_c(w, z) = (A_c w + p_c z,\ \ell_c^t w + t_c z,\ p_c^t w + \eta_c z), \quad A_c^t = A_c,$$

*and*

$$\eta_c = 0, \qquad p_c^t p_c = 0, \qquad p_c^t A_c p_c = 0. \tag{6}$$

*Proof.* A degenerate hyperplane in a nondegenerate $(d+1)$-space has a one-dimensional radical, so the stated Witt coordinates exist. Flatness with $b$ gives the displayed form of $T_c$. Suppress the subscript $c$. The blocks of $Q(b + \tau c)$ are the following exact multiplication, computation C2:

$$\begin{aligned} \mathcal{M} &= I + 2\tau A + \tau^2(A^2 + \ell p^t + p\ell^t), \\ \nu &= 2\tau p + \tau^2(Ap + tp + \eta\ell), \\ \omega &= 2\tau\eta + \tau^2(p^t p + 2t\eta). \end{aligned}$$

Since $\mathcal{M}(0) = I$, the formal Schur complement $\omega - \nu^t \mathcal{M}^{-1} \nu$ is defined. With $\nu = \tau\nu_1 + \tau^2\nu_2$, $\nu_1 = 2p$, $\nu_2 = Ap + tp + \eta\ell$ and $\mathcal{M}^{-1} = I - 2\tau A + O(\tau^2)$,

$$\nu^t \mathcal{M}^{-1} \nu = 4\tau^2 p^t p + \tau^3(2\nu_1^t \nu_2 - 2\nu_1^t A \nu_1) + O(\tau^4) = 4\tau^2 p^t p + \tau^3(-4p^t Ap + 4tp^t p + 4\eta p^t \ell) + O(\tau^4),$$

and $\omega$ has no $\tau^3$ term. Hence the first three coefficients of $\omega - \nu^t \mathcal{M}^{-1} \nu$ are

$$2\eta, \quad 2t\eta - 3p^t p, \quad 4p^t Ap - 4tp^t p - 4\eta p^t \ell.$$

Its vanishing follows from $\det Q(b + \tau c) = 0$. Successively setting these coefficients to zero proves (6). □

**Theorem 25** (Full-support rank-six exclusion)**.** *There is no complex Hankel multiplier on $\mathbb{C}^5 \otimes \mathbb{C}^4$ of rank six, supports $(5,4)$, and product-free image.*

*Proof.* Suppose such a multiplier exists. Use $s, U, Q$ from (1)–(3).

*Case (i): some $Q(b)$ is invertible.* Lemma 6, with $d = 5$, gives a contradiction.

*Case (ii): $s = 5$, but every $Q(b)$ is singular.* Apply Lemma 24; now $W = \mathbb{C}^4$. The vectors $p_c$, linear in $c$, span a totally isotropic space $P \subset W$ by (6) and polarization, so $\dim P \le 2$. If $P = 0$, then $T^* Je = 0$, contrary to surjectivity. If $\dim P = 1$, then

$$(T^* Je)(w, z; c) = p_c^t w$$

is a nonzero decomposable covector. It remains to exclude $\dim P = 2$.

The mixed $w, z$ coefficients of flatness give, for all $i, j$,

$$A_i p_j + t_j p_i = A_j p_i + t_i p_j.$$

Set $\widetilde{A}_i = A_i - t_i I$. The tensor $\mathcal{B}(i,j,k) = p_i^t \widetilde{A}_j p_k$ is symmetric in all three indices: symmetry of $A_j$ interchanges the first and third indices, and $\widetilde{A}_j p_k = \widetilde{A}_k p_j$ interchanges the last two. Its diagonal cubic is zero by (6) and total isotropy of $P$. Polarization therefore gives $\mathcal{B} = 0$. It follows that $\widetilde{A}_c P \subset P^\perp = P$, and hence $A_c P \subset P$. This polarization and inclusion are computation C3. For $D = P \oplus \mathbb{C}z \subset A$, of dimension three, the formula for $T_c$ now gives

$$T(D \otimes B) \subset P \oplus \mathbb{C}e.$$

The right-hand space is totally isotropic. Lemma 5 applies since $\dim(A/D) = 2$, giving a contradiction.

*Case (iii): $s \le 4$.* Since $\ker T_b \neq 0$, (2)–(3) force exactly the alternatives

$$s = 3, \quad \operatorname{rank} Q(b) = 1; \qquad s = 4, \quad \operatorname{rank} Q(b) = 2. \tag{7}$$

In the first alternative, Lemma 7 either gives first support one or writes $q(x,b) = gL(x,b)^2$ with $g$ constant and $L$ bilinear. A $5 \times 4$ bilinear form has first support at most four. Both conclusions contradict first support five.

