HOW MANY DYCK WORDS DO THE DIVISORS OF AN INTEGER MAKE? EXACT COUNTS AND QUASIPOLYNOMIAL GROWTH

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ABSTRACT. For an integer $n \geq 1$, Rodríguez Caballero's word $\langle\langle n \rangle\rangle_2$ records how the divisors of n interleave with the doubled divisors: read $D_n \triangle 2D_n$ in increasing order and write an opening bracket (for elements of D_n and a closing bracket) for elements of $2D_n$. It is a symmetric Dyck word whose shape reflects classical divisor statistics. Rodríguez Caballero asked which Dyck words arise in this way; we know of no answer in the literature. Let L_k be the set of these words of semilength k. Starting from the reduction to the odd part, we describe the words through the cells of a hyperplane arrangement in the logarithms of the primes, and show that L_k is produced by the single finest factorisation pattern $p_1^{q_1-1} \cdots p_r^{q_r-1}$, where $k = q_1 \cdots q_r$ is the prime factorisation; as a consequence, membership in L is decidable. For the pattern $p^a p'^b$ the number $R(a,b)$ of cells is a classical count: $8R(a,b) - 4a - 4b - 6$ is the number of two-dimensional threshold functions on a grid of $(2b+1) \times (2a+1)$ points, for which Alekseyev gave an exact formula, so $R(a,b) = R(b,a)$ and its leading coefficient C_a in b is asymptotic to $12a^2/\pi^2$; we give a direct proof in our coordinates. We show that the number $W(a,b)$ of words satisfies $(\frac{2}{3} - o(1))b^2 \leq W(a,b) \leq (C_a + o(1))b^2$. Together with a theorem of Woods on Presburger counting functions, this shows that for fixed a, $W(a,b)$ is eventually a quasipolynomial in b of degree exactly two; its period and threshold are not determined. We also prove three general mechanisms by which distinct cells give the same word. Finally, $|L_{2q}| = (9q^2 - 10q + 13)/4$ for every odd prime q, with a proof by hand, and $|L_{3q}| = \big(44q^2 - 69q + 58 - 3\big(\frac{q}{3}\big)\big)/6$ for every prime $q \geq 5$, where $\big(\frac{q}{3}\big) = \pm 1$ according as $q \equiv \pm 1 \pmod 3$. In the second count, one lemma (the listed coincidences are all of them, and are disjoint) is a computer-assisted proof using exact SMT queries (496 queries for completeness and 231 for disjointness, all unsatisfiable, after 38 of 69 branches were discarded as infeasible by the same solver), without a checked proof object. Whether almost every cell has its own word for $a \geq 2$ is left to a companion paper.

Results at a glance

- **I. Geometry** (Lemmas 2.1, 2.3). The words with odd part of pattern f are the words of the cells of the arrangement $\{\alpha \cdot x = 1\}$ in the positive orthant; every cell is realised by primes. written proof
- **I. Maximal refinement** (Theorem 3.1, Corollary 3.3). If $k = q_1 \cdots q_r$ (primes), L_k is the set of words of the pattern $(q_1 - 1, \ldots, q_r - 1)$; so $|L_q| = q$ and $|L_{q_1 q_2}| = W(q_1 - 1, q_2 - 1)$. written proof
- **I. Decidability** (Corollary 4.2 and the remark after it). Membership in L is decidable, in time $\exp\big(O((\log k)^2)\big)$. written proof
- **II. Number of cells** (Theorem 5.1, Corollary 5.2, Proposition 5.3). A known count: $R(a,b) = (N(2b,2a) + 4a + 4b + 6)/8$, with N the number of two-dimensional threshold functions (Acketa–Žunić, Alekseyev); a quasipolynomial of degree 2 in b with leading coefficient $C_a \sim 12a^2/\pi^2$. known result, direct proof here

Date: November 2026, version 2. Zenodo, doi:10.5281/zenodo.24541288. Companion site: karlesmarin.github.io/blueprints/dyck-divisors.

- **II. Growth** (Theorems 6.1, 6.3, Corollary 6.4). $(\frac{2}{3} - o(1))b^2 \le W(a,b) \le (C_a + o(1))b^2$; for fixed a, $W(a,b)$ is eventually a quasipolynomial in b of degree exactly two (existence only). written proof
- **II. Coincidences** (Lemmas 7.1, 7.2, 7.5, 7.7). Three mechanisms; the number of cells lost is at least linear in b. written proof
- **III. Theorem A** (Theorem 8.12). $W(1,b) = \lfloor (9b^2+8b+12)/4 \rfloor$ for $b \ge 2$; $|L_{2q}| = (9q^2-10q+13)/4$ for odd primes q. written proof
- **III. Theorem B** (Theorem 9.4). $W(2,b) = (44b^2+19b+e_{b \bmod 6})/6$ for $b \ge 3$; $|L_{3q}| = (44q^2 - 69q + 58 - 3(\frac{q}{3}))/6$ for primes $q \ge 5$. computer-assisted (SMT) (one lemma, §9.2), written proof (the rest)
- **III. Small k** (Table 1). Three or more primes: values only, $|L_8| = 40$, $|L_{12}| = 252$, $|L_{16}| = 1083$, $|L_{18}| = 1222$. exact enumeration
- **Open** (Section 11). Rigidity, $W(a,b) = C_a b^2 + O_a(b)$ for $a \ge 2$ (Question 11.1); the closed form of $W(3,b)$ (Question 11.3). conjecture (for $a = 3$)

written proof proof in the text

computer-assisted (SMT) one step decided by exact SMT queries, no proof object

exact enumeration exact enumeration in rational arithmetic

conjecture fitted to data, not proved

known result, direct proof here the count is in the literature; the text gives a direct proof

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Companion site. https://karlesmarin.github.io/blueprints/dyck-divisors/: every statement with its dependencies and its checks, its Isabelle/HOL proof where there is one, or the script that checks it. This paper: doi:10.5281/zenodo.24541288 (Zenodo), with its supplementary material.

1. Introduction

Rodríguez Caballero [6, 8, 5] attached to every integer $n \ge 1$ a word over a two-letter alphabet. Let D_n be the set of divisors of n and $2D_n = \{2d : d \in D_n\}$. Write the elements of the symmetric difference $D_n \triangle 2D_n$ in increasing order and replace each element by (if it lies in $D_n \setminus 2D_n$ and by) if it lies in $2D_n \setminus D_n$; the result is $\langle\langle n \rangle\rangle_2$ ([8, Definition 1] with $\lambda = 2$, letters $a = (, b =)$). Caballero proved that every $\langle\langle n \rangle\rangle_2$ is a symmetric Dyck word [8, Lemma 13] and related several of

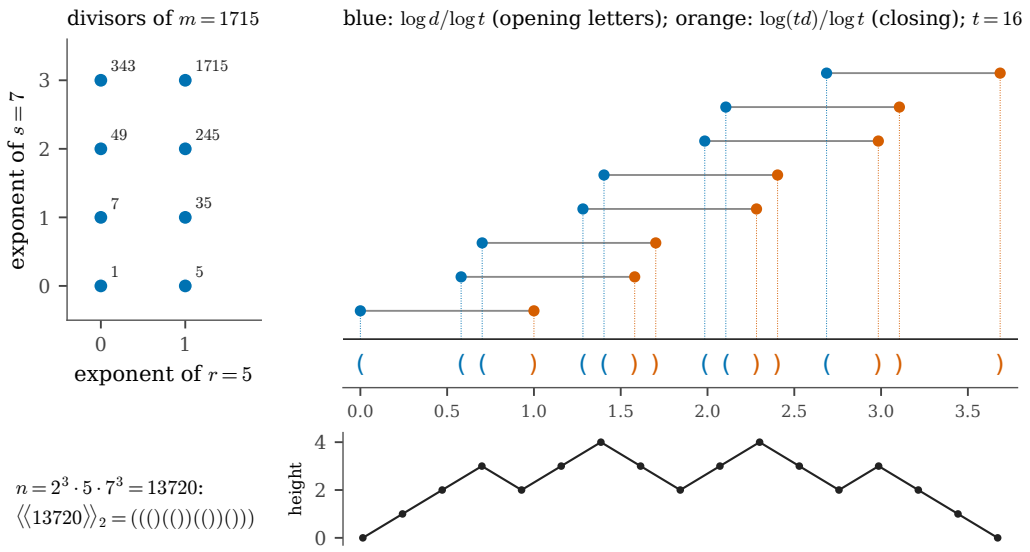


Figure 1. The word of $n = 2^3 \cdot 5 \cdot 7^3$. Left: the divisors of the odd part $m = 5 \cdot 7^3$ as the grid of exponents $\{0,1\} \times \{0,\dots,3\}$. Right: each divisor d projected to $\log d/\log t$ (blue) and joined to its partner $\log(td)/\log t$ (orange), one unit to the right ($t = 16$). Reading the line from left to right gives $\langle\langle n \rangle\rangle_2$; its Dyck path is drawn below. Live version in the blueprint.

its features to classical arithmetic: its height is the largest number of divisors in an interval $[d, 2d)$ [8, Lemma 19], which ties it to Hooley's Δ-function [3, 6]; it lies in $(())^*$ exactly when n is not the semi-perimeter of a Pythagorean triangle [8, Lemma 18]; it is irreducible exactly when n is 2-densely divisible [8, Lemma 27]. Through the Kassel–Reutenauer polynomials the same divisor pattern also controls the number of ideals of finite codimension of the Laurent polynomial ring in two variables over $\mathbb{F}_q$ [4, 9]; see Appendix A. Let $L = \{\langle\langle n \rangle\rangle_2 : n \geq 1\}$. The language L is Rodríguez Caballero's [8], and so is the question of describing it: in 2020 he asked publicly whether, for a given balanced word, it can be proved in first-order Peano arithmetic that it is, or is not, of the form $\langle\langle n \rangle\rangle_2$ [14]; in December 2021 he asked on the same list whether there is a balanced word w for which it cannot be determined whether the search for an n with $\langle\langle n \rangle\rangle_2 = w$ halts, and wrote that he thought the problem undecidable [15]; and he raised it again in a public discussion in November 2026. We do not know of a description of L in the literature. The paper [8], which we read in full, gives none and does not count words by divisor pattern. This note studies the problem one semilength at a time.

Definition 1.1. For $k \geq 1$ let L_k be the set of words of L of semilength k (length $2k$).

By Lemma 2.1 below, the semilength of $\langle\langle n \rangle\rangle_2$ is $\tau(m)$, where m is the odd part of n. So $L_k = \{\langle\langle n \rangle\rangle_2 : \tau(\text{odd part of } n) = k\}$. Figure 1 shows how the word of one integer is read off its divisors.

Approach and organisation. Two observations drive the note. First, after the reduction to the odd part (Lemma 2.1), the word only sees on which side of each hyperplane $\alpha \cdot x = 1$ the point $x = (\log p_i/\log t)_i$ lies; so the words are labels of the cells of a finite arrangement, and every cell is realised by primes (Lemma 2.3). Distinct cells may carry the same word, because the word forgets which divisor produced each letter. Second, the semilength k fixes a single finest pattern of exponents that already produces all of L_k (Theorem 3.1). So membership in L is decidable (Corollary 4.2), which answers algorithmically the decision problem posed by Rodríguez Caballero in 2020 [14], in the opposite sense to the undecidability he expected in 2021 [15]; his question about Peano arithmetic remains open.

For two primes the finest pattern is $p^a p'^b$, and counting L_{pq} splits into two problems of different nature. Counting the cells is a classical problem in disguise: after an affine rescaling, the number of cells is the number of two-dimensional threshold functions on a grid of $(2b+1)\times(2a+1)$ points, counted exactly by Alekseyev [16] (Proposition 5.3); we give a direct proof in our coordinates (Theorem 5.1). Controlling the cells that share a word is where the difficulty lies: one cell in ten is redundant for $a=1$, whereas for $a=2$ the proportion tends to 0 (Remark 9.5). This note counts the cells and words and describes how they coincide; whether almost every cell has its own word for every $a\ge 2$, the rigidity question (Question 11.1), is left to a companion paper in preparation.

Part I (Sections 2–4) holds for any number of primes. Part II (Sections 5–7) treats the pattern $p^a p'^b$ for general a: the number of cells, the growth of $W(a,b)$ and the mechanisms of coincidence. Part III (Sections 8–10) gives the exact counts for $a=1,2$ and a table of $|L_k|$. The box on the first page lists the results with references and the status of each. Section 11 lists open questions, and Section 12 what this note does not claim. Appendix A records, for completeness, a reformulation of the words through the Kassel–Reutenauer polynomials that is implicit in Caballero's work [9, 8]; we claim no novelty for it.

Part I. Structure of the language

2. Reduction to an arrangement of hyperplanes

Lemma 2.1. *Write $n=2^{e-1}m$ with m odd, $e\ge 1$, and put $t=2^e$. Then $D_n\setminus 2D_n=D_m$ and $2D_n\setminus D_n=tD_m$. Hence $\langle\!\langle n\rangle\!\rangle_2$ is obtained by merging D_m (letters $($) with tD_m (letters $)$), and its semilength is $\tau(m)$.*

The reduction is implicit in Rodríguez Caballero's original construction [14], which writes the divisors of $2n$ and deletes those d for which d and $2n/d$ are both even: what remains is exactly $D_m\cup 2^eD_m$. The identity in this form was noted in the public discussion that led to this note; we include the short proof.

Proof. $D_n=\{2^a d : 0\le a\le e-1,\ d\mid m\}$ and $2D_n=\{2^a d : 1\le a\le e,\ d\mid m\}$. The two sets share the elements with $1\le a\le e-1$, so the difference $D_n\setminus 2D_n$ consists of the elements with $a=0$ and $2D_n\setminus D_n$ of those with $a=e$. □

Since t is even and m odd, no element of D_m equals an element of tD_m. The map $d\mapsto td$ preserves order, so the j-th letter) corresponds to the j-th smallest divisor $d_{(j)}$ of m. The word is therefore determined by the multiset

$$C(n)=\{\, c_{d'}:=\#\{d\in D_m : d<td'\}\ :\ d'\in D_m\,\}. \tag{1}$$

Sorted increasingly, it is the number of letters (preceding each letter). On a logarithmic scale, each divisor d and its partner td are one unit $\log t$ apart; Figure 1 shows this for $n=13720$.

2.1. Log-coordinates and patterns. Let $m=p_1^{f_1}\cdots p_r^{f_r}$, where the p_i are distinct odd primes and the exponents $f_i\ge 1$. The multiset $\{f_1,\dots,f_r\}$ is the *pattern* (signature) of m, and $\tau(m)=\prod(f_i+1)$. Put

$$x_i=\frac{\log p_i}{\log t}>0,\qquad I_f=\prod_{i=1}^{r}\{0,\dots,f_i\},\qquad u_a(x)=\sum_i a_i x_i\quad (a\in I_f).$$

Then $u_a(x)=\log d/\log t$ for $d=\prod p_i^{a_i}$, and $d<td'$ if and only if $u_a(x)<1+u_{a'}(x)$.

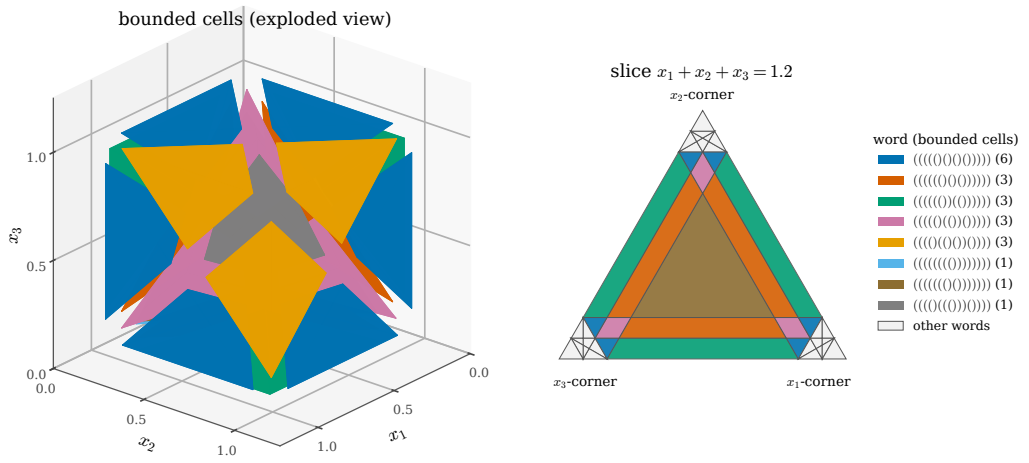


Figure 2. The arrangement $\mathcal{A}_{(1,1,1)}$ (pattern $pp'p''$, $k=8$) of the 19 planes $\alpha\cdot x=1$, $\alpha\in\{-1,0,1\}^3$, that meet the positive octant. Left: its 21 bounded cells, each shrunk towards its centre so that all are visible, coloured by word (8 words; the number of cells of each word is in the legend). Permuting the primes does not change the word, and the multiplicities 6, 3, 1 are consistent with orbits under permutation of the coordinates. Right: the slice $x_1+x_2+x_3=1.2$, cut exactly by the planes (43 pieces, 11 words); pieces of the eight words on the left keep their colours.

Definition 2.2. Let $\mathcal{A}_f$ be the finite set of hyperplanes $\{x:\alpha\cdot x=1\}$ with $\alpha\in\prod_i\{-f_i,\dots,f_i\}$, $\alpha\neq 0$. For $x\in\mathbb{R}^r_{>0}\setminus\bigcup\mathcal{A}_f$ let $w_f(x)$ be the word whose letters) are preceded by

$$c_b(x)=\#\{a\in I_f : u_a(x)<1+u_b(x)\}\qquad (b\in I_f)$$

letters (, sorted increasingly. Explicitly, if $c_{(1)}\le\cdots\le c_{(K)}$ is the sorted list, then $w_f(x)=(^{c_{(1)}})(^{c_{(2)}-c_{(1)}})\dots(^{c_{(K)}-c_{(K-1)}})$.

The function w_f depends only on the signs of $\alpha\cdot x-1$ with $\alpha=a-b$. It is therefore constant on every cell, that is, on every connected component of $\mathbb{R}^r_{>0}\setminus\bigcup\mathcal{A}_f$. For an integer n with odd part of pattern f, the point $x(n)$ avoids every hyperplane, because $u_a-u_b=1$ would give $d=td'$. By (1), $\langle\!\langle n\rangle\!\rangle_2=w_f(x(n))$. Figure 2 shows the arrangement for the pattern $(1,1,1)$.

The cell of x records all the comparisons $u_a<1+u_b$, that is, the strict relation $a\prec b\iff u_a+1<u_b$, which is a *semiorder* (unit interval order) on the exponent vectors: the order of the unit intervals $[u_a,u_a+1]$. The word keeps only the multiset (1) of counts and forgets the labels a; this is why distinct cells can give the same word. Arrangements of the hyperplanes $x_i-x_j=1$ and their relation to semiorders and interval orders go back to Stanley [12], and the polyhedron of numerical representations of a semiorder is studied in [1]. Unlabelled semiorders on m elements are counted by the Catalan numbers, a result of Wine and Freund, and those of length H are equinumerous with the Dyck paths of semilength m and height $H+1$ [23]. Here the points are constrained to the additive form $u_a=\sum_i a_ix_i$, which is what makes the arrangement $\mathcal{A}_f$ finite-dimensional in the number of primes.

Lemma 2.3 realisation by primes. *Let $x^0\in\mathbb{R}^r_{>0}\setminus\bigcup\mathcal{A}_f$. There are $e\ge 1$ and distinct odd primes $p_1,\dots,p_r$ such that $n=2^{e-1}p_1^{f_1}\cdots p_r^{f_r}$ satisfies $\langle\!\langle n\rangle\!\rangle_2=w_f(x^0)$.*

Proof. Let U be the cell containing x^0. It is open and w_f is constant on it. Points with pairwise distinct coordinates are dense, so pick one, $x^1\in U$. Then pick $\delta>0$ such that the box $B=\prod_i(x^1_i-\delta,x^1_i+\delta)$ lies in U, $x^1_i-\delta>0$ for every i, and the r intervals $(x^1_i-\delta,x^1_i+\delta)$ are pairwise

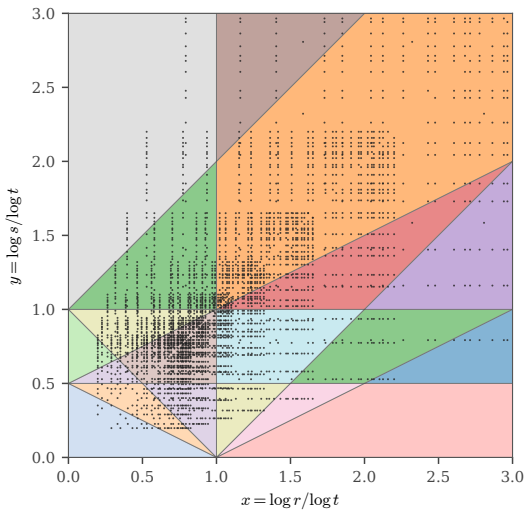


Figure 3. Pattern $(1, 2)$ ($m = rs^2$, $k = 6$): the cells of the lines $\alpha x + \beta y = 1$, $|\alpha| \le 1$, $|\beta| \le 2$, in the window $[0, 3]^2$, coloured by word (19 cells, 16 words; cells of equal colour have the same word), and the points $(\log r, \log s)/\log 2^e$ for distinct primes $3 \le r, s \le 100$ and $e \le 8$ (3590 points). Every one of the 19 cells in the window contains such a point.

disjoint. We use Bertrand's postulate in the form: for every real $y \ge 1$ there is a prime p with $y < p \le 2y$ [2]. Choose e so large that, for each i,

$$J_i = \left(2^{e(x_i^1 - \delta)},\ 2^{e(x_i^1 + \delta)}\right)$$

has left end $y_i \ge 2$ and ratio $2^{2e\delta} > 2$. Then $(y_i, 2y_i] \subset J_i$ contains a prime $p_i > 2$. The J_i are pairwise disjoint, so the p_i are distinct. Moreover $\log p_i / \log 2^e \in (x_i^1 - \delta, x_i^1 + \delta)$, so $x(n) \in B \subset U$ and $\langle\langle n \rangle\rangle_2 = w_f(x(n)) = w_f(x^0)$. □

Hence the set of words of pattern f is exactly $\{w_f(U) : U \text{ a cell of } \mathcal{A}_f\}$. There may be fewer words than cells: non-adjacent cells can give the same word. For instance, the pattern $(3, 3)$ has 191 cells but only 95 words. Figure 3 shows real primes populating the cells of the pattern $(1, 2)$.

