How a Tree Forgets Its Order

The Eulerian idempotent and the Adams spectrum on the commutative Connes–Kreimer–Butcher Hopf algebra of rooted trees, machine-checked in Lean 4

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Abstract

A companion paper built the *planar* (Foissy, noncommutative) Connes–Kreimer / Butcher Hopf algebra of rooted trees in Lean 4 and left two things open: the *commutative* (classical Butcher / Connes–Kreimer) quotient, and the spectral theory of its operators. This paper closes both, `sorry`-free. First, the commutative algebra $\mathcal{H}_{\mathrm{ab}} = \mathcal{H}/(ab - ba)$ is built as a Mathlib `HopfAlgebra` instance, by descending the planar structure through Mathlib's `RingQuot`; we are unaware of a previous machine-checked commutative renormalization Hopf algebra of rooted trees. Second, we run the Adams ladder $\Psi_n = \mathrm{id}^{\star n}$—which identifies the antipode as $\Psi_{-1} = S$ and acts by $\Psi_n(x) = n\,x$ on primitives—and prove its spectral payoff: the *first Eulerian idempotent* $e^{(1)} = \log_\star(\mathrm{id})$ is *idempotent*,

$$e^{(1)} \circ e^{(1)} = e^{(1)} \quad \text{on } \mathcal{H}_{\mathrm{ab}}.$$

This is exactly the property that *fails* on the planar algebra—we formalized that counterexample too (the cherry)—so commutativity is not a convenience here but the hypothesis that makes the theorem true. The proof avoids the Vandermonde inversion of the textbook argument: it routes through a small piece of *formal logarithm calculus* on power series ($\log(fg) = \log f + \log g$, $\log(f^n) = n \log f$), proved by antiderivative uniqueness with no analysis, and transported into the convolution algebra by an eigenvalue identity $\Psi^i \circ e^{(1)} = i \cdot e^{(1)}$ and a finite-difference collapse. The main technical by-product is independent of trees: a formal proof of logarithmic additivity and the power law for power series over a $\mathbb{Q}$-algebra ($\log(fg) = \log f + \log g$, $\log(f^n) = n \log f$), currently absent from Mathlib, obtained by uniqueness of formal antiderivatives. As with the companion, the mathematics is classical—Adams, Solomon, Loday–Reutenauer–Patras, Aguiar–Lauve; the contribution is the certified instance, the formalized planar counterexample that pins commutativity as the operative hypothesis, and the formal-logarithm lemmas.

Reader's map. This is a sequel to "How a Tree Remembers Its Cuts" [14], which built the planar Hopf algebra $\mathcal{H}$ and is assumed only as a black box: $\mathcal{H}$ is a Mathlib `HopfAlgebra` of rooted forests. If you read only two things here, read Theorem 1 (the commutative quotient $\mathcal{H}_{\mathrm{ab}}$ is a Hopf algebra) and Theorem 2 (the Eulerian idempotent is idempotent on it). Section 2 is the spectral setup—the Adams ladder, $\Psi_{-1} = S$, and the eigenvalue on primitives—and ends with the *planar obstruction* that forces the quotient. Section 3 builds the quotient; Section 4 proves $e^{(1)} \circ e^{(1)} = e^{(1)}$. The load-bearing formulas are framed; the honest limits (Section 6) are stated as plainly as the results.[1]

[1] *Reproducibility.* The Lean source is the single self-contained file `ConnesKreimer.lean`, built against Lean `v4.30.0-rc2` and the pinned Mathlib commit `701fb6e9c3`. Reproduce with `lake exe cache get && lake env lean ConnesKreimer.lean`; the file ends with `#print axioms` for the whole chain through `eulerian1_idem_ab`, each reporting `[propext, Classical.choice, Quot.sound]` and no `sorryAx`. The numerical identities (idempotency, the

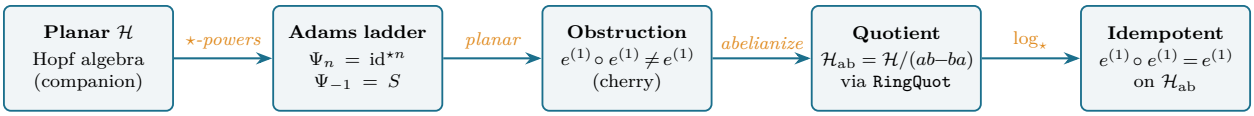


Figure 1: The spine of this paper. The planar Hopf algebra of the companion carries an Adams ladder whose first Eulerian idempotent is *not* idempotent (the cherry counterexample). Abelianizing—letting the tree forget the order of its branches—produces the commutative quotient $\mathcal{H}_{\mathrm{ab}}$, on which the idempotency $e^{(1)} \circ e^{(1)} = e^{(1)}$ becomes a theorem. Each arrow is a step formalized in Lean; $\mathcal{H}$, $\mathcal{H}_{\mathrm{ab}}$ are Mathlib Hopf-algebra instances.