For the second alternative, let $\mathbb{C}x(b) = \ker T_b$ generically, and let $N_b = U \cap U^\perp$, of dimension two. Equation (1) says $T_c x(b) \in N_b$ for every $c$; (2) makes the map $c \mapsto T_c x(b)$ have rank exactly two. Choose a rational local representative $x(b)$. Differentiating $T_b x(b) = 0$ gives

$$T_b(dx_b(c)) = -T_c x(b).$$

Because $T_b$ identifies $A/\mathbb{C}x(b)$ with $U$, the projective kernel map $\mathbb{P}(B) \dashrightarrow \mathbb{P}(A)$ has generic differential rank two. In characteristic zero its image closure $X$ is an irreducible surface. Its linear span $\mathbb{P}(D)$ has $\dim D \ge 3$.

For every $c$, $q(x(b), c) = 0$, since $N_b$ is isotropic. Also $\operatorname{rank} Q(c) \le 2$ for every $c$, by closedness of rank conditions. Each nonzero quadric $q(\,\cdot\,, c)$ therefore factors into two linear forms over $\mathbb{C}$. Irreducibility puts $X$, and hence its span, in one of the two hyperplanes. Thus $q(d, c) = 0$ for all $d \in D, c \in B$. At least one such quadric is nonzero, so $\dim D \le 4$. Polarizing first in $d$, then in $c$, and using (H), gives

$$H(D \otimes B, D \otimes B) = 0.$$

Consequently $T(D \otimes B)$ is totally isotropic, while $\dim(A/D) \le 2$. Lemma 5 supplies the contradiction. □

**Lemma 26** (Support bounds)**.** *A nonzero product-free multiplier with supports $a \le 5$, $b_0 \le 4$ satisfies*

$$a, b_0 \ge 2, \qquad \max(a, b_0) + 1 \le r \le (a-1)(b_0-1). \tag{8}$$

*Proof.* Restrict to complements of the factor radicals; this preserves rank and product-freeness. If a factor has dimension one, every nonzero image covector is decomposable. Otherwise the projective image, of dimension $r-1$, is disjoint from the Segre variety of dimension $a+b_0-2$ in $\mathbb{P}^{ab_0-1}$. The projective dimension theorem gives the upper bound in (8).

For the first lower bound, suppose $r \le a \le 4$. If generic $T_b$ is not injective, (2)–(3) contradict $\min(s-1, r-s) \le 1$. If it is injective, then $r = a$ and it is an isomorphism onto $E$. Identify $A = E$ by this map. Flatness makes the operators $T_b^{-1}T_c$ self-adjoint and pairwise commuting. A common eigenvector contradicts Lemma 3. Exchanging the two factors proves $r \ge b_0 + 1$, since $b_0 \le 4$.

For $a = 5$, suppose $r \le 5$. The injective case has just been settled. Otherwise (2)–(3) force $r = 5, s = 3$ and generic $\operatorname{rank} Q = 1$. Lemma 7 gives first support one or at most $b_0 \le 4$, a contradiction. This proves the repaired first-support-five lower bound. □

**Lemma 27** (Rank five in $4 \times 4$)**.** *A rank-five multiplier with supports $(4,4)$ cannot have product-free image.*

*Proof.* Restrict $H$ to $A' \otimes B$, with $A'$ a complement of the first radical ($\dim A' = 4$); rank, flatness and product-freeness are kept, as in the support lemma. An invertible slice is excluded by Lemma 6 with $d = 4$. If generic $T_b$ has rank four but every $Q(b)$ is singular, apply Lemma 24. The $p_c$ span a totally isotropic subspace of the three-dimensional nondegenerate space $W$, hence have span of dimension at most one. Span zero gives $T^*Je = 0$, contradicting surjectivity; span one makes $T^*Je$ decomposable and nonzero.