3. Maximal refinement

Theorem 3.1. *Let $k \ge 2$ with prime factorisation $k = q_1 q_2 \cdots q_r$, repetitions allowed. Then L_k is the set of words $\langle\langle n \rangle\rangle_2$ whose odd part has pattern $(q_1 - 1, \ldots, q_r - 1)$:*

$$L_k = \{\langle\langle n \rangle\rangle_2 : \text{ the odd part of } n \text{ has pattern } (q_1 - 1, \ldots, q_r - 1)\}.$$

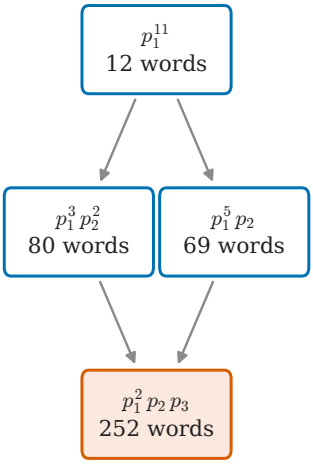
Equivalently, every word of every pattern f with $\prod(f_i + 1) = k$ is also a word of the finest pattern $(q_1 - 1, \ldots, q_r - 1)$.

Lemma 3.2 mixed radix; classical. *Let $u, v \ge 1$ and $F = (u + 1)(v + 1) - 1$. Every integer $N \in \{0, \ldots, F\}$ can be written in exactly one way as $N = i + (u + 1)j$ with $0 \le i \le u$, $0 \le j \le v$. Hence, for every real x_1, $\{Nx_1 : 0 \le N \le F\} = \{ix_1 + j(u + 1)x_1 : 0 \le i \le u,\ 0 \le j \le v\}$, and the map $(i, j) \mapsto i + (u + 1)j$ is a bijection between the index sets.*

Proof. Division with remainder by $u + 1$; $j = \lfloor N/(u + 1) \rfloor \le v$. □

Proof of Theorem 3.1. The words of the finest pattern lie in L_k by definition. For the converse it is enough to show: if $f = (f_1, \ldots, f_s)$ has $f_1 + 1 = (u+1)(v+1)$ with $u, v \ge 1$, and $f' = (u, v, f_2, \ldots, f_s)$,

(a) Absorption of signatures, $k = 12$



arrow: split one exponent F into (u, v) with $(u+1)(v+1) = F+1$
all words of L_{12} occur for the maximal signature: $|L_{12}| = 252$ (exact)

(b) How the results depend on each other

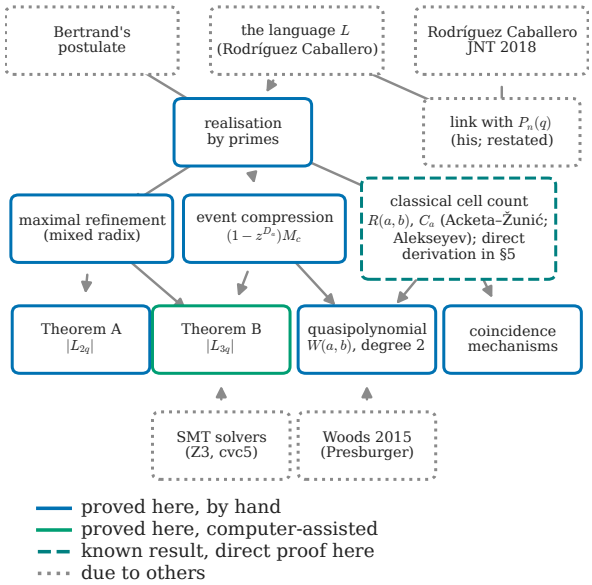


Figure 4. (a) Maximal refinement for $k = 12$: each arrow splits one exponent F into (u, v) with $(u + 1)(v + 1) = F + 1$, and every word of the coarser pattern is a word of the finer one. The counts 12, 69, 80 and 252 of words of the patterns p^{11}, p^5p', $p^3p'^2$ and $p^2p'p''$ are exact enumerations (SageMath over $\mathbb{Q}$), so $|L_{12}| = 252$. (b) How the results depend on each other. Boxes in dotted grey are due to others (Bertrand's postulate, the language L and the link with $P_n(q)$ of Rodríguez Caballero, Woods' theorem) or are external tools (the SMT solvers). The dashed teal box is a known result with a direct proof here (the cell count of Section 5, classical: Acketa–Žunić, Alekseyev). The colour of the frame of the other boxes gives the status of our results (proved by hand, or computer-assisted).

then every word of pattern f is a word of pattern f'. Repeating this step splits every composite $f_i + 1$ into primes. The step terminates because each application increases the number of parts.

Let $\langle\langle n \rangle\rangle_2$ have pattern f, and let $x = x(n) \in \mathbb{R}^s_{>0}$, so x lies on no hyperplane of $\mathcal{A}_f$ and $\langle\langle n \rangle\rangle_2 = w_f(x)$. Define $y = (x_1, (u + 1)x_1, x_2, \ldots, x_s) \in \mathbb{R}^{s+1}_{>0}$ and $\varphi : I_{f'} \to I_f$, $\varphi(i, j, a_2, \ldots, a_s) = (i + (u + 1)j, a_2, \ldots, a_s)$. By Lemma 3.2, φ is a bijection and $u_{a'}(y) = u_{\varphi(a')}(x)$ for all $a' \in I_{f'}$.

Now y lies on no hyperplane of $\mathcal{A}_{f'}$. Indeed, $u_{a'}(y) - u_{b'}(y) = 1$ would give $u_{\varphi(a')}(x) - u_{\varphi(b')}(x) = 1$, which is impossible. Moreover $c'_{b'}(y) = \#\{a' : u_{a'}(y) < 1 + u_{b'}(y)\} = c_{\varphi(b')}(x)$, so the two sorted multisets coincide and $w_{f'}(y) = w_f(x) = \langle\langle n \rangle\rangle_2$. The point y is degenerate, since its second coordinate is exactly $u + 1$ times its first and no pair of primes has that property. But Lemma 2.3 only needs y to avoid $\mathcal{A}_{f'}$. It supplies n' with odd part of pattern f' and $\langle\langle n' \rangle\rangle_2 = w_{f'}(y) = \langle\langle n \rangle\rangle_2$. □

The proof is a one-line identity of point sets. The only analytic input is that strict comparisons survive a small perturbation, followed by Bertrand's postulate. Figure 4(a) shows the absorption of patterns for $k = 12$, and Figure 4(b) how the results of this note depend on each other and on the work of others.

Corollary 3.3. *(a) $L_1 = \{()\}$. For q prime, $L_q = \{O_q(N) : 0 \le N \le q - 1\}$ with $O_q(N)$ the word whose j-th letter) is preceded by $\min(q, N + j)$ letters ($(j = 1, \ldots, q)$. Hence $|L_q| = q$.*

(b) For primes $p \le q$, L_{pq} is the set of words of pattern $(p - 1, q - 1)$. So $|L_{pq}| = W(p - 1, q - 1)$, where $W(a, b)$ denotes the number of words of pattern (a, b).

Proof. (b) is Theorem 3.1. For (a), let $m = p^{q-1}$ and $\Lambda = \log t/\log p \notin \mathbb{Z}$. In units of $\log p$, the letters (sit at $0, \dots, q-1$ and the letters) at $\Lambda+0, \dots, \Lambda+q-1$. The j-th) (at $\Lambda+j-1$, $j \geq 1$) is preceded by $\#\{i \leq q-1 : i < \Lambda+j-1\} = \min(q, \lfloor\Lambda\rfloor + j)$ letters (, so the word is $O_q(\min(\lfloor\Lambda\rfloor, q-1))$. The words $O_q(N)$, $0 \leq N \leq q-1$, are pairwise distinct: the first) is preceded by $N+1$ letters (. Each is realised, by Lemma 2.3 applied to $x^0 = 1/(N+\frac{1}{2})$. □

4. Decidability and complexity

By Theorem 3.1, deciding whether a word of semilength k lies in L reduces to listing the cells of one finite arrangement. The following bound confines every cell to an explicit box.

Proposition 4.1. *Let $\mathcal{A}$ be an arrangement of hyperplanes $\alpha \cdot x = 1$ in the positive orthant of $\mathbb{R}^r$, with $\alpha \in \mathbb{Z}^r$ and $|\alpha_i| \leq M$ ($M \geq 1$). Every open cell of $\mathcal{A}$ contains a point with $0 < x_i < r!\, M^{r-1}+1$ for all i.*

Proof. The closure P of a cell U is a full-dimensional polyhedron in the closed orthant, hence pointed; let v be a vertex. At v there are r linearly independent active constraints $Av = d$, each a hyperplane $\alpha \cdot x = 1$ or a face $x_i = 0$. So A is an integer matrix with entries of absolute value at most M, and $d \in \{0,1\}^r$. By Cramer's rule $v_i = \det A_i / \det A$ with $|\det A| \geq 1$, and expanding $\det A_i$ along the column d gives $|\det A_i| \leq r!\, M^{r-1}$. Points of U arbitrarily close to v satisfy the bound. □

The same bound makes the enumerations of Table 1 exact (last section of Part III).

Corollary 4.2 decidability. *Membership in L is decidable. The empty word is not in L, and $L_1 = \{()\}$. Given a balanced word w of semilength $k \geq 2$, let $k = q_1 \cdots q_r$. Then $w \in L$ if and only if w is the word of some cell of the arrangement of the pattern $(q_1 - 1, \dots, q_r - 1)$, and these finitely many cells can be listed exactly.*

Proof. By Theorem 3.1, L_k is the set of words of the cells of $\mathcal{A}_{(q_1-1,\dots,q_r-1)}$, and every cell is realised (Lemma 2.3). The arrangement is finite. Each sign vector of the hyperplanes and of the coordinate faces defines a strict rational linear system, whose feasibility is decided exactly, and a feasible system yields a rational point and hence the word of its cell. The search can be confined to the box of Proposition 4.1. □

Remark 4.3 (complexity). Let $k = q_1 \cdots q_r$, so $r \leq \log_2 k$. The maximal pattern has $f_i = q_i - 1$, and its arrangement has at most $H = \prod_i (2q_i - 1) - 1 \leq k^{\log_2 3}$ hyperplanes, since $2q - 1 \leq q^{\log_2 3}$ for $q \geq 2$. An arrangement of H hyperplanes in $\mathbb{R}^r$ has at most $\sum_{j \leq r} \binom{H}{j} \leq (H+1)^r$ cells, and they can be listed incrementally with one rational linear program per candidate. So membership is decided in time $\exp(O((\log k)^2))$, and in time polynomial in k when the number r of prime factors of k (with multiplicity) is bounded. Whether a uniformly polynomial algorithm exists is Question 11.7.

This answers algorithmically the decision problem posed by Rodríguez Caballero in 2020 [14]. In 2021 he conjectured that the problem is undecidable ("I think that it is undecidable" [15]); the corollary settles it in the other direction. Whether the argument can be carried out in first-order Peano arithmetic, as he asked in 2020, is a separate question and remains open (Question 11.7).

Part II. General enumeration

This part treats the patterns $p^a p'^b$ with two primes for general a. For integers $a, b \geq 1$, $W(a,b)$ is the number of distinct words $\langle\langle n \rangle\rangle_2$ with $n = 2^{e-1} r^a s^b$, where $r \neq s$ are odd primes and $e \geq 1$;

by Corollary 3.3(b), $|L_{pq}| = W(p-1, q-1)$ for primes $p \le q$. We measure in units of $\log s$, with $\Lambda = \log t/\log s$ and $\theta = \log r/\log s$, and write $\mathrm{clip}(v) = \min(b, \max(-b-1, v))$ and $f(v) = \min(b+1, \max(0, v+1))$, as in Section 8. The word of n depends only on the cell of (Λ, θ) in an arrangement of lines (Section 6, Steps 1 and 2). Part III, which treats $a = 1, 2$ in full, uses from this part only Theorem 5.1, for the number of cells when $a = 2$.

5. The number of cells

Throughout, $a, b \ge 1$. Let $\mathcal{L}_{a,b}$ be the set of lines $\ell_{h,\rho} : \Lambda + h\theta = \rho$ with $|h| \le a$ and $\rho \in \mathbb{Z} \cap [-b, b]$, and let $R(a, b)$ be the number of cells (connected components) of the open quadrant $\mathcal{Q} = \{\Lambda > 0, \theta > 0\}$ minus these lines. The vector $c = (\mathrm{clip}\lfloor \Lambda + h\theta \rfloor)_{|h| \le a}$ records, for each h, between which two consecutive lines of slope h the point lies, so it is the sign vector of the point with respect to $\mathcal{L}_{a,b}$. A nonempty set of points with a given sign vector is an intersection of open half-planes with $\mathcal{Q}$, hence convex. So $R(a, b)$ is the number of realisable vectors (Step 2 of Section 6) and $W(a, b) \le R(a, b)$. For $a = 1, 2$ the formula below specialises to Lemmas 8.3 and 9.2. We write φ for Euler's function and

$$\Phi_q(x) = \#\{1 \le p \le x : \gcd(p, q) = 1\} \qquad (x \ge 0).$$

5.1. A direct count of the cells. The number $R(a, b)$ is a known count. By Proposition 5.3 below, $8R(a, b) - 4a - 4b - 6$ is the number of two-dimensional threshold functions on a grid of $(2b+1) \times (2a+1)$ points, for which Alekseyev gave an exact formula as a sum over coprime pairs [16, Theorem 3]; the corresponding count of linear partitions of the grid goes back to Acketa and Žunić [17], whose Theorem 3 already gives the number of threshold functions in another formulation, as Haukkanen and Merikoski note [20, (1)]. Žunić [18] points out that some of the results of [16] are easy consequences of results known before. Theorem 5.1 and Corollary 5.2 are a direct proof of this count in the coordinates (Λ, θ) of this note. We keep them because Part III uses the count in this form (Lemma 9.2), and because the proof reads the vertices of the quadrant directly: abscissae $\theta = p/q$ with $q \le 2a$, and the term B_q for the pairs of lines that meet on the boundary line $\Lambda = 0$.

Theorem 5.1. *For $a, b \ge 1$,*

$$R(a, b) = 1 + (3a+1)b + a + \frac{1}{2} \sum_{q=1}^{2a} \Big[(2a + 1 - q)\, S_q(b) - B_q(a, b) \Big],$$

where

$$S_q(b) = \sum_{\substack{1 \le p \le 2b \\ \gcd(p,q)=1}} (2b - p + 1),$$

$$B_q(a, b) = \sum_{\substack{-a \le h \le a-q \\ q \mid h}} \Phi_q\Big(\Big\lfloor \frac{b}{\max(|h/q|,\ |h/q+1|)} \Big\rfloor \Big) = 2 \sum_{k=1}^{\lfloor a/q \rfloor} \Phi_q(\lfloor b/k \rfloor).$$

The maximum in B_q is never 0, since h/q and $h/q + 1$ are integers that do not both vanish; $B_q = 0$ for $q > a$.

For $a = 1$: $S_1 = b(2b+1)$, $S_2 = b(b+1)$, $B_1 = 2b$, $B_2 = 0$, and the formula gives $(5b+4)(b+1)/2$, as in Lemma 8.3.

Proof. Step 1: Euler's relation. Let U be an open convex set and $\ell_1, \dots, \ell_n$ distinct lines meeting U. Then

$$\#\{\text{cells of } U \setminus \bigcup \ell_i\} = 1 + n + \textstyle\sum_v (m_v - 1),$$

where v runs over the points of U lying on $m_v \geq 2$ of the lines. Indeed, add the lines one at a time. The set $\ell_k \cap U$ is an open segment (or ray, or line); the earlier lines cut it at d_k distinct points into $d_k + 1$ open pieces. Each piece lies in one cell of the earlier arrangement, which is convex, and cuts that cell into exactly two convex parts; distinct pieces lie in distinct cells, since a convex cell meets ℓ_k in a connected set containing no cut point. So the number of cells grows by $d_k + 1$. A point v on m_v lines is counted in d_k for each of these lines except the first one added, that is $m_v - 1$ times.

Step 2: the lines meeting $\mathcal{Q}$. If $h \geq 0$ and $\rho \leq 0$, then $\Lambda + h\theta > 0 \geq \rho$ on $\mathcal{Q}$. If $h \geq 0$ and $\rho \geq 1$, the point $(\rho/2, \rho/(2h))$ (or $(\rho, 1)$ when $h = 0$) lies on the line in $\mathcal{Q}$. If $h < 0$, the point $(\Lambda, \theta) = (1 + |\rho|, (1 + |\rho| - \rho)/|h|)$ does. So the lines meeting $\mathcal{Q}$ are the b lines with $h = 0$, $\rho \geq 1$; the ab lines with $h \geq 1$, $\rho \geq 1$; and all $a(2b + 1)$ lines with $h \leq -1$: in total $(3a + 1)b + a$.

Step 3: vertices and their lines. Two lines $\ell_{h,\rho}$, $\ell_{h',\rho'}$ with $h \neq h'$ meet at $\theta = (\rho - \rho')/(h - h')$; so every vertex $v = (\Lambda, \theta) \in \mathcal{Q}$ has $\theta = p/q$ with $p, q \geq 1$ coprime and $q \leq 2a$. The lines through v are the $\ell_{h,\rho}$ with $\rho = \Lambda + h\theta \in \mathbb{Z}$, $|h| \leq a$, $|\rho| \leq b$. Let I_v be the set of their indices h. If $h_1 \in I_v$, then for $|h| \leq a$ we have $\Lambda + h\theta \in \mathbb{Z}$ if and only if $(h - h_1)p/q \in \mathbb{Z}$, if and only if $q \mid h - h_1$. Moreover $h \mapsto \Lambda + h\theta$ is increasing, so $\{h \in \mathbb{R} : |\Lambda + h\theta| \leq b\}$ is an interval. Hence I_v is the intersection of the residue class $h_1 + q\mathbb{Z}$ with an interval: an arithmetic progression of difference q *without gaps*. Therefore $m_v - 1 = |I_v| - 1$ is the number of pairs $(h, h + q)$ with both h and $h + q$ in I_v.

Step 4: counting pairs. By Step 3, $\sum_v (m_v - 1) = \sum_{q=1}^{2a} N_q$, where N_q is the number of pairs (v, h) such that $\theta(v)$ has denominator q and $h, h + q \in I_v$. Such a pair determines p (by $\theta = p/q$) and $\rho = \Lambda + h\theta$; then $\Lambda + (h + q)\theta = \rho + p$. Conversely, given q, a $p \geq 1$ coprime to q, an integer h and an integer ρ with

$$-a \leq h \leq a - q, \qquad -b \leq \rho, \qquad \rho + p \leq b, \qquad \Lambda := \rho - hp/q > 0,$$

the point $(\Lambda, p/q)$ is a vertex of $\mathcal{Q}$ (it lies on $\ell_{h,\rho}$ and $\ell_{h+q,\rho+p}$), and $h, h + q \in I_v$. So $N_q = \#\{(p, h, \rho) \in \mathcal{T}_q : \ q\rho - hp > 0\}$, where $\mathcal{T}_q$ is the set of triples satisfying all conditions except the sign of Λ. Note $p \leq 2b$ on $\mathcal{T}_q$, and

$$|\mathcal{T}_q| = (2a + 1 - q) \sum_{\substack{1 \leq p \leq 2b \\ \gcd(p,q) = 1}} (2b - p + 1) = (2a + 1 - q)\, S_q(b),$$

since h takes $2a + 1 - q$ values and ρ takes the $2b - p + 1$ values in $[-b, b - p]$.

Step 5: the involution. The map $\sigma(p, h, \rho) = (p, -h - q, -\rho - p)$ is an involution of $\mathcal{T}_q$: $-a \leq h \leq a - q$ if and only if $-a \leq -h - q \leq a - q$, and $\{\rho, \rho + p\} \mapsto \{-\rho - p, -\rho\}$ preserves the condition $|\rho|, |\rho + p| \leq b$. It changes the sign of Λ: $(-\rho - p) - (-h - q)p/q = -\rho + hp/q = -\Lambda$. So $\mathcal{T}_q$ splits into the triples with $\Lambda > 0$, those with $\Lambda < 0$ (equally many) and those with $\Lambda = 0$. Hence $N_q = \frac{1}{2}(|\mathcal{T}_q| - |\mathcal{T}_q^0|)$ with $\mathcal{T}_q^0 = \{\Lambda = 0\}$.

Step 6: the triples on $\Lambda = 0$. $q\rho = hp$ with $\gcd(p, q) = 1$ forces $q \mid h$; write $h = kq$, so $\rho = kp$ and $\rho + p = (k + 1)p$. The conditions become $p \cdot \max(|k|, |k + 1|) \leq b$, and $\max(|k|, |k + 1|) \geq 1$. So $|\mathcal{T}_q^0| = B_q(a, b)$ in the first form. The range $-a \leq kq \leq a - q$ is $-\lfloor a/q \rfloor \leq k \leq \lfloor a/q \rfloor - 1$; the values $k = 0, \dots, \lfloor a/q \rfloor - 1$ give $\max = k + 1$ and the values $k = -1, \dots, -\lfloor a/q \rfloor$ give $\max = |k|$, which is the second form.

Steps 1, 2, 4, 5 and 6 together give the formula. □

The counting of cells through vertices is the planar case of Zaslavsky's face-count formula for arrangements [11], and counting regions of families of hyperplanes with integer translates, as functions of a dilation parameter, is the subject of the theory of inside-out polytopes [10]. Region counts of this kind, with vertices at fractions and totient sums, are classical: the number of linear partitions of a rectangular grid [17] and the number of two-dimensional threshold functions on it

[16], which has an exact formula as a sum over coprime pairs, count regions of arrangements of lines dual to grid points; and the number $1+\sum_{i=1}^{n}(n-i+1)\varphi(i)$ of factors of length n of Sturmian words [21] has a proof through such an arrangement [22]. (The vector $(\lfloor \Lambda + h\theta \rfloor)_h$ is a digital segment of this kind.) So the arithmetic of the vertices, with abscissae the fractions p/q ($q \le 2a$) and the totient sums it produces, is not new, and neither is the count itself: Proposition 5.3 deduces it from Alekseyev's formula, and conversely. We claim no novelty for Theorem 5.1 or Corollary 5.2; what is ours is the direct proof in the coordinates of this note.