1 Introduction: the spectrum a planar tree cannot have

The companion paper [14] formalized the Connes–Kreimer / Butcher / Foissy Hopf algebra of rooted trees in its *planar* (noncommutative) form: trees with ordered children, the admissible-cut coproduct, coassociativity, the disjoint-union product, and a connected-graded antipode, all assembled into a Mathlib `HopfAlgebra` instance and depending only on the three standard axioms of classical Mathlib. That paper was explicit about two things it did not do. It did not build the *commutative* algebra—the classical Butcher / Connes–Kreimer quotient by reordering children—and it did not develop the spectral theory of the algebra's natural operators. This paper does both, and shows they are the same task: the spectral theory is exactly what the quotient is *for*.

The operators are the *Adams operators* $\Psi_n = \mathrm{id}^{\star n}$, the convolution powers of the identity. They have been circulating through topology, numerical analysis and quantum field theory since the 1960s, and on any Hopf algebra they organize the antipode and the grading into a single $\mathbb{Z}$-indexed family. On the planar algebra we can already run the ladder—it puts the antipode at $\Psi_{-1} = S$ and gives the eigenvalue $\Psi_n(x) = n\,x$ on primitive elements (Section 2)—but the ladder's spectral payoff, the *Eulerian idempotent* $e^{(1)} = \log_\star(\mathrm{id})$ that should split the algebra into Adams eigenspaces, runs into a wall. On the planar algebra $e^{(1)}$ is not idempotent: $e^{(1)} \circ e^{(1)} \neq e^{(1)}$, with the smallest witness a root carrying two leaves (the *cherry*). We formalized this failure as a result in its own right. It is not a defect of the construction; it is the precise mark left by planarity, and it has a textbook cause: the orthogonality of the Eulerian idempotents needs commutativity (or cocommutativity), and the planar algebra has neither.

The failure of idempotency on $\mathcal{H}$ disappears after passing to the abelianization $\mathcal{H}_{\mathrm{ab}} = \mathcal{H}/(ab{-}ba)$, the classical commutative Connes–Kreimer / Butcher algebra: the obstruction is exactly the noncommutativity, and the quotient removes it. The contribution of this paper is to make this precise in Lean, in two steps:

- **The quotient is a Hopf algebra** (Section 3). We build $\mathcal{H}_{\mathrm{ab}}$ by descending the entire planar Hopf structure—product, coproduct, counit, antipode—through Mathlib's `RingQuot` by the commutator relation, and register $\mathcal{H}_{\mathrm{ab}}$ as a Mathlib `HopfAlgebra` instance (`instHabHopf`); we are unaware of a previous commutative combinatorial renormalization Hopf algebra of rooted trees in a proof assistant (Section 8).
- **The idempotent is idempotent** (Section 4). On $\mathcal{H}_{\mathrm{ab}}$ we prove

eigen-relation, the log power law) were additionally cross-checked in exact rational arithmetic before they were proved. A large language model was used as a coding assistant in the Lean development; every statement was independently checked by the Lean kernel, and all mathematical decisions and claims are the author's.

$$e^{(1)} \circ e^{(1)} \;=\; e^{(1)} \qquad (\texttt{eulerian1_idem_ab}).$$

The proof avoids the Vandermonde inversion of the classical argument. It routes through a small piece of *formal logarithm calculus* on power series—additivity $\log(fg) = \log f + \log g$ and the power law $\log(f^n) = n \log f$, proved by antiderivative uniqueness, no analysis—transported into the convolution algebra by an eigenvalue identity $\Psi^i \circ e^{(1)} = i \cdot e^{(1)}$ and closed by a finite-difference collapse.

None of the mathematics is new: the Adams operators are Adams [1] and, on Hopf algebras, Aguiar–Lauve [2]; the Eulerian idempotents are Euler through Solomon [18] and the free-Lie school of Loday–Reutenauer–Patras [10, 17, 16]; the commutative tree algebra is Butcher [4] and Connes–Kreimer [5]. The contribution is the certified instance, and the discipline of stating exactly which hypothesis (commutativity) makes the headline true—demonstrated by formalizing both the theorem and its planar failure. On the formalization side, the piece least specific to trees is a formal proof of logarithmic additivity for power series over a $\mathbb{Q}$-algebra, missing from Mathlib, which the proof needs and which we prove separately. More than the lemma in isolation, the reusable element is the route by which it reaches the operators—a formal power-series identity, polynomial evaluation at the generator J_{ab}, local nilpotency to make the truncation exact, then the Adams eigenvalue; the idempotency $e^{(1)} \circ e^{(1)} = e^{(1)}$ is the consequence this transport delivers (Section 4).