Otherwise (2)–(3), with $r = 5$, force $s = 3$ and generic $\operatorname{rank} Q = 1$. Lemma 7 either gives first support one or $q(x,b) = g(x^tPb)^2$, with $g \ne 0$ constant. Full first support makes the $4 \times 4$ matrix $P$ invertible. Change bases and rescale to make $q = (x^tb)^2$. Its Hankel polar map on $4 \times 4$ matrices is

$$X \longmapsto \tfrac{1}{2}(\operatorname{tr}(X)I + X^t). \tag{9}$$

This identity is computation C4. If its value is zero, $X^t = -\operatorname{tr}(X)I$; taking traces gives $5\operatorname{tr}(X) = 0$, and then $X = 0$. Thus its rank is sixteen, contradicting rank five. □

For $r \le 6$, the integer inequalities (8) and Lemma 27 leave precisely the following list, computation C5:

$$\begin{gathered}(a, b_0, r) = (5,4,6), (4,4,6), (5,3,6),\\ (4,3,5), (4,3,6), (3,4,5), (3,4,6), (3,3,4).\end{gathered} \tag{10}$$

Indeed $a = 2$ or $b_0 = 2$ violates (8); $a, b_0 \ge 3$, $a \le 5, b_0 \le 4, r \le 6$ then give this list and $(4,4,5)$, which Lemma 27 removes.

Assume now that $\Sigma_{14} \ne \Sigma$, and take $F, H, W$ from Proposition 10. Its support triple is on list (10). The triple $(5,4,6)$ is excluded by Theorem 25. For $a = 3$, (15) and $\mathrm{BSR}(2,4) \le 5$ give

$$\mathrm{SOS}_{\mathbb{R}}(F) \le 17 - r.$$

Thus the triples $(3,4,5), (3,4,6), (3,3,4)$ give bounds $12, 11, 13$, respectively, contradicting length fourteen. These three integer evaluations are computation C9.

**The triple** $(4,4,6)$**.** Write $K = \ker H$. Since $\dim K = 14 = \dim W$, we have $W = K$. The first radical is $\mathbb{R}x_0$, so $x_0 \otimes B \subset W$. Use Lemmas 11–12 with $a = n = 4, r = 6$. The alternating map has domain dimension $4\binom{4}{2} = 24$ and target dimension $4 \cdot 6 = 24$. Its annihilator has exactly four dimensions: an extra map would give a product-free multiplier with supports $(5,4)$ and rank six, forbidden by Theorem 25. Therefore $\operatorname{rank}\Psi = 24 - 4 = 20$. Since $H$ is indefinite, $p \le 5$, and (13) gives the contradiction

$$20 \le 4p - 4 \le 16. \tag{16}$$

**The triple** $(5,3,6)$**.** Again $W = K$. If $\mathbb{R}y_0$ is the second radical, then $A \otimes y_0 \subset W$. Exchange the factors in Lemmas 11–12. The coefficient support has dimension $a = 3$, while the indexed fiber has dimension $n = 5$. Domain and target both have dimension thirty: $3\binom{5}{2} = 30$, $5 \cdot 6 = 30$. The annihilator has dimension three, since an additional map would produce the excluded full-support $5 \times 4$ multiplier. Consequently

$$27 = \operatorname{rank}\Psi \le 5p - 3 \le 22. \tag{17}$$

The dimensions and inequalities (16)–(17) are computation C10.

**The triple** $(4,3,6)$**.** Both factor radicals are lines, with generators $x_0, y_0$, and $W = K$. Here and below $WK$ denotes the linear span in $R_2$ of the products $wk$ with $w \in W$, $k \in K$. Put $u = x_0 \otimes y_0$, and choose complements with active bases $a_1, \dots, a_4$, $b_1, b_2, b_3$. For every active tensor $v = \sum v_{ij} a_i \otimes b_j$, the rectangle identities give

$$uv = \sum_{ij} v_{ij}(a_i \otimes y_0)(x_0 \otimes b_j) \in WK. \tag{18}$$

Choose orthogonal summand coordinates with $h_1 = cu$, $c \neq 0$. For an active $v$ with $H(v,v) < 0$, write $-h_1 v = \sum_i h_i k_i$, $k_i \in K$, using (18). Then $D_i = \delta_{i1} v + k_i$ is a syzygy, and its second variation is $H(v,v) < 0$, contradicting (11). Here the active space has dimension twelve and maps onto the six-dimensional space $E$, so a negative $v$ exists because $J$ is indefinite.

**The triple** $(4,3,5)$**.** Now $\dim K = 15$, so $W = \ker\lambda$ for a nonzero real functional $\lambda$ on $K$. Let

$$u = x_0 \otimes y_0, \quad U = x_0 \otimes B + A \otimes y_0, \quad K = U \oplus K_0,$$

where $K_0$ is the kernel in the active $4 \times 3$ block. Thus $\dim U = 4 + 5 - 1 = 8$ and $\dim K_0 = 12 - 5 = 7$. We treat both positions of $u$.