5.2. The leading coefficient and the period.

Corollary 5.2. *(a) For each $a \ge 1$, $b \mapsto R(a,b)$ is a quasipolynomial of degree 2 on all of $b \ge 1$ (no threshold), with a period dividing $\operatorname{lcm}(1,2,\dots,2a)$, and*

$$R(a,b) = C_a b^2 + O_a(b), \qquad C_a = \sum_{q=1}^{2a} (2a+1-q)\frac{\varphi(q)}{q}.$$

So $C_1,\dots,C_4 = 5/2,\ 22/3,\ 73/5,\ 1709/70$. The leading coefficient is C_a in every residue class. Equivalently $C_a = (2a+1)\Psi(2a) - \sum_{q\le 2a}\varphi(q)$, with $\Psi(x) = \sum_{q\le x}\varphi(q)/q$.

(b) $C_a = \dfrac{12}{\pi^2}\,a^2 + O(a\log a)$ as $a \to \infty$.

Proof. (a) Write $\Phi_q(x) = \frac{\varphi(q)}{q}x + \pi_q(x)$ for integers $x \ge 0$. Since $\Phi_q(x+q) = \Phi_q(x) + \varphi(q)$, π_q is q-periodic, and $|\pi_q| \le \varphi(q)$. Then $S_q(b) = \sum_{y=1}^{2b}\Phi_q(y) = \frac{\varphi(q)}{q}\,b(2b+1) + \sum_{y=1}^{2b}\pi_q(y)$. The last sum is $2b\,\bar\pi_q$ + (a q-periodic function of $2b$), where $\bar\pi_q$ is the mean of π_q over a period. So S_q is a quasipolynomial in b of degree 2, leading coefficient $2\varphi(q)/q$, and period dividing q (in fact $q/\gcd(q,2)$). Next, $g(b) = \Phi_q(\lfloor b/k \rfloor)$ satisfies $g(b+kq) = g(b) + \varphi(q)$, because $\lfloor (b+kq)/k \rfloor = \lfloor b/k \rfloor + q$; so $g(b) - \frac{\varphi(q)}{kq}b$ has period kq, and g is a quasipolynomial of degree ≤ 1 with period dividing $kq \le a$. Every term of Theorem 5.1 is therefore a quasipolynomial for all $b \ge 1$ with period dividing $\operatorname{lcm}(1,\dots,2a)$, and the leading coefficient is $\frac12\sum_q (2a+1-q)\cdot 2\varphi(q)/q = C_a$; the terms $B_q \le 2\lfloor a/q \rfloor b$ and $(3a+1)b$ are $O_a(b)$. The proof gives the slightly finer period $\operatorname{lcm}(\operatorname{lcm}(1,\dots,a),\ \text{odd } q \le 2a)$. (Numerically, the minimal periods for $a = 1,\dots,5$ are $1, 3, 15, 105, 105$; we do not prove them.)

The second expression for C_a is $\sum_q (2a+1-q)\varphi(q)/q = (2a+1)\sum_q \varphi(q)/q - \sum_q \varphi(q)$.

(b) Since $\#\{x : q \le x \le 2a\} = 2a+1-q$, we have $C_a = \sum_{x=1}^{2a}\Psi(x)$. By $\varphi(q)/q = \sum_{d\mid q}\mu(d)/d$,

$$\Psi(x) = \sum_{d\le x}\frac{\mu(d)}{d}\Big\lfloor \frac{x}{d}\Big\rfloor = x\sum_{d\le x}\frac{\mu(d)}{d^2} + O\Big(\sum_{d\le x}\frac{1}{d}\Big) = \frac{6}{\pi^2}x + O(1+\log x),$$

using $\sum_d \mu(d)/d^2 = 6/\pi^2$ and $\sum_{d>x} d^{-2} = O(1/x)$. Summing over $x \le 2a$, $C_a = \frac{6}{\pi^2}\cdot\frac{2a(2a+1)}{2} + O(a\log a) = \frac{12}{\pi^2}a^2 + O(a\log a)$. □

So $C_a/a^2 \to 12/\pi^2 \approx 1.216$. This determines the leading coefficient of the number of cells. It says nothing about the leading coefficient of $W(a,b)$, which for $a = 1$ is $9/4 < C_1 = 5/2$; if the answer to Question 11.1 is positive, the leading coefficient of $W(a,b)$ is C_a for $a \ge 2$, as it is for $a = 2$ ($44/6 = 22/3$). The period bound for R does not by itself bound the period of W (Question 11.4).

5.3. The cells as threshold functions.

Following Alekseyev [16], for integers $m, n \ge 1$ put

$$V(m,n) = \sum_{\substack{1\le i\le m,\ 1\le j\le n\\ \gcd(i,j)=1}} (m+1-i)(n+1-j), \qquad N(m,n) = (2m+1)(2n+1) + 1 + 4V(m,n).$$

A *two-dimensional threshold function* on the grid $G_{m,n} = \{0,\dots,m\}\times\{0,\dots,n\}$ is a map $f : G_{m,n} \to \{0,1\}$ for which there is a line $\alpha x + \beta y + \gamma = 0$ with $f(x,y) = 0$ if and only if $\alpha x + \beta y + \gamma \le 0$.

Alekseyev proved that there are exactly $N(m,n)$ of them [16, Theorem 3]. Apart from the two constant functions, they come in complementary pairs, one pair for each partition of the grid into two nonempty parts by a line; so $(N(m,n)-2)/2$ is the number of linear partitions of the grid, the quantity studied by Acketa and Žunić [17] (sequence A115009 of [25]).

Proposition 5.3. *For $a,b \geq 1$,*

$$R(a,b) = 1 + a + b + 2ab + \tfrac{1}{2}V(2b,2a) = \frac{N(2b,2a) + 4a + 4b + 6}{8}.$$

In particular $R(a,b) = R(b,a)$, and $R(a,b)$ is the count of Alekseyev [16] after the affine rescaling $N \mapsto (N + 4a + 4b + 6)/8$.

First proof, from Theorem 5.1. Exchanging the names of the summation variables, $\sum_{q=1}^{2a}(2a+1-q)S_q(b) = \sum(2a+1-q)(2b+1-p)$ over the coprime pairs $1 \leq p \leq 2b$, $1 \leq q \leq 2a$, which is $V(2b,2a)$. For the boundary terms, the second form of B_q gives

$$\frac{1}{2}\sum_{q=1}^{2a} B_q(a,b) = \sum_{q=1}^{a}\sum_{k=1}^{\lfloor a/q \rfloor} \Phi_q(\lfloor b/k \rfloor) = \#\{(q,k,p) \in \mathbb{Z}^3_{\geq 1} :\ kq \leq a,\ kp \leq b,\ \gcd(p,q)=1\} = ab,$$

because $(q,k,p) \mapsto (kq,kp)$ is a bijection onto $\{1,\dots,a\} \times \{1,\dots,b\}$, with inverse $(x,y) \mapsto (x/g, g, y/g)$, $g = \gcd(x,y)$. Theorem 5.1 then reads $R(a,b) = 1 + (3a+1)b + a + \frac{1}{2}V(2b,2a) - ab = 1 + a + b + 2ab + \frac{1}{2}V(2b,2a)$. Since $(4b+1)(4a+1) + 1 + 4a + 4b + 6 = 8(1 + a + b + 2ab)$, this is $(N(2b,2a) + 4a + 4b + 6)/8$. The symmetry follows from $V(m,n) = V(n,m)$. □

The second proof does not use Theorem 5.1; it explains the identity, by showing that the cells of the quadrant are the regions of Alekseyev's arrangement in one octant.

Second proof, from [16, Theorem 3]. In $\mathbb{R}^3$ with coordinates (X,Y,Z), let $P_{h,\rho}$ be the plane $X + hY - \rho Z = 0$, and let $\mathcal{P}$ be the central arrangement of the $(2a+1)(2b+1)$ distinct planes $P_{h,\rho}$, $|h| \leq a$, $|\rho| \leq b$. In the chart $Z = 1$, $(X,Y) = (\Lambda,\theta)$, the plane $P_{h,\rho}$ is the line $\ell_{h,\rho}$, and the open octant $\mathcal{O} = \{X,Y,Z > 0\}$ is the cone over $\mathcal{Q}$; so the regions of $\mathcal{P}$ inside $\mathcal{O}$ are the cones over the cells of $\mathcal{Q}$, and there are $R(a,b)$ of them.

(i) $\mathcal{P}$ has $N(2b,2a)$ regions. For $(\alpha,\beta,\gamma) \in \mathbb{R}^3$ put $X = \alpha b + \beta a + \gamma$, $Y = \beta$, $Z = \alpha$, an invertible linear change of coordinates. Then $\alpha(b-\rho) + \beta(a+h) + \gamma = X + hY - \rho Z$, and $(h,\rho) \mapsto (b-\rho, a+h)$ is a bijection onto $G_{2b,2a}$. So $\mathcal{P}$ is the arrangement of the planes $\{\alpha x + \beta y + \gamma = 0\}$, $(x,y) \in G_{2b,2a}$, and the sign vector of a region is the threshold function represented by any coefficient vector (α,β,γ) in the region. As in the opening paragraph of this section, each nonempty set of points with a prescribed sign vector is an intersection of open half-spaces, hence convex, so it is exactly one region, and distinct regions have distinct sign vectors. Every threshold function arises in this way (replacing γ by $\gamma - \varepsilon$ with $\varepsilon > 0$ small puts no grid point on the line and changes no value). Hence $\mathcal{P}$ has $N(2b,2a)$ regions.

(ii) Eight octants. Let $\mathcal{P}' = \mathcal{P} \cup \{Y = 0\} \cup \{Z = 0\}$. It contains the three coordinate planes ($X = 0$ is $P_{0,0}$), and the sign changes $X \mapsto -X$, $Y \mapsto -Y$, $Z \mapsto -Z$ send $P_{h,\rho}$ to $P_{-h,-\rho}$, $P_{-h,\rho}$ and $P_{h,-\rho}$, so they preserve $\mathcal{P}'$. Hence every region of $\mathcal{P}'$ lies in one open octant, every octant contains as many regions as $\mathcal{O}$, and $\mathcal{P}'$ has $8R(a,b)$ regions.

(iii) Two more planes. Adding to a central arrangement a plane H not in it increases the number of regions by the number of regions that the traces of the arrangement cut in H (each of them splits one region in two) [11]. On $Y = 0$ the traces of $\mathcal{P}$ are the $2b+1$ distinct lines $X = \rho Z$, which cut it into $2(2b+1)$ regions. On $Z = 0$ the traces of $\mathcal{P} \cup \{Y = 0\}$ are the $2a+2$ distinct lines $X + hY = 0$ and $Y = 0$, which cut it into $2(2a+2)$ regions. So $8R(a,b) = N(2b,2a) + (4b+2) + (4a+4)$. □

We did not find this reduction in [16] or in the OEIS entries cited below; it is elementary, and we record it because it makes the identity transparent.

Remark 5.4 (the leading term). For $m \geq n$, Alekseyev also proves $N(m,n) = 2\big((n+1)\Psi(n) - \sum_{q\leq n}\varphi(q)\big)m^2 + O(mn^3)$ [16, Theorem 25]. With $m = 2b$, $n = 2a$ and Proposition 5.3, this is $R(a,b) = C_a b^2 + O(ba^3)$ with $C_a = (2a+1)\Psi(2a) - \sum_{q\leq 2a}\varphi(q)$, the second expression of Corollary 5.2(a). Part (b) of that corollary also follows from the asymptotics of Ψ and of $\sum_{q\leq x}\varphi(q)$ recalled there. For the asymptotics of the number of threshold functions when both sides of the grid grow, see Haukkanen and Merikoski [20], who improve the error terms of [17] and [16, Theorem 25]. The quasipolynomial structure of $b \mapsto R(a,b)$ (no threshold, the bound on the period) is not stated in [16]; it follows from the exact formula by the elementary argument of Corollary 5.2(a). Nor did we find there, or in the OEIS entries cited, the linear term of $R(a,b)$ for fixed a.

Remark 5.5 (the drawing of $K_{m,n}$). Extend V to zero arguments by the empty sum, $V(m,0) = V(0,n) = 0$. Let $T(m,n)$ be the number of bounded regions of the figure formed by m equally spaced points on a line, n equally spaced points on a parallel line, and the segments joining each point of one line to each point of the other (sequence A335678 of [25]). In a comment of 28 June 2020 recorded in that entry, Alekseyev gives $T(m,n) = V(m-1,n-1) + mn - 1$ for $m,n \geq 1$, with the following justification. Place the points at $(i,0)$ and $(j,1)$, $0 \leq i < m$, $0 \leq j < n$. The projective map $(x,y) \mapsto (u,v) = \big((1-y)/(x+1),\ y/(x+1)\big)$ sends the segment from $(i,0)$ to $(j,1)$, which lies on $x = i(1-y)+jy$, onto the segment of the line $(i+1)u+(j+1)v = 1$ between the two axes. So it sends the figure to the partition of the right triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ by the lines $x_1u + x_2v = 1$, $(x_1,x_2) \in \{1,\dots,m\} \times \{1,\dots,n\}$, except for one cell. The image of the convex hull of the figure is $\{u,v \geq 0,\ u+v \leq 1,\ mu+nv \geq 1\}$, and no line of the partition meets the region $u,v > 0$, $mu+nv < 1$ (there $x_1u+x_2v \leq mu+nv < 1$); so that region is a single triangular cell of the partition with no counterpart in the figure, which is the -1. For $m,n \geq 2$, Alekseyev, Basova and Zolotykh count the cells of the partition as $c(m,n) = f_1(m,n)/4 + (m+n)/2$ [19, Theorem 13], where $f_1(m,n)$ is the sum of $(m-|i|)(n-|j|)$ over $-m < i < m$, $-n < j < n$, $\gcd(i,j) = 1$. The four open quadrants contribute $4V(m-1,n-1)$ to $f_1(m,n)$, and the two axes, where the only primitive vectors are $(\pm1,0)$ and $(0,\pm1)$, contribute $2(m-1)n + 2m(n-1)$. Hence

$$f_1(m,n) = 4V(m-1,n-1) + 4mn - 2m - 2n, \qquad c(m,n) = V(m-1,n-1) + mn,$$

and $T(m,n) = c(m,n) - 1$. For $m = 1$ (or $n = 1$) the figure is a fan of n (or m) segments, and $T = n-1$ (or $m-1$) is the same formula with $V(0,\cdot) = V(\cdot,0) = 0$. For example $V(3,3) = 31$, $f_1(4,4) = 172$, $c(4,4) = 47$ and $T(4,4) = 46$. With the first expression of Proposition 5.3 this gives

$$R(a,b) = \frac{T(2a+1,2b+1)+2}{2}.$$

As an independent check, we counted exactly (rational arithmetic, Euler's relation) the cells of the figure and of the partition of the triangle for $2 \leq m,n \leq 7$, and the figure for $m = 1$ or $n = 1$, which agree with these formulas, and the identity for $a,b \leq 3$ (for instance $T(7,7) = 380$ and $R(3,3) = 191$).

6. Quasipolynomial growth for every fixed exponent

Theorems A and B (Part III) are the cases $a = 1,2$ of a general phenomenon. We show that for fixed a the function $b \mapsto W(a,b)$ is eventually a quasipolynomial of degree at most two, and then, with a quadratic lower bound, that the degree is exactly two (Corollary 6.4). The proof does not compute it: it shows that $W(a,b)$ counts the integer points of a family of sets defined in Presburger arithmetic, and then applies a theorem of Woods [13]. All the depth is in Woods' theorem; encoding a counting problem in Presburger arithmetic to obtain quasipolynomiality is a standard route. The only point specific to this problem is that equality of words, a comparison of multisets whose size grows with b, admits an encoding whose size is bounded in terms of a (Step 3).

Recall that a function $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is a *quasipolynomial* if there are a period $P \geq 1$ and polynomials $Q_0, \dots, Q_{P-1}$ with $g(b) = Q_{b \bmod P}(b)$ for all b. It is *eventually* a quasipolynomial if this holds for all $b \geq N$, for some N.

Theorem 6.1. *For every fixed $a \geq 1$ there are N_a, $P_a \geq 1$ and polynomials $Q_{a,0}, \dots, Q_{a,P_a-1}$ of degree at most two such that*

$$W(a,b) = Q_{a,\, b \bmod P_a}(b) \qquad \text{for all } b \geq N_a.$$

In particular $\sum_{b \geq 1} W(a,b)\, t^b = A_a(t)/(1-t^{P_a})^3$ for some polynomial $A_a(t)$.

For $a = 1$ and $a = 2$ Theorems A and B identify the quasipolynomial, with periods 2 and 6 and thresholds $N_1 = 2$ and $N_2 = 3$. For $a \geq 3$ the theorem gives existence only; the period, the threshold and the coefficients are not determined, beyond the bounds on the leading coefficient in Corollary 6.4 (see Section 11).

6.1. Step 1: a vector of bounded length. Fix a and b. Measure in units of $\log s$, with $\Lambda = \log t / \log s$ and $\theta = \log r / \log s$. The letters (sit at $i\theta + j$ and the letters) at $\Lambda + i\theta + j$, for $0 \leq i \leq a$ and $0 \leq j \leq b$. A letter) at $\Lambda + i'\theta + j'$ is preceded by

$$\#\{j : i\theta + j < \Lambda + i'\theta + j'\} = f(\lfloor \Lambda + (i'-i)\theta \rfloor + j')$$

letters (of row i, where $f(v) = \min(b+1, \max(0, v+1))$ as in Section 8. Put

$$c_h = \operatorname{clip}\lfloor \Lambda + h\theta \rfloor \quad (-a \leq h \leq a), \qquad H_{i'}(j') = \sum_{i=0}^{a} f(c_{i'-i} + j'),$$

$$Z(c) = \biguplus_{i'=0}^{a} \{H_{i'}(j') : 0 \leq j' \leq b\}.$$

The proof of Lemmas 8.1 and 9.1 applies verbatim for every a. No $\Lambda + h\theta$ is an integer: such an equality would read $2^e \cdot (\text{odd}) = \text{odd}$. Clipping does not change $f(c_h + j')$ for $0 \leq j' \leq b$, and $C(n) = Z(c)$. The word is determined by the multiset $C(n)$ and conversely, so two vectors give the same word if and only if $Z(c) = Z(d)$. The vector c has $2a+1$ entries, a number independent of b; each entry lies in $[-b-1, b]$.

6.2. Step 2: realisability is a Presburger condition. Say that $c \in \mathbb{Z}^{2a+1}$ is *realisable for b* if it arises from some (Λ, θ) with $\Lambda, \theta > 0$ off the lines $\Lambda + h\theta \in \mathbb{Z}$. The change of coordinates $x = (\theta/\Lambda, 1/\Lambda)$ is a homeomorphism of open quadrants. It maps the line $\Lambda + h\theta = \rho$ to the hyperplane $-hx_1 + \rho x_2 = 1$ of $\mathcal{A}_{(a,b)}$. So Lemma 2.3 (realisation by primes), applied to the open cell of (Λ, θ), shows that the realisable vectors are exactly the vectors that occur. Realisability means that the following system in the two real unknowns (Λ, θ) has a solution: $\Lambda > 0$, $\theta > 0$, and for each h,

$$\begin{cases} c_h < \Lambda + h\theta < c_h + 1 & \text{if } -b \leq c_h \leq b-1, \\ \Lambda + h\theta > b & \text{if } c_h = b, \\ \Lambda + h\theta < -b & \text{if } c_h = -b-1. \end{cases}$$

(A solution can be moved off the lines $\Lambda + h\theta \in \mathbb{Z}$ because the solution set is open.) Fix one of the 3^{2a+1} *clipping patterns*, which says for each h which of the three cases holds. The system is then $A\binom{\Lambda}{\theta} < u$, with a fixed integer matrix A and a right-hand side u whose entries are affine functions of (b, c) with integer coefficients. By Motzkin's transposition theorem (see [26, §7.8]), $A\binom{\Lambda}{\theta} < u$ is solvable if and only if $y \cdot u > 0$ for every nonzero $y \geq 0$ with $yA = 0$; it suffices to test y on a finite set Y of generators of the cone $\{y \geq 0 : yA = 0\}$ (Minkowski–Weyl; the cone is pointed). Since A is

rational, the generators can be chosen with nonnegative integer entries, so each condition $y \cdot u > 0$ is a linear inequality with integer coefficients. Y depends only on the pattern. (Equivalently, apply Fourier–Motzkin elimination to Λ and then to θ.) Hence

$$\mathrm{Real}_a(b,c) \iff \bigvee_{\text{patterns } \pi} \Big(\pi(b,c) \ \wedge \bigwedge_{y \in Y_\pi} y \cdot u_\pi(b,c) > 0 \Big),$$

where $\pi(b,c)$ is the conjunction of the conditions $-b \le c_h \le b-1$, $c_h = b$ or $c_h = -b-1$ that define the pattern. This is a quantifier-free formula of Presburger arithmetic in the integer variables (b,c).