2 The antipode is an Adams operator

The companion's Hopf structure singled out one extra map, the antipode S, as the datum distinguishing a bialgebra from a Hopf algebra. We show that S is the value at $n = -1$ of a single $\mathbb{Z}$-indexed family of operators $\Psi_n = \mathrm{id}^{\star n}$, and we run that family far enough to reach its first spectral payoff and the obstruction to the rest. The exposition is a chain of facts, each forced by the one before (Table 1); everything in this section is formalized on the planar algebra $\mathcal{H}$, where every Ψ_n is an explicit operation on forests. That concreteness also exposes the limit the rest of the paper removes: planarity obstructs the global Eulerian diagonalization, which we record as an explicit counterexample.

Fact 0. A Hopf algebra carries an antipode

The antipode is the linear endomorphism $S\colon \mathcal{H} \to \mathcal{H}$ satisfying the two convolution identities

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta \;=\; u \circ \varepsilon \;=\; \mu \circ (\mathrm{id} \otimes S) \circ \Delta.$$

Heinz Hopf introduced the algebraic skeleton of this notion in 1941 [8]; the antipode and the systematic theory are due to Milnor–Moore [15] and Sweedler [19]. In our setting these identities are theorems of the companion (`leftAntipode_eq`, `rightAntipode'_eq`, packaged as `instCKHopf`). Read as written, they say "S cancels the identity." The next fact makes "cancel" mean "invert."

Fact 1. The endomorphisms form a ring under convolution

For linear maps $f, g\colon \mathcal{H} \to \mathcal{H}$ the *convolution* product

$$f \star g \;:=\; \mu \circ (f \otimes g) \circ \Delta$$

is associative, with two-sided unit $u \circ \varepsilon$. Thus $\mathrm{End}_k(\mathcal{H})$ is an associative k-algebra. This is exactly Mathlib's convolution ring `LinearMap.convRing` on `WithConv`; the construction is classical [19]. Nothing about trees is used here—only that $\mathcal{H}$ is a bialgebra. (We flag now what becomes decisive in Section 4: when the carrier is *commutative*, this convolution ring is *commutative*—Mathlib's `convCommRing`—and that is the structural difference between $\mathcal{H}$ and $\mathcal{H}_{\mathrm{ab}}$.)

Fact 2 (forced by 0 and 1). The antipode is the convolution inverse of the identity

Facts 0 and 1 are the same equations in two alphabets. Inside the ring of Fact 1, the antipode identities read

$$S \star \mathrm{id} \;=\; 1 \;=\; \mathrm{id} \star S,$$

so S is the two-sided inverse of id. Having an element and its inverse in a ring, one is obliged to ask for its powers.

Fact 3 (forced by 1). The identity has convolution powers—the Adams operators

An element of a ring has powers, so the identity has convolution powers

$$\Psi_n \;:=\; \mathrm{id}^{\star n} \qquad (n \in \mathbb{N}).$$

These are the *Adams operators*. The name records a second lineage: Frank Adams introduced operations ψ^k on topological K-theory in 1962 [1]; on a Hopf algebra the same operators arise as convolution powers of the identity [2]. Their small values are familiar objects wearing an index (`adams`, `toConv_adams_zero`, `adams_one`, `toConv_adams_add`):

$$\Psi_0 = u \circ \varepsilon, \qquad \Psi_1 = \mathrm{id}, \qquad \Psi_m \star \Psi_n = \Psi_{m+n}.$$

The counit-unit and the identity are the $n = 0$ and $n = 1$ rungs of one ladder, and $n \mapsto \Psi_n$ is a homomorphism of monoids $(\mathbb{N}, +) \to (\mathrm{End}_k(\mathcal{H}), \star)$.