If $u \notin W$, then

$$W \cap (x_0 \otimes B) = x_0 \otimes L$$

for a three-dimensional subspace $L$ projecting isomorphically onto $B/\mathbb{R}y_0$. The restricted multiplier on $A' \otimes L$ retains rank five, first support four, and the same signature. Lemma 11 applies with $a = 4, n = 3$. Its alternating map has domain dimension $4\binom{3}{2} = 12$ and target dimension $3 \cdot 5 = 15$. The annihilator has exactly four dimensions: Lemma 12 would turn an additional map into a product-free multiplier with supports $(5,3)$ and rank five, contradicting (8). Consequently, since $p \le 4$,

$$11 = \operatorname{rank}\Psi \le 3p - 4 \le 8, \tag{19}$$

which is impossible. These counts are computation C11.

Suppose instead that $u \in W$. Arrange $h_1 = cu$, $c \neq 0$. For active $v = \sum v_{ij} a_i \otimes b_j$, let

$$P(v) = \sum_{ij} v_{ij}(a_i \otimes y_0)(x_0 \otimes b_j) \in \operatorname{Sym}^2 U.$$

Its product in $R_2$ equals $uv$, by (18). If $\lambda|_U = 0$, then $U \subset W$, so $uv \in WK$ for every $v$. Otherwise $U \cap W$ is a hyperplane of $U$, and

$$\operatorname{Sym}^2 U/(U \cap W)U$$

is one-dimensional, with quotient measured by $(\lambda|_U)^2$. Hence $uv \in WK$ whenever

$$\varphi(v) = \sum_{ij} v_{ij}\lambda(a_i \otimes y_0)\lambda(x_0 \otimes b_j) = 0. \tag{20}$$

The functional $\varphi$ is zero or decomposable on the active tensor space. If nonzero, it cannot annihilate $K_0$: otherwise it would lie in $K_0^{\perp} = \operatorname{im} H_{\text{active}}$, contradicting product-freeness. Start with a negative active $v$. If necessary add an element of $K_0$ to impose $\varphi(v) = 0$, without changing $H(v,v)$. Thus in every subcase a negative $v$ satisfies $h_1 v \in WK$. Write $-h_1 v = \sum_i h_i k_i$, $k_i \in K$. The syzygy $D_i = \delta_{i1} v + k_i$ again has second variation $H(v,v) < 0$, violating (11). Equations (18),(20) and the one-dimensional quotient calculation are computation C12.

*Proof of the main theorem for $m = 5$.* If $\Sigma_{14} \neq \Sigma$, Proposition 10 supplies a product-free indefinite multiplier of rank at most six. Its supports and rank lie in the complete list (10). Theorem 25 excludes the full-support case. Compression excludes the three cases with first support three. Equations (16)–(19) and the two mixed-radical arguments exclude all four remaining cases. This contradiction proves $\Sigma_{14} = \Sigma$, or $\mathrm{BSR}(5,4) \leq 14$. □

All computations C1–C12 above are displayed symbolic identities, dimension evaluations, or explicit pairings. No numerical optimization, finite-search classification, or unreported computer calculation enters the proof.

## Appendix B. Certificates for the negative-syzygy obstruction

Each row is one rational multiplier $H$ of the given subtype $(a,b,r)$ with inertia $(p,q)$, built from the family $H = \sum_t w_t z_t z_t^{\mathsf{T}}$ of rank-one tensors $z_t$ (with a product-free image, checked exactly), and one rational $q$-dimensional $N$ with $H|_N$ negative definite (checked by an exact $LDL^{\mathsf{T}}$).

- $i(D)$ is the exact rank deficiency over $\mathbb{Q}$.
- The bound on $i(K)$ reduces the integer basis of $K$ modulo the primes 1000003 and 998244353. The script checks that the reduced basis keeps full rank, and then uses $\operatorname{rank}_{\mathbb{F}_p} \leq \operatorname{rank}_{\mathbb{Q}}$.
- "Loss" bounds the middle term of Proposition 23 with $s = 13$. It is the binomial $\binom{k-13+q+1}{2}$, or, in the rows marked $^*$, that binomial minus $\binom{q+1}{2}$; there the $\operatorname{Sym}^2 N$-block vanishes exactly, and a nonzero minor at a rational $W$ shows that the bound is attained.