6.3. Step 3: equality of words is a Presburger condition. Encode a multiset of nonnegative integers by its generating polynomial: $Z(c) = Z(d)$ if and only if $M_c = M_d$, where

$$M_c(z) = \sum_{i'=0}^{a} \sum_{j'=0}^{b} z^{H_{i'}(j')} \in \mathbb{Z}[z].$$

The length of M_c grows with b; we compress it. Each $f(c_h + j')$ is, as a function of $j' \in [0,b]$, equal to 0, then to $c_h + j' + 1$, then to $b+1$, with breakpoints at $j' = -c_h - 1$ and $j' = b - c_h$. Hence each $H_{i'}$ is piecewise affine on $[0,b]$ with at most $2(a+1)$ breakpoints. Each breakpoint is an affine function of (b,c), and each slope lies in $\{0,1,\dots,a+1\}$. Fix one of the finitely many weak orderings (ties allowed) of all the breakpoints together with the endpoints 0 and $b+1$, a Presburger condition on (b,c). When comparing c with d below, the ordering is fixed jointly for the breakpoints of both. On it, replace each breakpoint by 0 if it lies below 0 and by $b+1$ if it lies above $b+1$, and let $0 = q_0 \le q_1 \le \cdots \le q_k \le q_{k+1} = b+1$ be the resulting values for one row. The half-open integer intervals $[q_m, q_{m+1}) = \{j' : q_m \le j' < q_{m+1}\}$, $0 \le m \le k$, partition $\{0,\dots,b\}$ (some may be empty), and on each of them every $f(c_h + j')$ is affine; at a breakpoint the two adjacent affine expressions agree, so ties may be ordered either way. The pieces, their first values and their lengths $N = q_{m+1} - q_m$ are affine functions of (b,c). Put $D_a = \mathrm{lcm}(1,2,\dots,a+1)$ and $K_c = (1 - z^{D_a}) M_c$. A piece with slope $s \ge 1$, first exponent u and $N \ge 0$ terms contributes

$$(1 - z^{D_a}) \sum_{n=0}^{N-1} z^{u+sn} = \sum_{\rho=0}^{D_a/s-1} (z^{u+s\rho} - z^{u+sN+s\rho}),$$

because $1 - z^{D_a} = (1-z^s) \sum_{\rho=0}^{D_a/s-1} z^{s\rho}$ and $(1-z^s) \sum_{n<N} z^{u+sn} = z^u - z^{u+sN}$. A piece with slope 0 contributes $Nz^u - Nz^{u+D_a}$. So, on each ordering, $K_c = \sum_{t=1}^{T} \kappa_t z^{e_t}$, with $T \le (a+1)(2a+3) \cdot 2D_a$ terms, a bound depending on a only. Each e_t is an affine function of (b,c), and each κ_t is ± 1 or $\pm N$, also affine in (b,c).

Since $\mathbb{Z}[z]$ is an integral domain, $M_c = M_d$ if and only if $K_c = K_d$, that is, if and only if $\sum_t \kappa_t(c) z^{e_t(c)} - \sum_t \kappa_t(d) z^{e_t(d)} = 0$. A sum of at most $2T$ monomials vanishes if and only if there is a set partition of its terms such that terms in the same block have the same exponent and the coefficients in each block sum to 0. Indeed, such a partition refines the partition by exponent, so the coefficient of each power of z is a sum of block sums. For a fixed partition both conditions are conjunctions of linear equalities. Taking the finite disjunction over orderings and partitions, $\mathrm{Same}_a(b,c,d) \iff Z(c) = Z(d)$ is a quantifier-free Presburger formula.

6.4. Step 4: counting with Woods' theorem. Let $\le_{\mathrm{lex}}$ be the lexicographic order on $\mathbb{Z}^{2a+1}$, which is Presburger-definable. Each word is represented by the lexicographically least realisable vector that gives it:

$$\mathrm{Rep}_a(b,c) \iff \mathrm{Real}_a(b,c) \wedge \neg \exists d \, (\mathrm{Real}_a(b,d) \wedge d <_{\mathrm{lex}} c \wedge \mathrm{Same}_a(b,c,d)).$$

This is a Presburger formula. Woods works over $\mathbb{N}$, so we shift every vector variable, $c' = c + (b+1)\mathbf{1}$ and $d' = d + (b+1)\mathbf{1}$. Both lie in $[0, 2b+1]^{2a+1} \subset \mathbb{N}^{2a+1}$, because Real_a forces every entry into $[-b-1, b]$. Linear conditions with negative coefficients are rewritten by moving terms across. Then

$W(a,b) = \#\{c' \in \mathbb{N}^{2a+1} : \mathrm{Rep}'_a(b,c')\}$ for the shifted formula Rep'_a, and this is finite for every b. So $b \mapsto W(a,b)$ is a Presburger counting function in the sense of [13, Def. 1.6]. By [13, Thm. 1.10, (A)⇒(B)] it is a piecewise quasipolynomial: there is a finite partition of $\mathbb{N}$ into sets $P_i \cap \mathbb{N}$, with P_i polyhedra, and a quasipolynomial on each piece. For one variable the P_i are intervals, and exactly one of them is unbounded above. So $W(a,b)$ agrees with a single quasipolynomial for all $b \geq N_a$. This proves the first statement of Theorem 6.1 without the bound on the degree.

6.5. Step 5: degree at most two. By Step 2, the realisable vectors correspond to the cells of the arrangement of the lines $\Lambda + h\theta = \rho$ ($|h| \leq a$, $\rho \in \mathbb{Z} \cap [-b, b]$) in the quadrant $\Lambda, \theta > 0$. Distinct cells give distinct vectors (Lemma 9.1). There are $L = (2a+1)(2b+1)$ such lines, so

$$W(a,b) \leq R(a,b) \leq 1 + L + \binom{L}{2} = O_a(b^2).$$

If some constituent $Q_{a,i}$ had degree at least 3, then along $b \equiv i \pmod{P_a}$ we would have $|Q_{a,i}(b)| \gg b^3$, contradicting $0 \leq W(a,b) \leq R(a,b) = O_a(b^2)$. So every constituent has degree at most two. A quasipolynomial of degree at most two with period P has generating function $A(t)/(1-t^P)^3$ with $\deg A < 3P$. This bound is for the quasipolynomial itself, summed over all $b \geq 0$. Replacing its finitely many values below the threshold by those of $W(a,b)$, and dropping the term $b = 0$, adds a polynomial to the generating function, that is, a polynomial multiple of $(1-t^P)^3$ to the numerator, so $A_a(t)$ in Theorem 6.1 may have degree $\geq 3P$. This proves Theorem 6.1. □

Remark 6.2. The proof gives no useful bound on N_a or P_a, and does not use Theorems A and B. What makes it work is that the multiset $Z(c)$, whose size grows with b, is described by a number of events bounded in terms of a alone. This is the same compression that makes the certificate of Section 9.2 independent of b. Whether the leading coefficient of $W(a,b)$ is the same on every residue class is open (Question 11.2).

6.6. A quadratic lower bound, and degree exactly two. The upper bound $W(a,b) \leq R(a,b)$ of Step 5 is quadratic. The following explicit family of cells with pairwise distinct words shows that the lower bound is quadratic too.

Theorem 6.3. *For all $a, b \geq 1$,*

$$W(a,b) \;\geq\; N(b) := \#\{(u,v) \in [0, b-1]^2 :\; 2u+v \geq b,\; u+2v \geq b\} \;\geq\; \lfloor 2b/3 \rfloor^2,$$

and $N(b) = \sum_{u=0}^{b-1} \min(2u, \lfloor (b+u)/2 \rfloor) = \frac{2}{3}b^2 + O(b)$. Hence $W(a,b) = \Theta_a(b^2)$; more precisely $(\frac{2}{3} - o(1))b^2 \leq W(a,b) \leq (C_a + o(1))b^2$, where the lower constant does not depend on a.

Proof. Fix (u,v) in the set and put $\Lambda = u + \frac{1}{3}$, $\theta = u + v + \frac{2}{3}$; both are positive. Then $\Lambda + h\theta = u + h(u+v) + \frac{1+2h}{3}$.

- $h = 0$: $\Lambda \in (u, u+1)$ and $0 \leq u \leq b-1$, so $c_0 = u$.
- $h = -1$: $\Lambda - \theta = -v - \frac{1}{3} \in (-v-1, -v)$ and $-b \leq -v - 1$, so $c_{-1} = -v - 1$.
- $h \geq 1$: $\Lambda + h\theta \geq \Lambda + \theta = 2u + v + 1 > b$, so $c_h = b$.
- $h \leq -2$: $\Lambda + h\theta \leq \Lambda - 2\theta = -u - 2v - 1 < -b$, so $c_h = -b - 1$.

The point lies on no line of $\mathcal{L}_{a,b}$: $\Lambda + h\theta$ is an integer only when $3 \mid 1 + 2h$, that is $h \equiv 1 \pmod 3$, and then $h \geq 1$ or $h \leq -2$, where $|\Lambda + h\theta| > b$ by the last two items. So the vector c above is realisable (its cell is open; by Lemma 2.3 it is realised by primes, after moving the point off the lines $\Lambda + h\theta \in \mathbb{Z}$ with $|\rho| > b$, which does not change c).

Now compute $Z(c)$. For $0 \leq i' \leq a$, the values $h = i' - i$ ($0 \leq i \leq a$) include i' values $h \geq 1$ ($f = b+1$), the value $h = 0$, and the value $h = -1$ exactly when $i' < a$; the others have $f = 0$.

Since $f(-v-1+j') = \max(0, j'-v)$ for $0 \le j' \le b$,

$$H_{i'}(j') = i'(b+1) + \min(b+1,\, u+j'+1) + [i'<a]\max(0,\, j'-v).$$

The minimum of Z is $H_0(0) = u+1$: H_0 is nondecreasing in j', and $H_{i'} \ge b+1 > u+1$ for $i' \ge 1$. The sum of Z is $F(u) + a\binom{b-v+1}{2}$, where $F(u)$ does not depend on v; since $a \ge 1$ and $b-v \ge 1$, it is strictly decreasing in v. Both are functions of the word. So the word determines u, then v, and the $N(b)$ points give $N(b)$ distinct words.

Counting: for fixed u, v ranges over $[\max(b-2u, \lceil (b-u)/2 \rceil),\, b-1]$, which has $\min(2u,\, b-\lceil (b-u)/2 \rceil) = \min(2u, \lfloor (b+u)/2 \rfloor)$ elements. Replacing the summand by $\min(2u, (b+u)/2)$ changes it by less than 1, and $\sum_{u<b} \min(2u, (b+u)/2) = \int_0^{b/3} 2u\,du + \int_{b/3}^{b} \frac{b+u}{2}\,du + O(b) = \frac{b^2}{9} + \frac{5b^2}{9} + O(b)$. The square $[\lceil b/3 \rceil, b-1]^2$ lies in the set (there $2u+v \ge 3\lceil b/3 \rceil \ge b$), which gives $N(b) \ge (b - \lceil b/3 \rceil)^2 = \lfloor 2b/3 \rfloor^2$. The upper bound is $W \le R$ and Corollary 5.2. □

Corollary 6.4. *In Theorem 6.1 every constituent $Q_{a,i}$ has degree exactly 2, and its leading coefficient lies in $[\frac{2}{3}, C_a]$.*

Proof. Along $b \equiv i \pmod{P_a}$, $b \ge N_a$, we have $\frac{2}{3}b^2 - O(b) \le Q_{a,i}(b) \le C_a b^2 + O_a(b)$. □

Remark 6.5. The argument uses only the constraints $0 \le u, v \le b-1$, $2u+v \ge b$ and $u+2v \ge b$. Restricted to the square $[\lceil 2b/3 \rceil, \lfloor 3b/4 \rfloor]^2$, which lies in this set, it already gives $W(a,b) \ge n_b^2$ for every $b \ge 1$, with $n_b = \lfloor 3b/4 \rfloor - \lceil 2b/3 \rceil + 1 \ge (b-5)/12$; among $b \le 5$ one has $n_b = 0$ exactly for $b \in \{1, 2, 5\}$. The larger set gives the constant $2/3$. For $a = 1, 2$ the true leading coefficients are $9/4$ and $22/3$, so the constant $\frac{2}{3}$ is far from sharp.

7. Three mechanisms of coincidence

Throughout, $a \ge 1$ and $b \ge 1$, $K = (a+1)(b+1)$, $c_h = \mathrm{clip}\lfloor \Lambda + h\theta \rfloor$ for $-a \le h \le a$, and $Z(c)$ is the multiset of Section 6, Step 1; two realisable vectors give the same word if and only if they have the same Z. For a word w let $\Phi(w)$ be the set of realisable vectors whose word is w (its *fibre*). For $a = 1$ we keep the order $(\nu, \beta, \gamma) = (c_0, c_{-1}, c_1)$ of Section 8.

The excess $R(a,b) - W(a,b)$ counts the cells lost to coincidences. Three mechanisms produce them (Figure 5): the empty graph, for every a; an exchange of arithmetic progressions between rows, for $a = 1$, in the notation of Section 8, where these coincidences were first listed; and a reflection inside integer levels, for $a \ge 2$, with a variant around rational slopes. Together they give a lower bound for the excess, linear in b.

We use one fact from Step 1 in pointwise form. Let P be the multiset of the K positions $i\theta + j$ ($0 \le i \le a$, $0 \le j \le b$), indexed by (i,j), and $N(x) = \#\{(i,j) : i\theta + j < x\}$. If (Λ, θ) is off the lines $\Lambda + h\theta = \rho$ ($|h| \le a$, $|\rho| \le b$), then

$$Z(c(\Lambda,\theta)) = \{\, N(q+\Lambda) : q \in P \,\} \qquad \text{(as multisets, indexed by } P\text{)}. \tag{2}$$

Indeed the count of row i below $\Lambda + i'\theta + j'$ is $\#\{j : j < \Lambda + (i'-i)\theta + j'\} = f(c_{i'-i} + j')$; if $\Lambda + (i'-i)\theta$ is an integer ρ with $|\rho| > b$, both sides are 0 or $b+1$.

7.1. The empty-graph fibre. Write $O(0)$ for the word $()^K$, that is $O_K(0)$ of Corollary 3.3(a). Let

$$F_{a,b} = \Big\{ \frac{j}{h} :\ 1 \le h \le a,\ 1 \le j \le b \Big\} \subset (0, b],$$

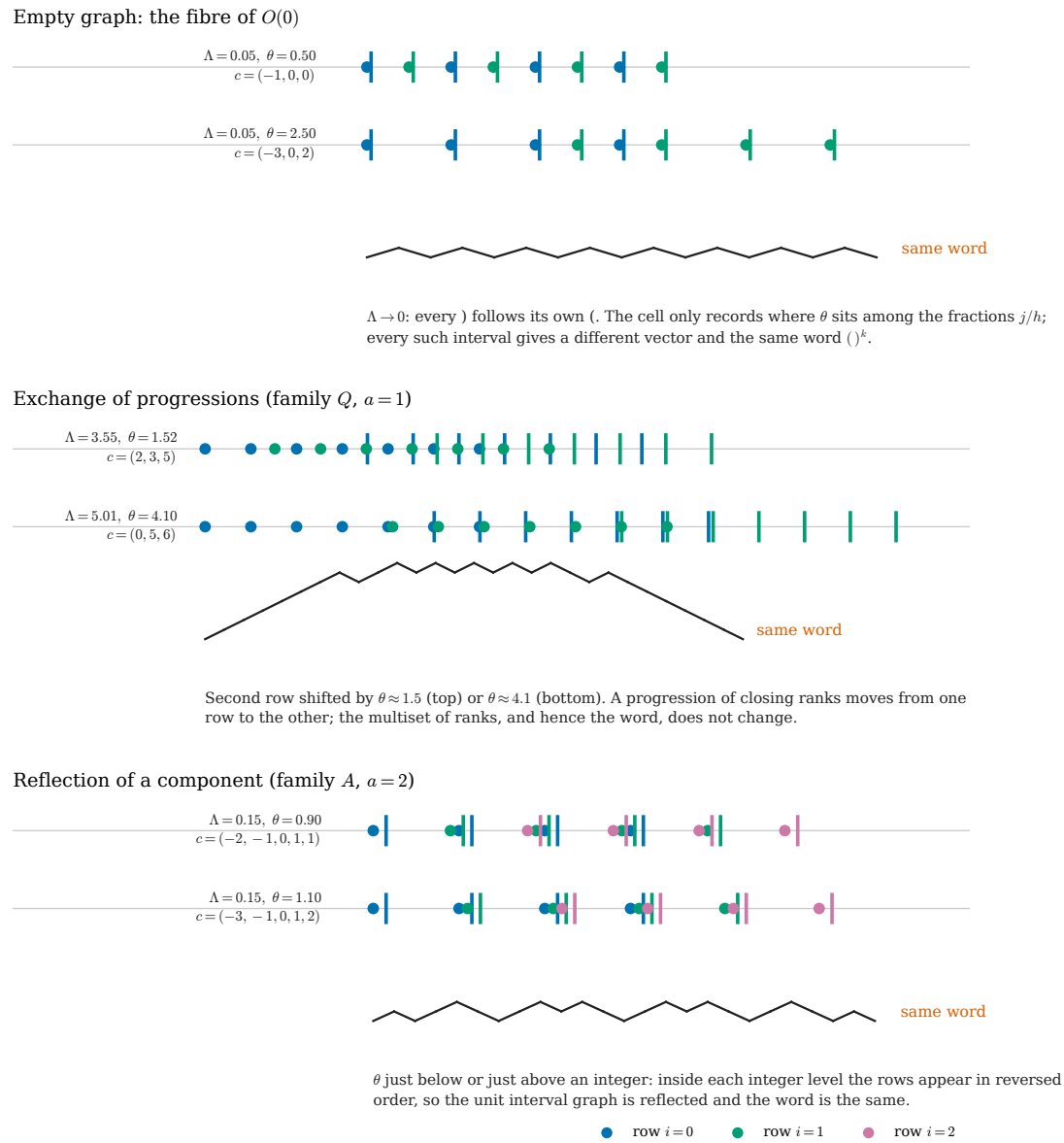


Figure 5. The three mechanisms of coincidence. In each row, two cells with different vectors c and the same word: the positions $i\theta+j$ (dots, coloured by the row i) and $\Lambda+i\theta+j$ (bars) on the logarithmic line, and the common Dyck path below; vectors are written as $(c_{-a}, \dots, c_a)$. Top: the empty graph (Lemma 7.1), $a = 1$, $b = 3$. Middle: exchange of an arithmetic progression of closing ranks between rows (Lemma 7.2), $a = 1$, $b = 6$: the vectors $(2, 3, 5)$ and $(0, 5, 6)$ are the triples $(\nu, \beta, \gamma) = (3, 2, 5) = I_1$ ($\beta = 2$, $d = 1$) and $(5, 0, 6) = J(7, 1)$ of the family Q (Lemma 8.7, Corollary 7.3(a)). Bottom: reflection of the rows inside each integer level (Lemma 7.5), $a = 2$, $b = 3$, $p = k = 1$: the pair A_1 of Section 9. Live version in the blueprint.

and for $\theta > 0$ let $\varphi(\theta) \in \mathbb{Z}^{2a+1}$ be

$$\varphi(\theta)_h = \begin{cases} \min\bigl(b, \lfloor h\theta \rfloor\bigr), & 0 \le h \le a, \\ -\min\bigl(b, \lfloor |h|\theta \rfloor\bigr) - 1, & -a \le h < 0. \end{cases}$$

Lemma 7.1. *Let $a, b \geq 1$. Then $\Phi(O(0)) = \{\varphi(\theta) : \theta \in (0,\infty) \setminus F_{a,b}\}$, and*

$$|\Phi(O(0))| = 1 + \sum_{h=1}^{a} \#\{1 \leq j \leq b : \ \gcd(j,h) = 1\}.$$

Proof. Step 1: when is the word $O(0)$. Fix (Λ,θ) off the lines and sort P as $p_1 \leq \cdots \leq p_K$. Then $N(p_k+\Lambda) \geq k$, and $N(p_k+\Lambda)$ is nondecreasing in k, so it is the k-th smallest element of Z. Hence $Z = \{1,\dots,K\}$ if and only if $N(p_k+\Lambda) = k$ for every k, that is $p_{k+1} \geq p_k + \Lambda$ for $k < K$. Equality is excluded: $p_{k+1} - p_k = \Lambda$ reads $\Lambda + h\theta = \rho$ with $|h| \leq a$, $|\rho| \leq b$. The differences of two positions with distinct indices are the numbers $\pm(h\theta - j)$ with $(h,j) \in \{0,\dots,a\} \times \{-b,\dots,b\} \setminus \{(0,0)\}$, all of which occur. So, by (2), the word at (Λ,θ) is $O(0)$ if and only if $\Lambda < g(\theta)$, where

$$g(\theta) = \min\{|h\theta - j| : \ 0 \leq h \leq a, \ |j| \leq b, \ (h,j) \neq (0,0)\}.$$

Coincident positions give $g(\theta) = 0$. For $\theta > 0$ we have $g(\theta) = 0$ if and only if $h\theta = j$ with $h \geq 1$ and $1 \leq j \leq b$, that is $\theta \in F_{a,b}$ (the terms with $h = 0$ are $|j| \geq 1$, since $b \geq 1$).

Step 2: the vector on this region. Let $0 < \Lambda < g(\theta)$. Then $\Lambda < 1$, so $c_0 = 0$. Let $1 \leq h \leq a$ and $m = \lfloor h\theta \rfloor$.

- $h\theta < b$: then $m+1 \leq b$ and $\Lambda < m+1-h\theta$, so $\Lambda + h\theta \in (m, m+1)$ and $c_h = m$. If $h\theta > b$: $\Lambda + h\theta > b$ and $c_h = b$. ($h\theta = b$ is excluded, since $b/h \in F_{a,b}$.) In both cases $c_h = \min(b,m)$.
- $h\theta < b+1$, so $m \leq b$: if $m \geq 1$, then $\Lambda < h\theta - m$; if $m = 0$, then $\Lambda < h\theta$ (the term $(h,0)$ of g). Either way, since also $\Lambda > 0$ and $h\theta < m+1$, we get $-m-1 < -h\theta < \Lambda - h\theta < -m$, so $\Lambda - h\theta \in (-m-1, -m)$ and $\lfloor \Lambda - h\theta \rfloor = -m-1 \geq -b-1$, so $c_{-h} = -\min(b,m) - 1$. If $h\theta \geq b+1$: $\Lambda - h\theta < 1 - (b+1) = -b$, so $c_{-h} = -b-1 = -\min(b,m) - 1$.

So $c(\Lambda,\theta) = \varphi(\theta)$ at every such point, on the lines or not. The case $\theta > b$, where every c_h with $h \neq 0$ is clipped, is included.

Step 3: the fibre. If c is realisable with word $O(0)$, take a point of its cell; by Step 1 it has $\Lambda < g(\theta)$ and $\theta \notin F_{a,b}$, and by Step 2, $c = \varphi(\theta)$. Conversely let $\theta_0 \notin F_{a,b}$, and let I be the component of $(0,\infty) \setminus F_{a,b}$ containing it. Since φ is constant on I and g is continuous and positive on I, the set $\{(\Lambda,\theta) : \theta \in I, \ 0 < \Lambda < g(\theta)\}$ is open and nonempty. The lines $\Lambda + h\theta \in \mathbb{Z}$ ($|h| \leq a$) form a locally finite family, so this set contains a point off all of them. By Steps 1 and 2 that point has word $O(0)$ and vector $\varphi(\theta_0)$, which is therefore realisable.

Step 4: counting. The coordinates φ_h with $h > 0$ are nondecreasing in θ (those with $h < 0$ are nonincreasing, and determined by them). Let $x \in F_{a,b}$, written in lowest terms as $x = j/h$. Then $h \leq a$ and $j \leq b$, because any representation J/H with $H \leq a$, $J \leq b$ has $h \leq H$ and $j \leq J$. When θ crosses x, $\lfloor h\theta \rfloor$ goes from $j-1$ to $j \leq b$, so φ_h increases strictly. Hence φ takes different values on different components of $(0,\infty) \setminus F_{a,b}$, and $|\Phi(O(0))| = |F_{a,b}| + 1$. Writing each element of $F_{a,b}$ in lowest terms is a bijection onto the pairs (h,j) with $1 \leq h \leq a$, $1 \leq j \leq b$, $\gcd(j,h) = 1$. □

For $a = 1$ the lemma gives $b+1$, the triples $(0, -\rho, \rho-1)$ of Lemma 8.6(a); for $a = 2$ it gives $1 + b + \lceil b/2 \rceil$, the class O of Section 9. In general $|\Phi(O(0))| = b\sum_{h=1}^{a} \varphi_{\mathrm{E}}(h)/h + O_a(1)$, with φ_{E} Euler's function, since $\#\{j \leq b : \gcd(j,h) = 1\} = b\,\varphi_{\mathrm{E}}(h)/h + O(h)$.

7.2. Exchange of progressions ($a = 1$). Write $\mathrm{AP}(x;n) = \{x, x+2, \dots, x+2(n-1)\}$ (empty if $n \leq 0$). Lemma 8.4(i) extends to $p = -1$: for $-1 \leq p \leq p' \leq b$, with $\tau = p + p' + 2$,

$$S(p,p') = \mathrm{AP}(\tau;\, b-p') \uplus [\tau + 2(b-p'),\, 2b+1] \uplus \{T\}^{p+1}. \tag{3}$$

Indeed $f(p+j) + f(p'+j) = \tau + 2j$ for $0 \leq j \leq b-p'-1$; then $b+p+j+2$ for $b-p' \leq j \leq b-p-1$ (the values $2b+2+p-p', \dots, 2b+1$); and T for the remaining $p+1$ values of j.

Lemma 7.2. *Let $b \geq 1$, and let $\beta \geq 0$, $d \geq 0$, $\iota \in \{0,1\}$ be integers with*

$$\beta + 2d + \iota \leq b-1 \qquad \text{and} \qquad 2\beta + 2d + \iota \geq b-1.$$

Put $\Delta = b - \beta - 2d - \iota$ (≥ 1) *and*

$$\xi = (\beta + d + \iota,\ \beta,\ \beta + 2d + 1), \qquad T(\xi) = (\nu + \Delta,\ \beta - \Delta,\ b) = (b - d,\ 2\beta + 2d + \iota - b,\ b).$$

Then ξ *and* $T(\xi)$ *are distinct realisable triples and* $Z(\xi) = Z(T(\xi))$.

In the notation $\gamma = 2\nu - \beta + \varepsilon$ with $\varepsilon \in \{-1, 1\}$, $\iota = (1 - \varepsilon)/2$ and $\Delta = b - 2\nu + \beta + \iota$; the two hypotheses say $\Delta \geq 1$ and $\beta - \Delta \geq -1$.

Proof. Write $\xi = (\nu, \beta, \gamma)$, $\nu' = \nu + \Delta = b - d$ and $\beta' = \beta - \Delta \geq -1$.

Realisability. For ξ take $\nu_0 = \nu$, $u = d + \iota$ and $\varepsilon = 1 - 2\iota$ in Lemma 8.2: $u + \varepsilon = d + 1 - \iota \geq 0$ and $\nu + u + \varepsilon = \beta + 2d + 1 \leq b$. For $T(\xi)$ take $\nu_0 = b - d$, $u = \nu' - \beta' = \nu - \beta + 2\Delta \geq 0$ and $\varepsilon = 1$: then $\nu_0 + u + 1 = 3b - 4d - 2\beta - \iota + 1 \geq b$, because $2\beta + 4d + 2\iota \leq 2b - 2$; so $\gamma = b$. All coordinates lie in $[-1, b]$, so nothing else is clipped.

Distinct. If $\gamma < b$ the third coordinates differ. If $\gamma = b$, then $\iota = 0$ (since $\beta + 2d + 1 + \iota \leq b$) and $\Delta = 1$, so the first coordinates differ.

Equal multisets. $Z(\nu, \beta, \gamma) = S(\nu, \beta) \uplus S(\nu, \gamma)$ and S is symmetric. Put $\tau = \nu + \beta + 2$, $\sigma = \nu + \gamma + 2$, $U = b - \nu$, $V = b - \gamma$. By (3), using $0 \leq \beta \leq \nu \leq \gamma \leq b$,

$$Z(\xi) = \mathrm{AP}(\tau; U) \uplus [\tau + 2U, 2b + 1] \uplus \mathrm{AP}(\sigma; V) \uplus [\sigma + 2V, 2b + 1] \uplus \{T\}^{\nu+\beta+2}.$$

For $T(\xi)$ we have $-1 \leq \beta' \leq \nu' \leq b$, $\nu' + \beta' = \nu + \beta$ and $b - \nu' = d = U - \Delta$, so

$$Z(T(\xi)) = \mathrm{AP}(\tau; d) \uplus [\tau + 2d, 2b + 1] \uplus [\nu' + b + 2, 2b + 1] \uplus \{T\}^{\nu+\beta+2},$$

where the second row $S(\nu', b) = [\nu' + b + 2, 2b + 1] \uplus \{T\}^{\nu'+1}$ is (3) with $p' = b$. Now $\tau + 2d + 1 = \nu + \beta + 2d + 3 = \sigma$, and $\tau + 2U = 2b + 2 + \beta - \nu \leq 2b + 2$. So the interval $[\tau + 2d, \tau + 2U - 1]$ splits into its even and odd offsets:

$$[\tau + 2d, 2b + 1] = \mathrm{AP}(\tau + 2d; \Delta) \uplus \mathrm{AP}(\sigma; \Delta) \uplus [\tau + 2U, 2b + 1],$$

and $\mathrm{AP}(\tau; d) \uplus \mathrm{AP}(\tau + 2d; \Delta) = \mathrm{AP}(\tau; U)$. After cancelling the common part it remains to show

$$\mathrm{AP}(\sigma; \Delta) \uplus [\nu' + b + 2, 2b + 1] = \mathrm{AP}(\sigma; V) \uplus [\sigma + 2V, 2b + 1].$$

Here $\sigma + 2V = 2b + 1 - d + \iota$ and $\nu' + b + 2 = 2b + 2 - d$. If $\iota = 1$, then $\Delta = V$ and $\nu' + b + 2 = \sigma + 2V$: the two sides coincide. If $\iota = 0$, then $\Delta = V + 1$, the left side is $\mathrm{AP}(\sigma; V) \uplus \{\sigma + 2V\} \uplus [\sigma + 2V + 1, 2b + 1]$, and $\sigma + 2V = 2b + 1 - d \leq 2b + 1$, so this is the right side. □

In words: passing from ξ to $T(\xi)$ shortens the step-two progression of the first row from U to d terms; the Δ odd ranks $\sigma, \sigma + 2, \ldots$ thus freed in the first row are exactly the step-two ranks that the second row loses when γ is pushed to b, and Δ copies of T move from the first row to the second. The word forgets which row produced each rank.

Corollary 7.3. *Let* $b \geq 2$. *The map* T *of Lemma 7.2 is injective, and:*

- *(a) for* $d \geq 1$, ξ *is the triple* I_1 $(\iota = 0)$ *or* I_2 $(\iota = 1)$ *of Lemma 8.7 with parameters* (β, d), *and* $T(\xi) = J(\tau, d)$ *with the same* $\tau = 2\beta + d + 2 + \iota$. *The pairs* $\{\xi, T(\xi)\}$ *with* $d \geq 1$ *are exactly the pairs* $\{I, J(\tau, d)\}$ *with* $I \neq J$ *of Lemma 8.7; there are* $\sum_{d=1}^{\lfloor (b-1)/2 \rfloor} (b - 2d) = \lfloor (b-1)^2/4 \rfloor$ *of them;*
- *(b) for* $d = 0$, $\xi = (\beta, \beta, \beta + 1)$ *or* $(\beta + 1, \beta, \beta + 1)$ *(Lemma 8.6(b),(c)) and* $T(\xi) = J_0(s)$ *with* $s = 2\beta + 1 + \iota$; *these are the* b *pairs of* $O(s)$ *with* $b \leq s \leq 2b - 1$.

Proof. From $T(\xi) = (b - d, 2\beta + 2d + \iota - b, b)$ one reads d, then $2\beta + \iota$, hence β and ι by parity: T is injective. (a) The formulas of Lemma 8.7 give $I_1 = (\beta + d, \beta, \beta + 2d + 1)$, $I_2 = (\beta + d + 1, \beta, \beta + 2d + 1)$, and $J(\tau, d) = (b - d, \tau - b + d - 2, b)$ with $\tau - b + d - 2 = 2\beta + 2d + \iota - b = \beta - \Delta$. So the hypothesis $\beta - \Delta \geq -1$ is exactly $\tau \geq b - d + 1$, the lower end of the J-range; the upper end $\tau \leq 2b - 3d + 2$ holds because $\tau \leq 2b - 3d + 1$ on the I-range. The hypothesis $\Delta \geq 1$ is automatic for I_1 $(\Delta = b - \gamma + 1)$

and, for I_2 ($\Delta = b-\gamma$), excludes only $\gamma = b$, the case $I_2 = J$ noted in the proof of Theorem 8.12. The count is the one in that proof. (b) $J_0(s) = (b, s-1-b, b)$ for $s \geq b$, and $s-1-b = 2\beta+\iota-b$. The two hypotheses read $b \leq s$ and $s \leq 2b-1-\iota$; as (β,ι) varies, s covers $[b, 2b-1]$ once. □

Remark 7.4 (what the exchange adds). Lemma 7.2 finds no coincidence that Lemmas 8.6 and 8.7 did not already list, and by Lemma 8.8 there is none to find. What it adds is a single affine map, $(\nu,\beta,\gamma) \mapsto (b-d,\ \beta-\Delta,\ b)$ on two sheets $\iota \in \{0,1\}$, that produces all the pairs of the family Q and the b pairs of the classes $O(s)$ with $b \leq s \leq 2b-1$, together with a proof that does not go through the profiles. Its domain has two free parameters, β and d, for each b; heuristically, this is why the excess for $a = 1$ is quadratic. The pairs of $O(s)$ with $1 \leq s \leq b-1$, whose partner $J_0(s)$ has $\beta \leq -2$, and the class $O(0)$ are not of this form.

7.3. Reflection inside integer levels.

Lemma 7.5. *Let $a \geq 2$, and let p, k be integers with $1 \leq p \leq a-1$, $k \geq 1$ and $(p+1)k \leq b$. Define $u = u^{(p,k)}$ and $v = v^{(p,k)}$ in $\mathbb{Z}^{2a+1}$ by*

$$u_h = \mathrm{clip}(hk - [h < -p]), \qquad v_h = \mathrm{clip}(hk - [h > p]) \qquad (-a \leq h \leq a).$$

Then:

- *(i) u and v are realisable and $u \neq v$;*
- *(ii) $Z(u) = Z(v)$;*
- *(iii) the common word begins with $()^k(($ and its height is $p+1$. Hence different pairs (p,k) give different words, and none of them is $O(0)$.*

So the lemma gives $\sum_{p=1}^{a-1} \lfloor b/(p+1) \rfloor = \sum_{j=2}^{a} \lfloor b/j \rfloor$ coincident pairs, in pairwise distinct fibres different from $\Phi(O(0))$. For $a = 2$ ($p = 1$) they are the pairs A_k, $1 \leq k \leq \lfloor b/2 \rfloor$, of Section 9.

Proof. Choose $\eta > 0$ with $(a+p+1)\eta < 1$ and Λ with $p\eta < \Lambda < (p+1)\eta$ (for instance $\Lambda = (p+\frac{1}{2})\eta$); then $\Lambda + a\eta < 1$. Let $\theta_\pm = k \pm \eta > 0$.

(i) $\Lambda + h\theta_+ = hk + (\Lambda + h\eta)$. For $h \geq -p$ we have $\Lambda + h\eta \in (0,1)$, and for $h < -p$, $\Lambda + h\eta \in (-1,0)$, because $\Lambda < (p+1)\eta \leq |h|\eta$ and $|h|\eta - \Lambda < 1$. So no $\Lambda + h\theta_+$ is an integer and $c(\Lambda, \theta_+) = u$. Symmetrically $\Lambda + h\theta_- = hk + (\Lambda - h\eta)$ gives $c(\Lambda,\theta_-) = v$. Both points are off the lines, so u and v are realisable. At $h = p+1 \leq a$: $u_{p+1} = (p+1)k \leq b$, while $v_{p+1} = (p+1)k - 1$, so $u \neq v$.

(ii) Use (2). The position (i,j) sits at $q \pm i\eta$ with $q = ik + j \in \mathbb{Z}$, the *level* of (i,j). For two positions of levels $q \neq q'$, $|q - q'| \geq 1$ while all the perturbations, together with Λ, are smaller than 1 in absolute value: $|(i-i')\eta \mp \Lambda| \leq (a+p+1)\eta < 1$. So the closing letter of (i',j'), at level q', is preceded by all positions of lower levels and by none of higher levels. Within level q' it is preceded, for θ_+, by the (i,j) with $(i-i')\eta < \Lambda$, that is $i \leq i' + p$; for θ_-, by those with $i \geq i' - p$. The indices i of the positions on level q form an integer interval $I_q = [i_0, i_1]$ (the conditions $0 \leq i \leq a$ and $0 \leq q - ik \leq b$). The reflection $i \mapsto i_0 + i_1 - i$ of I_q carries $\#\{i \in I_q : i \geq i' - p\}$ to $\#\{i \in I_q : i \leq (i_0 + i_1 - i') + p\}$. So, level by level, the multisets of counts for θ_+ and θ_- coincide, and $Z(u) = Z(v)$.

(iii) Take θ_+. Levels $0, \dots, k-1$ contain only the positions $(0,q)$, since $i \geq 1$ gives $q \geq k$; level k contains $(0,k)$ and $(1,0)$, at distance $\eta < \Lambda$. Positions of different levels are more than Λ apart, so the word begins with $()^k(($. The height of the word is the largest number of positions in a half-open window of length Λ. Such a window meets one level only, and within a level consecutive positions are η apart, so it holds at most $p+1$ of them; level pk contains the $p+1$ positions $(i,(p-i)k)$, $0 \leq i \leq p$ (here $pk \leq b$ and $p \leq a$), so the height is $p+1$. Thus the word determines k and p. It is not $O(0)$, whose height is 1. □

Remark 7.6. The lemma does not say that the fibres of $u^{(p,k)}$ have exactly two elements; for $a \leq 5$, $b \leq 8$ they do, except in the square patterns $a = b \in \{4,5\}$, where the pair $(p,k) = (1,2)$ lies in

a fibre of size 4 together with its image under the exchange $r \leftrightarrow s$. Together with Lemma 7.1 it gives, for $a \geq 2$, the lower bound

$$R(a,b) - W(a,b) \;\geq\; \sum_{h=1}^{a} \#\{1 \leq j \leq b : \gcd(j,h) = 1\} + \sum_{j=2}^{a} \Big\lfloor \frac{b}{j} \Big\rfloor,$$

which is linear in b. For $a = 2$ it gives $2b$, against the exact excess $E(b) \sim 43b/6$ of Theorem 9.4 ($b \geq 3$; its upper half rests on the computer-assisted Lemma 9.3).

7.4. Reflection around a rational slope. Lemma 7.5 reflects the rows inside integer levels near $\theta = k$. The same works near any rational m/q.

Lemma 7.7. *Let $m, q \geq 1$ be coprime and $p \geq 1$, with $(p+1)q \leq a$ and $(p+1)m \leq b$. Let $\eta > 0$ satisfy $(a + (p+1)q)\eta < 1/q$ (which implies $\eta < m/q$), let $pq\eta < \Lambda < (p+1)q\eta$, and put $\theta_\pm = m/q \pm \eta$. Then the vectors $c^\pm = c(\Lambda, \theta_\pm)$ are realisable and distinct, $Z(c^+) = Z(c^-)$, and the common word has height $p+1$. Explicitly,*

$$c_h^+ = \mathrm{clip}(\lfloor hm/q \rfloor - [\, q \mid h,\ h < -pq\,]), \qquad c_h^- = \mathrm{clip}(\lfloor hm/q \rfloor - [\, q \mid h,\ h > pq\,]),$$

so the vectors do not depend on the choice of η and Λ. For $q = 1$, $m = k$, this is Lemma 7.5.

Proof. Put $\varepsilon = (a + (p+1)q)\eta < 1/q$. Note $\Lambda < \varepsilon$ and $a\eta < \varepsilon$.

Off the lines, and the vectors. $\Lambda + h\theta_+ = hm/q + (\Lambda + h\eta)$ with $|\Lambda + h\eta| < \varepsilon < 1/q$. If $q \nmid h$, then hm/q is at distance at least $1/q$ from $\mathbb{Z}$ (as $\gcd(m,q) = 1$), so $\Lambda + h\theta_+$ is not an integer and its floor is $\lfloor hm/q \rfloor$. If $h = sq$, then $\Lambda + h\theta_+ = sm + (\Lambda + sq\eta)$, and $\Lambda + sq\eta \in (0,1)$ for $s \geq -p$ (since $\Lambda > pq\eta$), while $\Lambda + sq\eta \in (-1,0)$ for $s \leq -p-1$ (since $\Lambda < (p+1)q\eta \leq |s|q\eta$ and $|s|q\eta \leq a\eta < 1$). For θ_- replace $h\eta$ by $-h\eta$: the case $q \mid h$ now gives $(0,1)$ for $s \leq p$ and $(-1,0)$ for $s \geq p+1$. Both points therefore lie on no line $\Lambda + h\theta \in \mathbb{Z}$, $\theta_- > 0$ because $\eta < m/q$, and $c^\pm$ are as displayed, hence realisable. At $h = (p+1)q \leq a$: $c_h^+ = (p+1)m \leq b$ and $c_h^- = (p+1)m - 1$, so $c^+ \neq c^-$.

Levels. The position (i,j) ($0 \leq i \leq a$, $0 \leq j \leq b$) is at $i\theta_\pm + j = \ell/q \pm i\eta$, where $\ell = mi + qj \in \mathbb{Z}$ is its *level*. For positions (i,j), (i',j') of levels $\ell \neq \ell'$, the difference between the closing letter of (i',j') and the opening letter of (i,j) is $(\ell' - \ell)/q + (\Lambda \pm (i'-i)\eta)$, where $|\ell' - \ell|/q \geq 1/q$ and $|\Lambda \pm (i'-i)\eta| < \Lambda + a\eta < \varepsilon < 1/q$. So by (2) the count of the closing letter of (i',j') is the number of positions of level $< \ell'$, plus the number of positions (i,j) of level ℓ' with $\pm(i-i')\eta < \Lambda$.

Inside a level. On level ℓ, $mi \equiv \ell \pmod q$; since m is invertible modulo q, i lies in one residue class modulo q. The conditions $0 \leq i \leq a$ and $0 \leq (\ell - mi)/q \leq b$ define an interval of i. So the indices i on level ℓ are $i_0 + tq$, $t = 0, \ldots, N_\ell - 1$, without gaps. For two of them, $i - i' = q(t - t')$, and $pq\eta < \Lambda < (p+1)q\eta$ gives: $(i-i')\eta < \Lambda$ if and only if $t \leq t' + p$; $-(i-i')\eta < \Lambda$ if and only if $t \geq t' - p$. The reflection $t \mapsto N_\ell - 1 - t$ maps $\#\{t : t \geq t' - p\}$ to $\#\{t : t \leq (N_\ell - 1 - t') + p\}$. So, level by level, the multisets of counts at θ_+ and at θ_- coincide, and $Z(c^+) = Z(c^-)$.

Height. At a point off the lines, the height of the prefix of the word ending just before x is $\#\{\text{positions} < x\} - \#\{\text{positions} < x - \Lambda\}$, the number of positions in $[x - \Lambda, x)$; so the height of the word is the largest number of positions in a half-open window of length Λ. Positions of different levels are more than $1/q - a\eta > \Lambda$ apart, so a window meets one level, where consecutive positions are $q\eta$ apart; since $\Lambda < (p+1)q\eta$ it contains at most $p+1$ of them. The level pmq contains the $p+1$ positions $(tq, m(p-t))$, $0 \leq t \leq p$ (here $pq \leq a$, $pm \leq b$), spanning $pq\eta < \Lambda$. So the height is $p+1$. □

Remark 7.8. The height separates different p, but we have not proved that different (m,q) with the same p give different words, nor that the words for $q \geq 2$ differ from those of Lemma 7.5. So the lemma is not added to the lower bound for $R(a,b) - W(a,b)$ given above. In the exact data for $a \leq 5$, $b \leq 8$, the pairs with $q \geq 2$ give words that are not words of pairs with $q = 1$ (one or two per (a,b) for $a \in \{4,5\}$, except $a = b$; none for $a \leq 3$, where $(p+1)q \leq a$ forces $q = 1$), and different triples (m,q,p) give different words, with one exception: for the square patterns $a = b \in \{4,5\}$,

$(m,q,p) = (2,1,1)$ and $(1,2,1)$ give the same word, in a fibre of size 4 (the exchange $r \leftrightarrow s$, $(\Lambda,\theta) \mapsto (\Lambda/\theta, 1/\theta)$, pairs the four vectors $c^{\pm}(2,1,1)$, $c^{\pm}(1,2,1)$ two by two). Every other fibre met has size 2.

Part III. Explicit counts

This part computes $W(1,b)$ and $W(2,b)$ in closed form, hence $|L_{2q}|$ and $|L_{3q}|$, and ends with a table of $|L_k|$ for small k. It uses Part I and, for the number of cells when $a = 2$, Theorem 5.1. In both cases $W(a,b)$ is the number of realisable cells minus the number lost to coincidences, and the work is to list the coincidences and to prove that the list is complete.