Every bound is valid for every 13-dimensional $W \subseteq K$, but only at the multiplier shown. The script (`datos_perspectivas.py`) and its log are in the supplementary material.

| $(a,b,r)$ | $(p,q)$ | $k$ | $q$ | $i(D)$ | $i(K)\leq$ | loss $\leq$ | $i(D)-i(K)-\text{loss}\geq$ |
|---|---|---|---|---|---|---|---|
| $(3,3,4)$ | $(1,3)$ | 16 | 3 | 40 | 16 | 21 | 3 |
| $(4,3,5)$ | $(2,3)$ | 15 | 3 | 26 | 9 | 15 | 2 |
| $(4,3,5)$ | $(1,4)$ | 15 | 4 | 40 | 9 | 21 | 10 |
| $(4,4,6)$ | $(2,4)$ | 14 | 4 | 22 | 4 | 15 | 3 |
| $(5,3,6)$ | $(4,2)$ | 14 | 2 | 10 | 3 | 6 | 1 |
| $(5,3,6)$ | $(3,3)$ | 14 | 3 | 15 | 3 | 10 | 2 |
| $(5,3,6)$ | $(2,4)$ | 14 | 4 | 21 | 3 | 15 | 3 |
| $(4,4,6)$ | $(4,2)$ | 14 | 2 | 8 | 4 | $3^*$ | 1 |
| $(4,4,6)$ | $(3,3)$ | 14 | 3 | 12 | 4 | $4^*$ | 4 |
| $(4,3,5)$ | $(3,2)$ | 15 | 2 | 18 | 9 | $7^*$ | 2 |
| $(3,3,4)$ | $(2,2)$ | 16 | 2 | 29 | 16 | $12^*$ | 1 |

## Appendix C. Formula sheet

| quantity | formula |
|---|---|
| main bound (Theorem 1) | $\boxed{\mathrm{BSR}(m,4)\leq 3m-1}$ for $5\leq m\leq 16$; previous $\mathrm{BSR}(m,n)\leq m+\mathrm{BSR}(m,n-1)$, so $3m$ |
| dimensions $(m\times n)$ | $\dim V=mn$, $\dim R_2=M=\binom{m+1}{2}\binom{n+1}{2}$; for $m\times 4$, $M=5m(m+1)$, $N=3m-1$ |
| boundary inertia (Lemma 14) | $N(q-1)\leq M-\binom{N+2}{2}$; for $m\times 4$: $2(3m-1)(q-1)\leq m(m+7)$, i.e. $q\leq\lfloor(m+13)/6\rfloor$ $(m\geq 5)$; $\leq 4$ iff $m\leq 16$, $\leq 5$ iff $m\leq 22$ |
| Barvinok–Pataki (case $q=1$) | $\Sigma_N=\Sigma$ for $N\geq r_P=\max\{r:\binom{r+1}{2}\leq M\}$ |
| rank of the multiplier | $\operatorname{rank}H\leq\dim V-N=m+1$; product-free image: $\operatorname{rank}H\leq(a-1)(b-1)$ (Parthasarathy) |
| compression (Lemma 13) | $\mathrm{SOS}_{\mathbb{R}}(F)\leq(4a-r)+\mathrm{BSR}(m-a,4)\leq 3m-2$ for $a\leq m-2$, $r\geq a+1$ |
| low-index lemma (Lemma 20) | $q\leq 4$: $r\geq a+1$, and $r\geq a+2$ when $b=4$ |
| remaining triples | $(m-1,4,m+1)$, $(m,3,m+1)$, $(m-1,3,m+1)$, $(m-1,3,m)$ (Figure 5) |
| singular incidence | dimension $\leq M-2(m+3)+(2m+4)=M-2$, codimension two (Proposition 10, proof of Theorem 22); used through Lemma 9 |

| quantity | formula |
|---|---|
| slopes | $\mathrm{BSR}(m,2) = m+1$, $\mathrm{BSR}(m,3) = 2m$, here $3m-1$; Pataki slope $\sqrt{n(n+1)/2}$; $(n-1)^2 > n(n+1)/2$ iff $n \geq 5$ |

## The companion site

This paper has a companion website, its *blueprint*, at https://karlesmarin.github.io/blueprints/bsr-m-four/. There one finds:

- *The statements.* Every theorem, proposition, lemma and corollary, copied from this paper, with its proof and with the part of it that is formalised in Isabelle/HOL or certified by exact computation.
- *The dependencies.* What each statement uses and where it is used, as a list and as a graph.
- *The map.* The result, the tools it rests on and the question it answers.
- *The paper itself.* Its PDF, opened at the right page from every statement.

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(Carles Marín) Independent researcher
*Email address*: karlesmarin@gmail.com

(Liqun Qi) School of Science, Zhejiang University of Science and Technology, Hangzhou, Zhejiang, China; Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
*Email address*: maqilq@polyu.edu.hk

(Chen Ouyang) School of Computer Science and Technology, Dongguan University of Technology, Dongguan 523000, China
*Email address*: oych26@163.com