8. Theorem A: semilength $2q$

Fix $b \geq 2$. Let $m = rs^b$ with $r \neq s$ odd primes and $t = 2^e$; the semilength is $2(b+1)$. Put $T = 2b+2$. Measure everything in units of $\log s$: put $\Lambda = \log t/\log s$ and $\theta = \log r/\log s$. Note that $\theta \notin \mathbb{Z}$ (indeed θ is irrational, since $r^{q'} \neq s^{p'}$ for all positive integers p', q'), so θ has nonzero fractional part. The letters (then sit at $\{0,\dots,b\} \cup \{\theta,\dots,\theta+b\}$, and the letters) at the same set shifted by Λ. Let

$$f(v) = \min(b+1, \max(0, v+1)) = \#\{i \in \{0,\dots,b\} : i \leq v\},$$

$$S(p,p') = \{f(p+j) + f(p'+j) : 0 \leq j \leq b\}$$

(a multiset), and let $\mathrm{clip}(v) = \min(b, \max(-b-1, v))$.

Lemma 8.1 triples. *Put $\nu = \mathrm{clip}\lfloor\Lambda\rfloor$, $\beta = \mathrm{clip}\lfloor\Lambda-\theta\rfloor$, $\gamma = \mathrm{clip}\lfloor\Lambda+\theta\rfloor$. The multiset $C(n)$ of (1) is $Z(\nu,\beta,\gamma) := S(\nu,\beta) \uplus S(\nu,\gamma)$. In particular the word depends only on the triple (ν,β,γ).*

Proof. None of Λ, $\Lambda \pm \theta$ is an integer. Each such equality has the form $2^e \cdot (\text{odd}) = \text{odd}$:

- $t = s^k$;
- $t = rs^k$, or $ts^{|k|} = r$ when $k < 0$;
- $tr = s^k$.

All are impossible by parity. Take the letter) at $\Lambda + j'$. It is preceded by $\#\{i : i < \Lambda + j'\} = f(\lfloor\Lambda\rfloor + j')$ letters (of the row $\{0,\dots,b\}$, and by $\#\{i : \theta + i < \Lambda + j'\} = f(\lfloor\Lambda-\theta\rfloor + j')$ of the row $\theta + \{0,\dots,b\}$. Similarly, the letter) at $\Lambda + \theta + j'$ is preceded by $f(\lfloor\Lambda+\theta\rfloor + j') + f(\lfloor\Lambda\rfloor + j')$ letters (. Clipping does not change $f(v+j)$ for $0 \leq j \leq b$: if $v \leq -b-1$, then $f(v+j) = 0$, and if $v \geq b$, then $f(v+j) = b+1$. □

Lemma 8.2 realisable triples. *The triples that occur are exactly the clippings of $(\nu_0, \nu_0-u, \nu_0+u+\varepsilon)$ with integers $\nu_0 \geq 0$, $u \geq 0$, $\varepsilon \in \{-1,0,1\}$ and $u + \varepsilon \geq 0$.*

Proof. Write $\Lambda = \nu_0 + \lambda$ and $\theta = \kappa + \phi$ with $\nu_0, \kappa \geq 0$ integers and $\lambda, \phi \in (0,1)$ (both fractional parts are nonzero: $\Lambda \notin \mathbb{Z}$ by Lemma 8.1, $\theta \notin \mathbb{Z}$ as noted). Then

$$\lfloor\Lambda-\theta\rfloor = \nu_0 - \kappa - [\lambda < \phi], \qquad \lfloor\Lambda+\theta\rfloor = \nu_0 + \kappa + [\lambda + \phi > 1].$$

The four open regions $\{\lambda \gtrless \phi\} \cap \{\lambda + \phi \gtrless 1\}$ of $(0,1)^2$ are nonempty. They give $(u,\varepsilon) = (\kappa,0), (\kappa,1), (\kappa+1,-1), (\kappa+1,0)$. As κ ranges over $\mathbb{Z}_{\geq 0}$, these are all pairs with $u \geq 0$ and $u+\varepsilon \geq 0$. Each region is an open set of (Λ,θ); in the coordinates $x = (\log r, \log s)/\log t = (\theta/\Lambda, 1/\Lambda)$ it is an open set avoiding the lines of $\mathcal{A}_{(1,b)}$. Lemma 2.3 then realises it by primes $r \neq s$. □

Throughout, $\nu_0 = \lfloor \Lambda \rfloor \geq 0$ denotes the unclipped value and $\nu = \mathrm{clip}(\nu_0) = \min(\nu_0, b)$ the clipped one. For a realisable triple put $g = b - \nu$, $c = \gamma - \nu$ and $A = \nu - \beta$ (with β clipped). From the lemma, $c = \min(g, A + \varepsilon)$ for some $\varepsilon \in \{-1, 0, 1\}$ with $A + \varepsilon \geq 0$. If β is clipped at $-b - 1$, both sides equal g. Writing $\delta = c - A$ this gives

$$0 \leq c \leq g, \qquad \delta \leq 1, \qquad \delta \leq -2 \;\Rightarrow\; c = g. \tag{V}$$

This is immediate from the definitions.

Lemma 8.3. *There are $R(1, b) = (5b + 4)(b + 1)/2$ realisable clipped triples.*

Proof. We count clipped triples according to the unclipped parameters (ν_0, u, ε) of Lemma 8.2. If $\nu_0 \geq b$, then $\nu = b$ and the clipped triple is (b, β, b) with $\beta \in [-b - 1, b]$, giving $2b + 2$ triples. Fix $\nu_0 = \nu \leq b - 1$ (no clipping of the first coordinate) and write $\beta = \nu - u$ before clipping. The case $u = 0$ gives $\gamma \in \{\nu, \nu + 1\}$. Each $1 \leq u \leq b - \nu - 1$ gives three values $\gamma = \nu + u + \varepsilon$. The case $u = b - \nu$ gives $\gamma \in \{b - 1, b\}$. The range $b - \nu + 1 \leq u \leq \nu + b$ gives $\gamma = b$, for 2ν triples. All $u \geq \nu + b + 1$ give the single triple $(\nu, -b - 1, b)$. The total is $3b - \nu + 2$, and $\sum_{\nu=0}^{b-1}(3b - \nu + 2) + 2b + 2 = (5b + 4)(b + 1)/2$. □

8.1. Block form. Every $Z = Z(\nu, \beta, \gamma)$ is a multiset in $[\tau, T]$, where $\tau = \min Z$. We encode Z by τ and its *profile* $\mu(x)$, the multiplicity of $\tau + x$ in Z, for $0 \leq x < \ell := T - \tau$. The multiplicity of T is then $2b + 2 - \sum_x \mu(x)$. Profiles are written as strings; X and Y denote the profiles of $S(\nu, \beta)$ and $S(\nu, \gamma)$ on the same window, so $\mu = X + Y$ entrywise.

Conventions. An integer interval $[x, y]$ with $y < x$ is empty; so is an arithmetic progression $\{x, x + 2, \ldots, y\}$ with $y < x$; a string w^k with $k \leq 0$ is empty. Unions of multisets are written $\uplus$ and add multiplicities; $\{T\}^j$ is the multiset with j copies of T.

Lemma 8.4. *(i) If $0 \leq p \leq p' \leq b$, then $S(p, p') = \{p + p' + 2, p + p' + 4, \ldots, 2b + p - p'\} \uplus [2b + 2 + p - p', 2b + 1] \uplus \{T\}^{p+1}$.*

(ii) If $p = -\rho < 0 \leq p' \leq b$ and $\rho \leq b - p'$, then $S = [p' + 1, p' + \rho] \uplus \{p' + \rho + 2, p' + \rho + 4, \ldots, 2b - p' - \rho\} \uplus [2b + 2 - \rho - p', 2b + 2 - \rho]$.

(iii) If $p = -\rho$ and $\rho > b - p'$, then $S = [p' + 1, 2b + 2 - \rho] \uplus \{b + 1\}^{\rho - 1 - b + p'}$.

Proof. $f(p' + j) = p' + j + 1$ for $j \leq b - p'$ and $b + 1$ afterwards; $f(-\rho + j) = 0$ for $j < \rho$ and $j - \rho + 1$ afterwards. Add the two and split the range of j at the break points; a range that is empty contributes nothing, by the conventions above. □

Split the realisable triples into two families. Family P' has $\beta \geq -1$, so $A = \nu - \beta \geq 0$. Family N' has $\beta = -\rho \leq -2$; within it, N'_1 has $\rho \leq g$ and N'_2 has $\rho > g$, and in N'_2 we always have $c = g$.

Corollary 8.5 profiles.
- *In P': $\tau = \nu + \beta + 2$, $\ell = 2g + A$, $X = (10)^g 1^A$ and $Y = 0^{A+c}(10)^{g-c}1^c$.*
- *In N': $\tau = \nu + 1$, $\ell = 2g + \nu + 1$ and $Y = 0^{\nu+c+1}(10)^{g-c}1^c$. Moreover $X = 1^\rho(01)^{g-\rho}0\,1^{\nu+1}0^{\rho-1}$ in N'_1, and $X = 1^{\ell-\rho+1}0^{\rho-1}$ plus $\rho - g - 1$ at position $x = g$ in N'_2.*

Proof. Translate Lemma 8.4 by $-\tau$. When $\beta = -1$ the N' formula with $\rho = 1$ coincides with the P' formula with $A = \nu + 1$, so the triples with $\beta = -1$ may be placed in P'. □

8.2. The coincidences. Define two families of words by their profiles. The word $O(s)$, $0 \leq s \leq 2b + 1$, has $Z = [s + 1, 2b + 1] \uplus \{T\}^{s+1}$ and profile 1^ℓ. (This is the one-prime word $O_T(s)$ of Corollary 3.3.) For $d \geq 1$, the word $Q(\tau, d)$ has profile $(10)^d 1^{\ell - 3d} 2^d$. Figure 6 shows both families for $b = 3$.

Lemma 8.6. *Each of the following triples has word $O(s)$:*

(a) $s = 0$: $(0, -\rho, \rho - 1)$ *for* $\rho = 1, \dots, b + 1$;
(b) $s = 2k + 1$: $(k, k, k + 1)$ *for* $0 \le k \le b - 1$;
(c) $s = 2k + 2$: $(k + 1, k, k + 1)$ *for* $0 \le k \le b - 1$;
(d) $s = 2b + 1$: (b, b, b);
(e) $J_0(s) := (\min(b, s), s - 1 - b, b)$ *for* $0 \le s \le 2b$. *This triple is realisable: take* $u = \nu - \beta$ *and* $\varepsilon = 1$ *in Lemma 8.2; then* $\nu + u + 1 = 2\nu - \beta + 1 \ge b$, *so* $\gamma = b$.

Proof. All cases are read off Corollary 8.5.

- (a): the union of $S(0, -\rho)$ and $S(0, \rho - 1)$ is $[1, 2b + 1]$ plus copies of T, so $\tau = 1$ and the profile is 1^ℓ.
- (b): P' with $A = 0$, $c = 1$: $X + Y = (10)^g + 0(10)^{g-1}1 = 1^{2g}$.
- (c): P' with $A = 1$, $c = 0$: $(10)^g 1 + 0(10)^g = 1^{2g+1}$.
- (d): $\ell = 0$.
- (e) with $s \ge b$: P' with $g = c = 0$ gives $X + Y = 1^A + 0^A$ and $\tau = s + 1$.
- (e) with $s < b$: N_2' with $\nu = s$ and $\rho = g + 1$ gives $X + Y = 1^{\nu+g+1}0^g + 0^{\nu+g+1}1^g = 1^\ell$.

□

Lemma 8.7. *Let* $d \ge 1$. *The following triples have word* $Q(\tau, d)$:

- $I_1 = (\beta + d, \beta, \beta + 2d + 1)$, *with* $\tau = 2\beta + d + 2$;
- $I_2 = (\beta + d + 1, \beta, \beta + 2d + 1)$, *with* $\tau = 2\beta + d + 3$;
- $J(\tau, d) = (b - d, \tau - b + d - 2, b)$ *for* $1 \le d \le b$ *and* $\tau \in [b - d + 1, 2b - 3d + 2]$.

For I_1 *and* I_2 *the conditions are* $\beta \ge 0$ *and* $\beta + 2d + 1 \le b$. *They give exactly one* I*-triple for each* $\tau \in [d + 2, 2b - 3d + 1]$.

Proof. For I_1, $X = (10)^g 1^d$ and $Y = 0^{2d+1}(10)^{g-d-1}1^{d+1}$ with $g \ge d + 1$, and the sum is $(10)^d 1^{2g-2d} 2^d$. I_2 is the same with one more 1. For J with $\beta \ge 0$ we have $g = c = d$ and $A \ge d$, so $X + Y = (10)^d 1^A + 0^{A+d}1^d$. For J with $\beta = -1$ the same computation applies with $A = \nu + 1 \ge d$. □

For a realisable triple ξ, define its *label* $\lambda(\xi)$ as follows: $O(s)$ if ξ is in the list of Lemma 8.6; $Q(\tau, d)$ if ξ is in the list of Lemma 8.7; ξ itself otherwise. The label is well defined, because O-triples have profile without zeros while Q-triples have exactly $d \ge 1$ zeros.

Lemma 8.8. *Let* $b \ge 2$. *Two realisable triples have the same word if and only if they have the same label.*

The implication ($\Leftarrow$) is Lemmas 8.6 and 8.7. We prove ($\Rightarrow$) by decoding the profile.

8.3. Proof of ($\Rightarrow$). Let $K(Z) = (\tau, f_0, n_0)$, where n_0 is the number of zeros of the profile and f_0 is the position of the first zero ($f_0 = -1$ if $n_0 = 0$). K is a function of Z. Split the triples by $\delta = c - A$ into types P^+ ($\delta = 1$), P^0 ($\delta = 0$), P^- ($\delta = -1$) and P^c ($\delta \le -2$, hence $c = g$), and likewise N^+, N^0, N^-, N^c inside N_1'.

Lemma 8.9 zeros. *Let* $\mathcal{Z}_0$ *be the set of zeros of the profile. Then:*

type	n_0	$\mathcal{Z}_0$
P^+	A	$\{1, 3, \dots, 2n_0 - 1\}$
P^-	$A - 1$	*same*
P^0, P^c	g	*same*
N^-	ν	$\{\rho + 2i : 0 \le i < n_0\}$
N^0, N^c	$g - \rho + 1$	*same*
N^+	$\nu + 1$	*same*
N_2'	$\rho - g - 1$	$[2g + \nu + 2 - \rho,\ \nu + g + 1)$

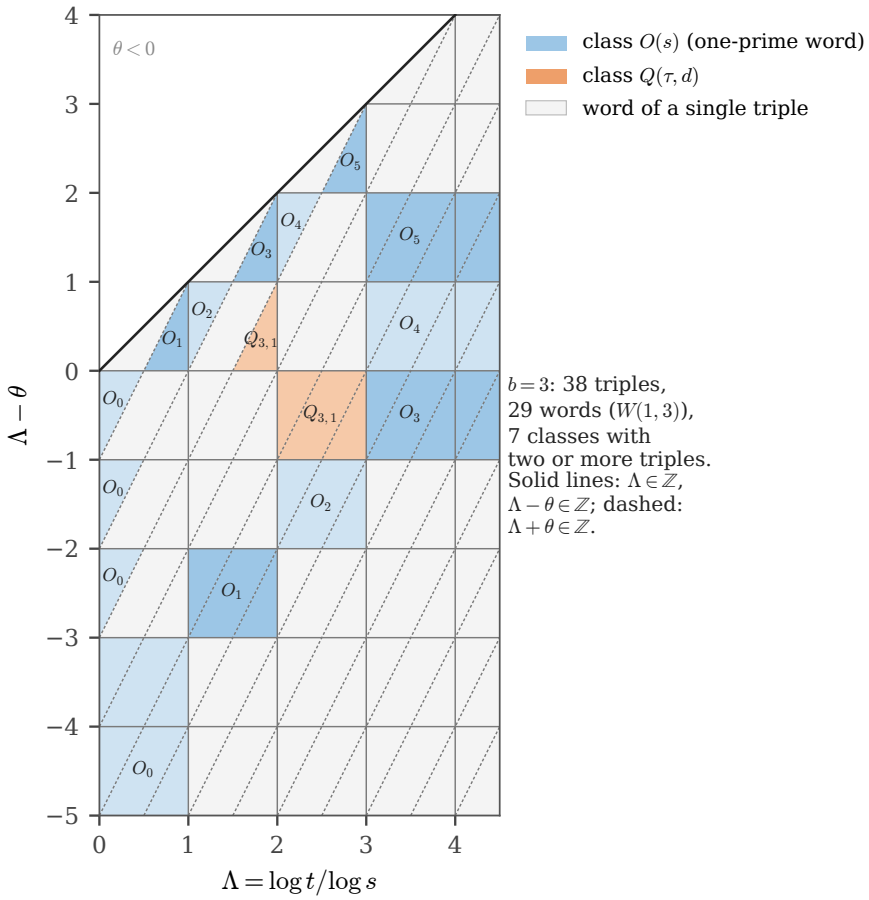


Figure 6. Pattern $(1,3)$ ($m = rs^3$, $b = 3$) in the coordinates $(\Lambda, \Lambda - \theta)$ of Lemma 8.1: the lines $\Lambda \in \mathbb{Z}$, $\Lambda - \theta \in \mathbb{Z}$ (solid) and $\Lambda + \theta \in \mathbb{Z}$ (dashed) cut the region $\theta > 0$ into pieces; the triple (ν, β, γ) is constant on each piece. Pieces whose word is shared by two or more triples are coloured: $O(s)$ in blue (label O_s), $Q(\tau, d)$ in orange (label $Q_{\tau,d}$); one label per triple. The 38 triples give $29 = W(1,3)$ words, and every class with two or more triples is an O or a Q class (Lemma 8.8 for $b = 3$). Live version in the blueprint.

In particular, when $n_0 > 0$: $f_0 = 1$ on P', $f_0 = \rho$ (≥ 2) on N_1', and $f_0 = 2g + \nu + 2 - \rho \geq g + 1$ on N_2'.

Proof. P'. We have $X(x) = 0$ iff x is odd and $x < 2g$. Put $\sigma = A + c$. Then $Y(x) = 0$ iff $x < \sigma$, or $\sigma \leq x < \ell - c$ with $x - \sigma$ odd. By type:

- $\delta = 1$: $\sigma = 2A + 1 \leq 2g - 1$ is odd, so the zeros are the odd $x < \sigma$ and $n_0 = A$.
- $\delta = -1$: $\sigma = 2A - 1 \leq 2g + 1$, so the zeros are the odd $x < \min(\sigma, 2g)$ and $n_0 = A - 1$.
- $\delta = 0$: $\sigma = 2c \leq 2g$ is even and $\ell - c = 2g$, so the zeros are the odd $x < 2g$ and $n_0 = g$.
- $\delta \leq -2$: $c = g$ and $\sigma \geq 2g + 2$, so the zeros are the odd $x < 2g$ and $n_0 = g$.

N_1'. The zeros of X are $x_i = \rho + 2i$ $(0 \leq i \leq g - \rho)$ together with $[\ell - \rho + 1, \ell)$. Since $c \geq \rho - 1$, the tail 1^c of Y covers the latter. Y vanishes on $[0, w)$, where $w = \nu + c + 1$, and at $w+1, w+3, \ldots, \ell-c-1$. Write $c = \nu + \rho + \delta$. Then $x_i < w$ iff $i \leq \nu + \lfloor \delta/2 \rfloor$. For $x_i \geq w$, $Y(x_i) = 0$ iff δ is even and $x_i \leq 2g - \rho - \delta$. By type:

- $\delta = 0$: every $i \leq g - \rho$ qualifies, so $n_0 = g - \rho + 1$.
- $\delta = 1$: only $i \leq \nu$ qualifies, and $\nu \leq g - \rho - 1$, so $n_0 = \nu + 1$.
- $\delta = -1$: only $i \leq \nu - 1$ qualifies, so $n_0 = \nu$.

- $\delta \le -2$: $c = g$ and $\nu \ge g - \rho + 2$, so $\nu + \lfloor \delta/2 \rfloor = \lfloor (\nu + g - \rho)/2 \rfloor \ge g - \rho + 1$ and every i qualifies.

N_2'. Here $Y = 0^{\nu+g+1}1^g$. X is 1 on $[0, \ell - \rho + 1)$ (with the excess at $x = g$) and 0 afterwards. Since $\rho \le b + 1$, we have $\ell - \rho + 1 \ge g + 1$, and the zeros are $[\ell - \rho + 1, \nu + g + 1)$. □

Lemma 8.10. *$n_0 = 0$ if and only if the triple is in the list of Lemma 8.6. In that case the label is $O(\tau - 1)$.*

Proof. By Lemma 8.9, $n_0 = 0$ exactly in the following cases:

- P^+ with $A = 0$: the triple $(k, k, k + 1)$, case (b).
- P^- with $A = 1$: $(k + 1, k, k + 1)$, case (c), or $(0, -1, 0)$, case (a).
- P^0 with $g = 0$: case (d).
- P^c with $g = 0$: $(b, \beta, b) = J_0(\beta + 1 + b)$.
- N^- with $\nu = 0$: case (a).
- N_2' with $\rho = g + 1$: $J_0(\nu)$.

All other types have $n_0 \ge 1$. □

Lemma 8.11. *Let $b \ge 2$. If $K(\xi) = K(\eta)$ and $n_0 \ge 1$, then either $\xi = \eta$, or $\{\xi, \eta\} = \{I, J(\tau, d)\}$ with I an I_1/I_2-triple and $d = n_0$.*

Proof. Equal τ gives equal ℓ. Between two triples of N' it also gives equal ν, hence equal g.

(1) Different families. By Lemma 8.9, $f_0 = 1$ on P', $2 \le f_0 = \rho \le g$ on N_1' and $f_0 \ge g + 1$ on N_2'. Since two triples of N' with equal τ have equal g, the ranges $[2, g]$ and $[g + 1, \infty)$ of f_0 are disjoint, so N_1' and N_2' never collide; and N_1' never collides with P', since $f_0 \ge 2$ there. So P' and N' can only collide when $f_0 = 1$ in N_2', which forces $g = 0$, $\nu = b$, $\rho = b + 1$, $\ell = b + 1$, $n_0 = b$. No P' triple has $\ell = b + 1$ and $n_0 = b$ when $b \ge 2$:

- P^0/P^c: $n_0 = g = b$ and $\ell = 2g + A$ would need $A = 1 - b < 0$;
- P^+: $n_0 = A = b$ and $\ell = 2g + A$ would need $2g = 1$;
- P^-: $n_0 = A - 1 = b$ gives $A = b + 1$, and $\ell = 2g + A = b + 1$ gives $g = 0$; then $c = A + \delta = A - 1 = b > 0 = g$, contradicting $c \le g$ in (V).

(For $b = 1$ such a pair exists, which is why $b \ge 2$ is assumed.)

(2) N_2' with N_2'. n_0 fixes ρ, and $c = g$. So $\xi = \eta$.

(3) N_1' with N_1'. ν and $\rho = f_0$ agree. The types are constrained by $N^- : \nu + \rho - 1 \le g$, $N^0 : \nu + \rho \le g$, $N^+ : \nu + \rho + 1 \le g$ and $N^c : \nu + \rho \ge g + 2$. Comparing the values of n_0 in Lemma 8.9: N^- against N^0/N^c forces $\nu = g - \rho + 1$, which neither allows; N^+ against N^0/N^c forces $\nu = g - \rho$, which neither allows; and $N^\pm$ differ by 1. Within a type, c is determined. So $\xi = \eta$.

(4) P' with P'. Here $A \equiv \ell \pmod 2$ and $g = (\ell - A)/2$. P^0 and P^c both give $g = d$ and $A = \ell - 2d$, but P^0 needs $A \le g$ and P^c needs $A \ge g + 2$, so they cannot both occur. P^+ gives $A = d$ and P^- gives $A = d + 1$, which have different parities, so these two never collide. Within one type, (ℓ, d) determines the triple.

If $\xi \in P^+$ and $\eta \in P^0 \cup P^c$, then $A_\eta = 2g_\xi - d \ge d + 2 = g_\eta + 2$, so $\eta \in P^c$ and $\beta_\eta = b - 2g_\xi \ge -1$. Then $\beta_\xi = b - g_\xi - d \ge 0$ and $\xi = I_1(\beta_\xi, d)$. Also $\eta = (b - d, b - 2g_\xi, b) = J(\tau, d)$, since $\tau - b + d - 2 = 2\beta_\xi + 2d - b = b - 2g_\xi$.

If $\xi \in P^-$ and $\eta \in P^0 \cup P^c$, then $g_\eta = d$, $A_\eta = \ell - 2d$, and $\ell = 2g_\xi + A_\xi = 2g_\xi + d + 1$, so $A_\eta = 2g_\xi + 1 - d$. In P^- we have $c_\xi = A_\xi - 1 = d \le g_\xi$. If $g_\xi = d$, then $A_\eta = d + 1 = g_\eta + 1$, which is allowed neither in P^0 (needs $A \le g$) nor in P^c (needs $A \ge g + 2$); so this case does not occur. Hence $g_\xi \ge d + 1$, $A_\eta \ge g_\eta + 3$ and $\eta \in P^c$. In that case $\beta_\eta = b - 2g_\xi - 1 \ge -1$, $\xi = I_2(\beta_\xi, d)$ with $\beta_\xi = b - g_\xi - d - 1 \ge 0$, and $\eta = J(\tau, d)$. □

Proof of Lemma 8.8, ($\Rightarrow$). Equal words give equal Z, hence equal K. If $n_0 = 0$, both labels are $O(\tau - 1)$ by Lemma 8.10. If $n_0 > 0$, Lemma 8.11 applies. □

8.4. Count.

Theorem 8.12 Theorem A. *For $b \geq 2$,*

$$W(1,b) = \Big\lfloor \frac{9b^2+8b+12}{4} \Big\rfloor.$$

Consequently, for every odd prime q,

$$|L_{2q}| = \frac{9q^2-10q+13}{4} \qquad (q = 3,5,7:\ 16,47,96).$$

Proof. By Lemmas 8.1, 8.2 and 8.8, $W(1,b)$ is the number of labels. This equals the number of triples (Lemma 8.3) minus the excess $\sum(|\text{class}|-1)$.

- *O-classes.* $O(0)$ has $b+1$ triples ($J_0(0)$ is case (a) with $\rho = b+1$). For $1 \leq s \leq 2b-1$, $O(s)$ has two triples. $O(2b)$ has one, since $J_0(2b) = (b, b-1, b)$ is case (c). $O(2b+1)$ has one. The excess is $3b-1$.
- *Q-classes.* For $2d \leq b-1$, the I- and J-ranges of τ overlap in $[b-d+1, 2b-3d+1]$. At the right end, $I_2 = J$. The excess is $b-2d$. If $2d \geq b$ there is no I-triple.

So $E(b) = 3b-1+\sum_{d=1}^{\lfloor (b-1)/2 \rfloor}(b-2d)$, which is $3b-1+k(k-1)$ for $b=2k$ and $3b-1+k^2$ for $b=2k+1$. Then $W(1,b) = (5b+4)(b+1)/2 - E(b)$ equals $9k^2+4k+3$ for $b=2k$ and $9k^2+13k+7$ for $b=2k+1$. Equivalently,

$$W(1,b) = 3 + \frac{9b(b+1)-2\lceil b/2 \rceil}{4} = \Big\lfloor \frac{9b^2+8b+12}{4} \Big\rfloor,$$

since $(9b^2+8b+12)/4$ equals $9k^2+4k+3$ for $b=2k$ and $9k^2+13k+7+\frac{1}{4}$ for $b=2k+1$. For $|L_{2q}|$, apply Corollary 3.3(b) with $b = q-1$ (even): $3+(q-1)(9q-1)/4 = (9q^2-10q+13)/4$. □

Remark 8.13 (status of Theorem A). The proof above is by hand. Each lemma was also checked by computer for $b \leq 20$, with negative controls; the direction ($\Rightarrow$) was checked by Z3 on a separate encoding (9 cases unsatisfiable, controls satisfiable) and on the event encoding of §9.2 (55/55 unsatisfiable); and $|L_6| = 16$ was confirmed from actual integers.

9. Theorem B: semilength $3q$

This section gives the statements and the structure of the proof. The explicit class lists, the hand computations, the realisability witnesses, the derivation of the event encoding and the content and status of every SMT query are in the separate supplement to Theorem B.

Let $m = r^2 s^b$ with $b \geq 3$; the semilength is $T = 3b+3$. In units of $\log s$, the letters (sit at $i\theta + j$ ($i \in \{0,1,2\}$, $0 \leq j \leq b$). Put $c_h = \mathrm{clip}\lfloor \Lambda + h\theta \rfloor$ for $h = -2, \ldots, 2$, and

$$H_i(j) = \sum_{l=0}^{2} f(c_{i-l}+j) \quad (0 \leq j \leq b), \qquad Z(c) = H_0 \uplus H_1 \uplus H_2.$$

Lemma 9.1 vector. *$C(n) = Z(c)$. The vectors c that occur are exactly the points of the open quadrant $\Lambda, \theta > 0$ that avoid the lines $\Lambda + h\theta = \rho$ ($h \in \{-2, \ldots, 2\}$, $\rho \in [-b, b]$), taken up to the cell they lie in. Every such cell is realised by primes.*

Proof. This is the argument of Lemmas 8.1, 8.2 and 2.3. A letter) at $\Lambda + i'\theta + j'$ is preceded by $f(c_{i'-i}+j')$ letters (of row i. Realisability of c is an open linear system, decided by Fourier–Motzkin elimination of Λ. The region of a vector is convex, and distinct cells give distinct vectors. □

Lemma 9.2 cells. *The number of realisable vectors is $R(2,b) = (22b^2+31b+9+[b \equiv 1\ (3)])/3$.*

Proof. This is Theorem 5.1 with $a = 2$. There $(3a+1)b + a = 7b + 2$, $S_1(b) = b(2b+1)$, $S_2(b) = S_4(b) = b(b+1)$ (the $p \le 2b$ coprime to 2, or to 4, are the odd ones), $B_1(2,b) = 2(b + \lfloor b/2 \rfloor)$, $B_2(2,b) = 2\lceil b/2 \rceil$ and $B_3 = B_4 = 0$. Since $\lfloor b/2 \rfloor + \lceil b/2 \rceil = b$, the formula becomes $R(2,b) = 6b^2 + 9b + 3 + S_3(b)$. With $J = \lfloor 2b/3 \rfloor$,

$$S_3(b) = \sum_{p=1}^{2b} (2b - p + 1) - \sum_{j=1}^{J} (2b + 1 - 3j) = (2b+1)(b - J) + \tfrac{3}{2} J(J+1),$$

which equals $(4b^2 + 4b + [b \equiv 1\ (3)])/3$ in each residue class: $12m^2 + 4m$, $12m^2 + 12m + 3$ and $12m^2 + 20m + 8$ for $b = 3m$, $3m+1$, $3m+2$ (with $J = 2m$, $2m$, $2m+1$). Substituting gives the formula. □

9.1. The coincidences (all by hand). Write $O(s)$ $(0 \le s \le 3b+2)$ for the one-prime word $Z = [s+1, 3b+2] \uplus \{T\}^{s+1}$. Let $G(s) = (\lfloor (s+2h)/3 \rfloor)_h$, $O_1(s) = (\lfloor (s+h)/3 \rfloor)_h$ and $O_2(s) = (s + h(b+1))_h$, all clipped. Two tools are used:

- *Lemma M.* Changing one coordinate c_h by ± 1 changes $H_i(j)$ by ± 1 exactly for $0 \le i - h \le 2$ and j in an explicit interval, because $f(v+1) - f(v) = [-1 \le v \le b-1]$.
- *Lemma P.* At a point (Λ, θ) off all lines with distinct positions, Z is the multiset of counts $\#\{\text{positions} < p + \Lambda\}$. The useful points are $\theta = 1/3$, $2/3$ and $b+1$.

The classes of size ≥ 2 are:

O. $O(0)$, which contains $b + 1 + \lceil b/2 \rceil$ vectors (limit $\Lambda \to 0^+$). For $1 \le s \le 3b$, $O(s) \supseteq \{O_1(s) \ne O_2(s)\}$ (Lemma P at $\theta = 1/3$ and $\theta = b+1$).

A. $A_k = \{(-2k-1, -k, 0, k, 2k), (-2k, -k, 0, k, 2k-1)\}$ for $1 \le k \le \lfloor b/2 \rfloor$ (Lemma M: the two changes cancel as multisets).

B. Only for $b = 3w + 2$, $w \ge 1$, three pairs:

$$B_1 = \{(-w-2, -1, w, 2w+2, b),\ (-w-1, -1, w, 2w+1, b)\},$$
$$B_2 = \{(-w-2, -1, w+1, 2w+2, b),\ (-w-1, 0, w, 2w+1, b)\},$$
$$B_3 = \{(-w-2, 0, w+1, 2w+2, b),\ (-w-1, 0, w+1, 2w+1, b)\}.$$

B_1 and B_3 follow from Lemma M; B_2 from a direct block computation.

C. $\{(-b, -1, b-1, b, b), G(b)\}$ and $\{(-b, 0, b-1, b, b), G(b+1)\}$. Lemma P at $\theta = 2/3$ gives $Z(G(s)) = \{s\} \uplus [s+2, 3b+2] \uplus \{3b+2\} \uplus \{T\}^s$.

D. For $0 \le i \le (b-5)/3$ and $\varepsilon \in \{0, 1\}$, with $x = b - 6 - 3i$, $y = b - 4 - 2i + \varepsilon$, $z = b - 1 - i$: $\{(x, y, z, b, b), (x+1, y, z-1, b, b)\}$.

E. For $0 \le i \le (b-5)/2$, with $x = b - 6 - 2i$, $y = b - 2 - i$: $\{(x, y, b, b, b), (x+1, y-1, b, b, b)\}$.

F. For $-1 \le x \le b - 3$: $\{(x, b-1, b, b, b), G(2b + 2 + x)\}$. For $x = b - 4$ the class also contains $(b-3, b-2, b, b, b)$.

Each vector listed is realisable (explicit witnesses). Each class has a single word; we prove this by hand by exact multiset cancellation. The written hand computations for B_2, D, E, F and the explicit list of the vectors of $O(0)$ are not reproduced here; they are in the supplement to Theorem B, together with realisability witnesses for every vector as functions of the parameters. The realisability witnesses were checked by computer for $3 \le b \le 80$.

Lemma 9.3. *Let $b \ge 3$. The classes of size ≥ 2 of the relation "same word" on realisable vectors are exactly the classes listed above, and they are pairwise disjoint.*

($\Leftarrow$) is proved by hand (above); ($\Rightarrow$) and the disjointness are computer-assisted (§9.2), uniformly in $b \ge 3$. No hand proof of ($\Rightarrow$) is known to us. The scalar invariants that sufficed for $a = 1$ do not separate the words here: no subset of seven natural scalar invariants does for $4 \le b \le 14$, while τ together with the full zero set does for $b \le 16$.

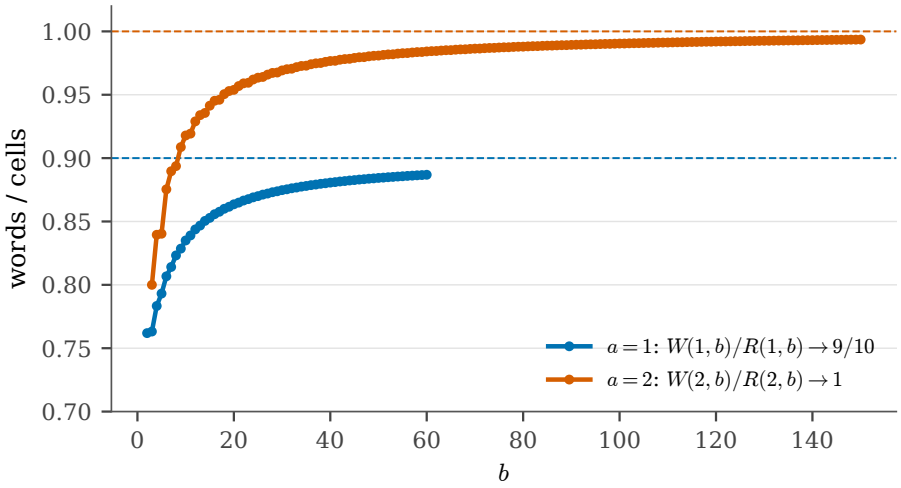


Figure 7. Words per realisable cell, $W(a,b)/R(a,b)$, for $a=1$ ($2 \le b \le 60$) and $a=2$ ($3 \le b \le 150$), from exact enumeration; every value agrees with the closed forms of Lemmas 8.3, 9.2 and Theorems 8.12, 9.4. Live version in the blueprint.

Theorem 9.4 Theorem B; computer-assisted in Lemma 9.3. *For $b \ge 3$,*

$$W(2,b) = \frac{44b^2 + 19b + e_{b \bmod 6}}{6}, \qquad (e_0,\dots,e_5) = (30,33,8,27,36,5).$$

Consequently, for primes $q \ge 5$, with $\left(\frac{q}{3}\right) = \pm 1$ according as $q \equiv \pm 1 \pmod 3$ (the Legendre symbol of q modulo 3, which equals $\left(\frac{-3}{q}\right)$ by quadratic reciprocity),

$$|L_{3q}| = \frac{44q^2 - 69q + 58 - 3\left(\frac{q}{3}\right)}{6} \qquad (q = 5, 7: \ 136, \ 288).$$

Proof. $W(2,b) = R(2,b) - E(b)$, where $E(b)$ is the total excess of the classes of Lemma 9.3. Reading off the class sizes by hand: O contributes $(b + \lceil b/2 \rceil) + 3b$, A contributes $\lfloor b/2 \rfloor$, B contributes $3[b \equiv 2\ (3)]$, C contributes 2, D contributes $2\lfloor (b-2)/3 \rfloor$, E contributes $\lfloor (b-3)/2 \rfloor$ and F contributes b. Summing,

$$E(b) = 6b + \lceil b/2 \rceil + 2\lfloor (b-2)/3 \rfloor + 3[b \equiv 2\ (3)].$$

Subtracting from Lemma 9.2, one residue class mod 6 at a time, gives the formula. For $|L_{3q}|$, take $b = q - 1 \equiv 0, 4 \pmod 6$ in Corollary 3.3(b). This gives

$$|L_{3q}| = \begin{cases} (44q^2 - 69q + 55)/6, & q \equiv 1 \pmod 6, \\ (44q^2 - 69q + 61)/6, & q \equiv 5 \pmod 6, \end{cases}$$

which is the stated formula, since $\left(\frac{q}{3}\right) = 1$ for $q \equiv 1 \pmod 6$ and $\left(\frac{q}{3}\right) = -1$ for $q \equiv 5 \pmod 6$. (For $q = 3$, $W(2,2) = 28$, whereas the formula for $W(2,b)$ would give 37.) □

Remark 9.5 (how many cells are redundant). By Lemma 8.3 and Theorem 8.12, $W(1,b)/R(1,b) \to (9/4)/(5/2) = 9/10$: asymptotically one cell in ten repeats a word. By Lemma 9.2 and Theorem 9.4, $W(2,b)/R(2,b) \to (44/6)/(22/3) = 1$: the coincidences for $a=2$ are only linearly many in b (about $43b/6$), whereas for $a=1$ they are quadratically many (about $b^2/4$). Figure 7 shows the two ratios.

9.2. The computer-assisted step. *Exact encoding.* Let $M_c(z) = \sum_{i,j} z^{H_i(j)}$. It determines the word, and so does $K_c = (1 - z^6)M_c$, because $1 - z^6$ is not a zero divisor in $\mathbb{Z}[z]$. Fix a *branch*: a

category for each coordinate (clipped below, negative, nonnegative, clipped above) and an order of the break points of each row. On each piece of such a row, H_i is an arithmetic progression with common difference $\Delta \in \{0,1,2,3\}$. If $\Delta \in \{1,2,3\}$, then $\Delta \mid 6$ and the contribution of a progression $u, u+\Delta, \dots, u+(N-1)\Delta$ to K_c telescopes to a signed sum of at most $2 \cdot 6/\Delta$ monomials. If $\Delta = 0$ (a constant piece with N equal values u), its contribution is $Nz^u - Nz^{u+6}$. In both cases the exponents and coefficients of these "events" are affine in (b, c). Realisability is linear by Fourier–Motzkin. There are 69 syntactic branches, 31 of them feasible for $b \geq 3$.

Queries. For each pair of feasible branches $i \leq j$ (496 pairs), with $b \geq 3$ and $c, d \in \mathbb{Z}^5$ free, Z3 4.16.0 decided the quantifier-free linear integer formula "c, d realisable $\wedge\ K_c = K_d \wedge \kappa(c) \neq \kappa(d)$". Here κ is the canonical label of the full predicted relation. All **496 queries returned UNSAT**, none unknown, in 608 s total. The reverse query (κ equal, K different) was also UNSAT in all 496 cases, which re-checks ($\Leftarrow$). Disjointness: each of the 231 pairs of label cases with different tuples was UNSAT.

Validation and controls.

- Validation: for $b \leq 12$, the evaluated events equal $(1 - z^6) \cdot$ (real word) in all 6091 (branch, vector) cases, and the branch constraints equal realisability.
- Controls: removing any family (A, B, D, E, F) from κ makes the query SAT, each time with a genuine collision checked against the real word (for instance $b = 159\,827$ for B). The same encoding for $a = 1$ with labels O, Q gives 55/55 UNSAT, and dropping Q gives SAT on real collisions.

What this step trusts. It is a computer-assisted proof using exact SMT queries, not a checked certificate:

(i) the correctness of Z3 4.16.0; no proof object was produced or checked. A second solver, cvc5 1.4.2, agrees on all $496 + 496 + 231$ queries (all unsatisfiable); on a sample of 20 queries it also produced proofs and checked them internally. No proof was checked by an independent checker. The 38 syntactic branches discarded as infeasible rest on Z3 alone;

(ii) the hand derivation of the event formulas and of branch coverage, validated numerically for $b \leq 12$; that the fibres of the label κ are exactly the listed classes is argued by hand and checked numerically for $b \leq 70$ (supplement, §6);

(iii) Lemma 9.1.

The 496 queries are stored in SMT-LIB2 (26 MB compressed); a re-read of one in 25 from file gave 20/20 UNSAT. They are deposited with this note (file `smt2_k3q_a2_directa.tar.gz`).

10. The values of $|L_k|$ for small k

Table 1 lists $|L_k|$ for $k \leq 18$ and for the two-prime values $k = 21, 25, 35, 49$, with the finest pattern of Theorem 3.1 and the status of each value. For one prime the value is k (Corollary 3.3(a)); for two primes it comes from Theorems A and B or from an exact enumeration of cells; for three or more primes only exact enumerations are available.

The enumerations for three or more primes are exact because of Proposition 4.1. For the maximal patterns of $k = 8, 12, 16, 18$, namely $(1,1,1)$, $(2,1,1)$, $(1,1,1,1)$ and $(2,2,1)$, the bound $r!\,M^{r-1}+1$ equals 7, 25, 25 and 25. Every cell therefore meets the box $x_i \leq 50$ used in Table 1, and the four enumerations for $k = 8, 12, 16, 18$ are exact without restriction. Since cells and the box are convex, each cell meets the box in one piece.

Remark 10.1. Earlier counts posted in a public online discussion, 27 for $k = 9$ and 42 for $k = 10$, came from a search over $m \leq 10^5$ and are only lower bounds. The exact values are 28 and 47.

k	$\lvert L_k\rvert$	finest pattern	k	$\lvert L_k\rvert$	finest pattern
1	1	1	12	252	$p^2p'p''$
2	2	p	13	13	p^{12}
3	3	p^2	14	96	pp'^6
4	6	pp'	15	136	$p^2p'^4$
5	5	p^4	16	1083	$pp'p''p'''$
6	16	pp'^2	17	17	p^{16}
7	7	p^6	18	1222	$p^2p'^2p''$
8	40	$pp'p''$	21	288	$p^2p'^6$
9	28	$p^2p'^2$	25	255	$p^4p'^4$
10	47	pp'^4	35	971	$p^4p'^6$
11	11	p^{10}	49	1083	$p^6p'^6$

Table 1. $|L_k|$ for small k. Every value is exact; the colour gives how it is established: written proof (Corollary 3.3(a) for k prime or 1; Theorem A for $k = 2q$), Theorem B (computer-assisted step), exact enumeration in rational arithmetic. Two-prime values are exact enumerations of the cells of the arrangement of §2, and agree with Theorems A and B where those apply. The values for $k = 8, 12, 16, 18$ were enumerated exactly with SageMath polyhedra over $\mathbb{Q}$ in the box $x_i \le 50$, which meets every cell (Proposition 4.1). For $k = 16, 18$ they agree with an earlier floating-point enumeration.

Questions and scope

11. Open questions

The questions are ordered from the most structural to the most specific. For each we state what is known, what the data say, and what an answer would give.

Rigidity: where do the coincidences live? Let $E(a,b) = R(a,b) - W(a,b)$ be the number of cells lost to coincidences. For $a = 1$ it is quadratic, $E(1,b) \sim b^2/4$, and the loss comes from one mechanism, the family Q. For $a = 2$ it is linear, $E(2,b) \sim \frac{43}{6}b$ (Theorem B). For $a = 3$ the exact counts for $b \le 110$ give $E(3,b) \approx 10.15\,b$, again linear. The mechanisms of Section 7 suggest where the lost cells lie. The fibre of $O(0)$ is the strip $\Lambda < g(\theta)$ along the boundary $\Lambda = 0$ (Lemma 7.1), and the reflection pairs sit next to the lines $\theta = m/q$, with Λ small (Lemmas 7.5 and 7.7); each of these families has $O_a(b)$ members. The only mechanism we know whose domain has two free parameters for each b is the exchange of progressions, which is proved here for $a = 1$ only (Remark after Corollary 7.3).

Question 11.1. Is $E(a,b) = \Theta_a(b)$ for every fixed $a \ge 2$? Does almost every cell of the arrangement give its own word once the pattern has three rows, that is, $W(a,b)/R(a,b) \to 1$? Equivalently for the first question, by Corollary 5.2 and the linear lower bound for $a \ge 2$ after Lemma 7.5: is $W(a,b) = C_a b^2 + O_a(b)$? Where in the arrangement do the cells that share their word lie?

A positive answer would say that, beyond one exceptional row count, the word of n almost determines the relative position of $\log r$, $\log s$ and $\log t$ up to the cell. Then the leading term of $W(a,b)$ would be a purely geometric quantity (next question). By Theorem 6.1 for $W(a,b)$ and Corollary 5.2(a) for $R(a,b)$, the difference $E(a,b)$ is eventually a quasipolynomial in b of degree at most two (with period the least common multiple of the two periods). So the question asks

whether the quadratic coefficient vanishes on every residue class, which is a finite question for each fixed a. In particular the two questions are one: since $R(a,b) \sim C_a b^2$ with $C_a > 0$, $W/R \to 1$ means $E = o(b^2)$; for an eventual quasipolynomial of degree at most two this forces every quadratic coefficient to vanish, so $E = O_a(b)$; with the linear lower bound for $a \geq 2$ after Lemma 7.5, $W(a,b)/R(a,b) \to 1$ if and only if $E(a,b) = \Theta_a(b)$. This question is the subject of a companion paper in preparation, which treats the rigidity of the generic cells by finite affine certificates.

Leading coefficient. By Corollary 6.4 the leading coefficient of every constituent of $W(a,b)$ lies in $[\frac{2}{3}, C_a]$. A positive answer to Question 11.1 would give $W(a,b) \sim C_a b^2$ for every $a \geq 2$; for $a = 2$ this is Theorem B, and for $a = 3$ the conjectured coefficient $73/5$ of Question 11.3 is C_3.

Question 11.2. Does $\lim_b W(a,b)/b^2$ exist for every a, that is, is the leading coefficient of $W(a,b)$ the same on all residue classes? Is it C_a for every $a \geq 2$?

The case $a = 3$. The exact values of $W(3,b)$ suggest a closed form.

Question 11.3 (conjecture)**.** Find $W(a,b)$ for $a \geq 3$. For $a = 3$ the exact values for $b \leq 110$ fit, for $b \geq 5$, the quasi-polynomial

$$W(3,b) = \tfrac{73}{5} b^2 + \tfrac{253}{30} b + c_{b \bmod 12} + u_{b \bmod 5}$$

with $(c_0, \ldots, c_{11}) = (5, \frac{31}{6}, \frac{16}{3}, -\frac{25}{2}, \frac{20}{3}, \frac{17}{6}, 6, -\frac{65}{6}, \frac{13}{3}, \frac{7}{2}, \frac{23}{3}, -\frac{79}{6})$ and $(u_0, \ldots, u_4) = (0, -\frac{1}{5}, \frac{2}{5}, -\frac{1}{5}, 0)$. The fit used $b \in [6, 66)$ and was validated on $b = 66, \ldots, 110$ (45 points, no failure). The period is 60, with structure "12 plus 5", whereas for $a = 2$ the minimal period is 6. This is a **conjecture**: less than two full periods are observed. The leading coefficients are $9/4$, $22/3$ and $73/5$ for $a = 1, 2, 3$. The last two equal the leading coefficients C_2 and C_3 of the number of cells (Corollary 5.2), as rigidity predicts; $9/4 < C_1 = 5/2$ because of the family Q.

Periods. The minimal periods of $W(a,b)$ are 2 and 6 for $a = 1, 2$, and 60 for $a = 3$ (conjecturally, Question 11.3). All three divide $\operatorname{lcm}(1, 2, \ldots, 2a)$, the common denominator of the abscissae of the vertices of the arrangement.

Question 11.4. Is the minimal period of the quasipolynomial $W(a,b)$ a divisor of $\operatorname{lcm}(1, \ldots, 2a)$? Is it equal to it for $a \geq 3$? Is the threshold N_a linear in a?

A positive answer would make the computation of $W(a,b)$ finite and explicit for each a: the formula would be fixed by $3 \operatorname{lcm}(1, \ldots, 2a)$ exact values beyond the threshold.

Fibres. Write $\mathcal{F}(w)$ for the set of cells with word w. One fibre is always large: the word $O(0) = ()^k$ (each opening immediately followed by a closing) of a point with $\Lambda \to 0$, for which Lemma 7.1 gives $|\mathcal{F}(O(0))| = 1 + \sum_{h=1}^{a} \#\{1 \leq j \leq b : \gcd(j,h) = 1\}$, so the excess is at least linear in b (Section 7). Outside $O(0)$, in all 23 cases with $a \leq 5$, $b \leq 9$ that we enumerated exactly, the largest fibre has 2 cells for $a = 1$, 3 cells for $a = 2$, and 4 cells for $a = 3, 4, 5$.

Question 11.5. Is there an absolute constant bounding $|\mathcal{F}(w)|$ for all $w \neq O(0)$, for every a and b? Is it 4? Classify the coincidence mechanisms. Three are proved in Section 7: the empty graph ($O(0)$, Lemma 7.1), exchange of arithmetic progressions between rows (family Q, Lemma 7.2), and reflection inside integer levels (family A, Lemma 7.5, with the variant of Lemma 7.7). Are there others?

We expect a dichotomy for fixed a: the fibre sizes are either bounded or of order b along an arithmetic progression. This should follow from the Presburger description of Theorem 6.1, but we have not written the argument. A bounded answer, together with a list of mechanisms, would turn Theorems A and B into instances of one classification valid for all a, and would replace the SMT step by a proof.

Several primes. By Theorem 3.1, L_{4q}, L_{6q} and L_{8q} are the words of the patterns $(1,1,q-1)$, $(1,2,q-1)$ and $(1,1,1,q-1)$. The proof of Theorem 6.1 uses only that the number of rows is fixed and that each row count is monotone in j, so it should extend to any number of primes.

Question 11.6. For fixed $a_1,\dots,a_{r-1}$, is $W(a_1,\dots,a_{r-1},b)$ eventually a quasipolynomial in b of degree exactly r? Compute $|L_{4q}|$ in closed form. Is there a formula for the square-free pattern $(1,1,\dots,1)$, that is, for $|L_{2^r}|$?

Membership and proof. Corollary 4.2 decides membership, against the undecidability expected in [15]; Rodríguez Caballero's question about first-order Peano arithmetic [14] remains open. The running time of the algorithm is $\exp(O((\log k)^2))$, and polynomial in k only when the number of prime factors of k is bounded (remark after Corollary 4.2): the number of cells grows like a power of the number of hyperplanes, with exponent the number of prime factors of k.

Question 11.7. Is membership in L decidable in time polynomial in k? Can the algorithm above be carried out inside first-order Peano arithmetic, which would answer Rodríguez Caballero's question [14]? Is there an intrinsic description of L_{pq}, for instance a finite list of forbidden factors, or a characterisation of the unit interval graphs on a product of two chains that admit an additive representation with arithmetic progressions in each factor?

Frequencies. The word of n is a function of the point $x(n) = (\log p_i/\log t)_i$. Counting $n \le N$ instead of words leads to analytic questions in the spirit of Hooley's Δ-function.

Question 11.8. For a fixed pattern f and a word $w \in L$, what is the asymptotic proportion of $n \le N$ with odd part of pattern f and $\langle\!\langle n \rangle\!\rangle_2 = w$? Is it governed by a natural measure of the cells of $\mathcal{A}_f$ in the coordinates $\log p_i/\log t$, and which word is the most frequent?

This would connect the combinatorics of the cells with the distribution of the prime factors of n. The word is then not only an invariant that n has, but a statistic that can be predicted.

12. What this note does not claim

- It does not describe L or L_k for any k with three or more prime factors. The values for $k = 8, 12, 16, 18$ in Table 1 are exact enumerations, not formulas.
- Theorem B is not proved by hand: Lemma 9.3 is computer-assisted, and what it trusts is listed in §9.2.
- For two primes we count L_{pq}; we describe it only for $p \in \{2,3\}$, and then through the class lists rather than by an intrinsic property of words.
- The novelty claim is limited to the counts of words, to Theorem 3.1 and its consequence Corollary 4.2, to Theorem 6.1 as stated (an application of Woods' theorem) with Theorem 6.3 and Corollary 6.4, and to the mechanisms of Section 7.
- It claims no novelty for the number of cells. By Proposition 5.3, $R(a,b)$ is, after an affine rescaling, the number of two-dimensional threshold functions on a grid, whose exact formula and leading term are Alekseyev's [16], and whose count of linear partitions goes back to Acketa and Žunić [17]. Theorem 5.1 and Corollary 5.2 are a direct proof of that count in our coordinates. Counting cells of line arrangements through totient sums is classical in general [17, 16, 21, 22].
- We did not find the rows or diagonals of $W(a,b)$ in the OEIS (searched 25 Nov 2026). The connection with the Kassel–Reutenauer polynomials is due to Caballero [9, 8].

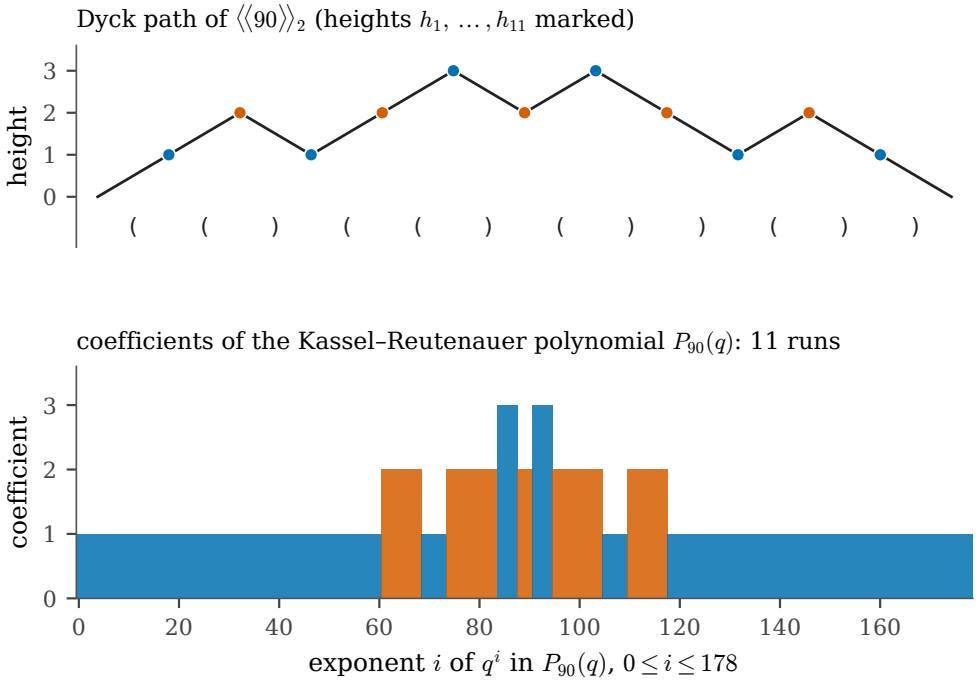


Figure 8. Top: the Dyck path of $\langle\langle 90\rangle\rangle_2 = (()(()()))()$ with the heights $h_1, \ldots, h_{11}$ marked alternately in blue and orange. Bottom: the 179 coefficients of $P_{90}(q)$ (from [4, Theorem 1.2]); the 11 runs carry the same colours and the values $h_1, \ldots, h_{11}$.

Appendix A. A reformulation through the Kassel–Reutenauer polynomials

Kassel and Reutenauer [4] study $C_n(q)$, the number of ideals of codimension n of $\mathbb{F}_q[x^{\pm 1}, y^{\pm 1}]$, and $P_n(q) = C_n(q)/(q-1)^2$. The latter is a palindromic polynomial of degree $2n-2$; write it as $P_n(q) = a_{n,0}q^{n-1} + \sum_{i=1}^{n-1} a_{n,i}(q^{n-1+i} + q^{n-1-i})$. Their identity [4, (3.2)] reads

$$\frac{P_n(q)}{q^{n-1}} = \sum_{\substack{1 \le k < m',\ km' = 2n \\ k \not\equiv m' \pmod 2}} (-1)^{k-1} [\![-\tfrac{m'-k-1}{2}, \tfrac{m'-k-1}{2}]\!], \tag{4}$$

where $[\![a, b]\!] = q^a + \cdots + q^b$. (In this section q is the variable of P_n, not a prime.) Rodríguez Caballero showed that the largest coefficient of $P_n(q)$ is the function $F(n)$ of Erdős and Nicolas [9], and that, for $\lambda = 2$, $\langle\langle n\rangle\rangle_2$ encodes the nonzero coefficients of $C_n(q)$ [8, Introduction]; see also [7]. The statement below makes the correspondence between the coefficient sequence and the Dyck path explicit (Figure 8). It is implicit in [9, 8]; we include a short proof for completeness and claim no novelty.

Observation A.1. *Let $n \ge 1$ and let $h_1, \ldots, h_{2K-1}$ be the heights of the Dyck path $\langle\langle n\rangle\rangle_2$ after its first $1, \ldots, 2K-1$ letters, where $K = \tau(\text{odd part of } n)$. Then the coefficient sequence of $P_n(q)$, from q^0 to q^{2n-2}, consists of a nonempty run of h_1, then a nonempty run of h_2, and so on up to a nonempty run of h_{2K-1}. More precisely, $a_{n,i} = h_{J(i)}$, where $J(i) = \#\{k \in D_n \triangle 2D_n : k < \sqrt{2n},\ 2n/k - k \ge 2i+1\}$.*

Proof. The pairs (k, m') in (4) are the factorisations $2n = km'$, $k < m'$, into factors of opposite parity. Let $n = 2^{e-1}m$. The odd factor o divides m and the even factor is $2n/o = t \cdot (m/o)$, so the pairs are $\{o, tm/o\}$ with $o \in D_m$. Their members are exactly $D_m \cup tD_m = D_n \triangle 2D_n$

(Lemma 2.1), and each element lies in exactly one pair. As $km' = 2n$ with $k \neq m'$, the smaller members $k_1 < \cdots < k_K$ are the K elements below $\sqrt{2n}$, that is, the first K letters of $\langle\langle n \rangle\rangle_2$. Put $\epsilon_j = (-1)^{k_j - 1}$; it is $+1$ if k_j is odd, i.e. the j-th letter is (, and -1 otherwise. So $h_j = \epsilon_1 + \cdots + \epsilon_j$ for $j \leq K$.

Comparing coefficients of q^i in (4) gives $a_{n,i} = \sum_{j:\, d_j \geq 2i+1} \epsilon_j$, where $d_j = m'_j - k_j = 2n/k_j - k_j$. The d_j are odd and strictly decreasing, so $\{j : d_j \geq 2i+1\} = \{1, \ldots, J(i)\}$ and $a_{n,i} = h_{J(i)}$. Here $d_1 = 2n - 1$ (from $k_1 = 1$), so $J(i) \geq 1$ for $i \leq n-1$, and $J(0) = K$. The function J takes the value $j \in [1, K]$ exactly for $i \in [(d_{j+1}+1)/2, (d_j - 1)/2]$ (with $d_{K+1} = -1$). This interval is nonempty because $d_j \geq d_{j+1} + 2$. Reading $i = n-1, \ldots, 0$ gives runs of $h_1, \ldots, h_K$. Palindromy of P_n and the symmetry $h_{2K-j} = h_j$ of the symmetric Dyck word [8, Lemma 13] give the second half. □

Example A.2. For $n = 3$: $D_3 \triangle 2D_3 = \{1, 2, 3, 6\}$ and $\langle\langle 3 \rangle\rangle_2 = ()()$, with heights $h_1, h_2, h_3 = 1, 0, 1$. Here $C_3(q)/(q-1)^2 = P_3(q) = 1 + q + q^3 + q^4$, whose coefficient sequence $1, 1, 0, 1, 1$ consists of runs of 1, 0, 1. The zero coefficient is part of the profile and must not be dropped.

The first half $(h_1, \ldots, h_K)$ of the heights also appears in the OEIS: it is row n of sequence A249223 (the widths of the symmetric representation of $\sigma(n)$) after merging equal neighbours [24].

Consecutive heights differ by ± 1, so merging equal neighbours in the coefficient sequence recovers $(h_1, \ldots, h_{2K-1})$ exactly, and therefore $\langle\langle n \rangle\rangle_2$. Figure 8 shows $n = 90$. Hence:

Corollary A.3. *L_k is in bijection with the set of compressed coefficient profiles of the polynomials $P_n(q)$ with τ(odd part of n) $= k$.*

The companion site

This paper has a companion website, its *blueprint*, at `https://karlesmarin.github.io/blueprints/dyck-divisors/`. It shows the paper as a whole (what it proves, how its results depend on each other, and what exactly has been checked) and lets the reader go from any statement to its proof, its dependencies and its checks. There one finds:

- *The statements.* Every numbered statement of the paper, copied verbatim with its number (6 theorems, 2 propositions, 20 lemmas, 7 corollaries, 2 definitions, 11 remarks, 8 questions, 1 observation and 1 example, in 12 sections), each with its proof, where it has one, a click away.
- *The dependencies.* For each statement, what it uses and where it is used, as a list and as a graph of the whole paper: one can follow a theorem down to the lemmas it rests on, or a lemma up to what it serves.
- *The map.* One page that places the main results, the tools they rest on (the branches of mathematics they borrow from) and the fields they feed.
- *What was checked, and how.* Besides the written proof, the site says for each statement what else was checked: 7 statements have named pieces machine-checked in Isabelle/HOL (23 theorems, linked by name to the public repository karlesmarin/formalizations; the site says, statement by statement, what they cover); 25 statements are corroborated by programs of the supplementary material, each named with what it computes.
- *Labs.* Next to Lemma 2.1, Theorem 5.1 and Lemma 7.1, a form runs the paper's own computations in the reader's browser (plain JavaScript, nothing to install) for values the reader chooses, and shows the exact results.
- *The paper itself.* Its PDF, opened at the right page from every statement.

To read it, one can start from the map, or from a main theorem and follow its dependencies down; the side panel lists the sections of the paper, and every statement has its own link. The site is a reading aid, not a part of the proofs: the statements and proofs are those of this paper, whose

text prevails, and a formal or computational check is reported for exactly the part it covers. It is generated from the source of the latest version of the paper (here version 2).

Versions

Version 2 corrects the novelty claim for the cell count $R(a,b)$: it equals $(N(2b,2a)+4a+4b+6)/8$, where $N(m,n)$ is the number of two-dimensional threshold functions on a grid of $(m+1)\times(n+1)$ points (Alekseyev [16]; Acketa–Žunić [17]; see also Žunić [18]), which I had overlooked. It adds Proposition 5.3 and Remarks 5.4 and 5.5; Theorem 5.1 and Corollary 5.2 are kept as a direct proof of a known count, and the abstract, the introduction, the box on the first page, Figure 4(b) and Section 12 say so. It also notes, after Question 11.1, that its two questions are equivalent, and words Steps 3 and 5 of Section 6 more precisely (half-open intervals; the degree of the numerator), with the same argument. All other statements, proofs and numbers are those of version 1 (doi:10.5281/zenodo.24536373).

Acknowledgements

This note grew out of questions posed by José Manuel Rodríguez Caballero on Mathstodon in November 2026. I thank him for sharing the problem, for his explanations and references, for turning the decidability argument into a Python implementation in that discussion, and for encouraging me to pursue it.

Use of AI. I worked with an AI assistant.

Verification and data. Every result with a written proof was also tested numerically against exact enumerations of the cells; the one computer-assisted step is described in §9.2. Every figure is produced by a script that recomputes its data and stops if a value disagrees with the text. The checks are these.

- Figure 1: the word equals the word computed from the definition.
- Figure 2: the word read from the coordinates agrees with the definition for 300 integers $n = 2^{e-1}p_1p_2p_3$. An incremental enumeration of the cells finds 181 cells and exactly the 40 words of L_8.
- Figure 3: for all 18 098 points with $r,s \le 200$ and $e \le 11$ in the window, the word of $n = 2^{e-1}rs^2$ is the word of its cell.
- Figure 8: the identity $a_{90,0} + 2\sum_{i\ge1} a_{90,i} = \sigma(90)$ holds, and the runs of the coefficients match the heights of the word.

The scripts, the exact enumerations, the generator of the SMT-LIB2 queries and the recorded logs of the full runs are deposited with this note (doi:10.5281/zenodo.24541288): Supplement B and its package of scripts.

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