Fact 4 (forced by 2 and 3). The antipode is Ψ_{-1}

To descend one rung below $\Psi_1 = \mathrm{id}$ is to invert id in the convolution ring—and Fact 2 has named that inverse. Therefore

$$\Psi_{-1} \;=\; S.$$

The antipode, the datum that distinguished a Hopf algebra from a bialgebra, is the $n = -1$ member of a family whose $n = 0, 1$ members are the counit and the identity (`adamsUnit` exhibits Ψ_1 as a unit, `adamsUnit_inv_val` identifies its inverse with S). *Being a Hopf algebra is precisely the statement that the rung Ψ_1 is invertible.*

Step 5. The eigenvalue on primitives

A distinguished element of a ring, with all its powers, asks for its spectrum. Write $J := \mathrm{id} - u \circ \varepsilon = \mathrm{id} - \Psi_0$; on a basis forest J keeps the nonempty cuts and kills the empty one. The decisive structural fact is *local nilpotency*: in degree m the convolution powers of J run out,

$$\boxed{J^{\star k}\big|_{\mathcal{H}_m} = 0 \quad \text{for } k > m,}$$

because each of the k convolution factors must absorb at least one node (Lean: `Jc_pow_ofConv_sf_eq_zero`). A *primitive* x (one with $\Delta x = x \otimes 1 + 1 \otimes x$) automatically has $\varepsilon(x) = 0$ (`IsPrimitive.counit_zero`), so $J(x) = x$ and $J^{\star k}(x) = 0$ for $k \geq 2$; feeding this into $\Psi_{n+1} = \mathrm{id} \star \Psi_n$ gives the eigenvalue

$$\boxed{\Psi_n(x) = n \cdot x \qquad (x \text{ primitive})}$$

(Lean: `adams_primitive`). The primitives are the eigenspace of every Adams operator with eigenvalue n; in particular Ψ_{-1} acts as -1, recovering $S(x) = -x$.

Step 6. The first Eulerian idempotent, and what planarity breaks

Local nilpotency makes the convolution logarithm

$$\boxed{e^{(1)} \;=\; \log_\star(\mathrm{id}) \;=\; \sum_{k \geq 1} \frac{(-1)^{k+1}}{k}\, J^{\star k}}$$

a well-defined endomorphism of $\mathcal{H}$ (the series is *finite in each degree*), defined over any $\mathbb{Q}$-algebra of coefficients (Lean: `eulerian1`). The series descends from Euler through Solomon's descent algebra [18] and the free-Lie school of Loday [10], Reutenauer [17] and Patras [16]. On a primitive only the $k = 1$ term survives, so $e^{(1)}$ *fixes the primitives*, $e^{(1)}(x) = x$ (Lean: `eulerian1_primitive`).

Here planarity asserts itself, and the honest statement is negative. In the classical (commutative or cocommutative) theory the Eulerian idempotents $e^{(j)}$ are *orthogonal idempotents*, $e^{(i)} \circ e^{(j)} = \delta_{ij}\, e^{(i)}$, diagonalizing every Ψ_n into the Hodge–Eulerian decomposition [16, 2]. The planar (Foissy) algebra $\mathcal{H}$ is *neither* commutative *nor* cocommutative, and the first instance of the property already fails: $e^{(1)}$ is *not* idempotent. The smallest witness is the cherry (a root with two leaves), where $e^{(1)} \circ e^{(1)}$ and $e^{(1)}$ differ precisely in the $\bullet \cdot \ell_2$ versus $\ell_2 \cdot \bullet$ terms—exactly the non-commutativity of planar forests (verified by computation in the artifact, and consistent with idempotency holding on the single generator $\bullet$, `eulerian1_idem_dot`).

This locates the missing theory precisely. The Aguiar–Lauve characteristic polynomial [2] holds at the level of *eigenvalues* in general, but the full *diagonalization*—and already its first instance, the idempotency of $e^{(1)}$—lives on the commutative quotient $\mathcal{H}/(ab - ba)$, the classical Connes–Kreimer / Butcher algebra. Building that quotient, and proving the idempotency on it, is the business of the rest of this paper.

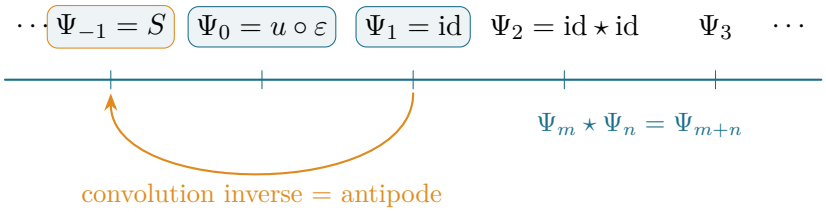


Figure 2: The Adams ladder of $\mathcal{H}$: the counit-unit, the identity and the antipode are the rungs $n = 0, 1, -1$ of the $\mathbb{Z}$-indexed family $\Psi_n = \mathrm{id}^{\star n}$. Descending from Ψ_1 to Ψ_{-1} inverts id in the convolution ring, and that inverse is S (Fact 4).

Fact	Statement	Forced by	Lean certificate
0	S is the two-sided antipode	—	`instCKHopf`
1	$\mathrm{End}_k(\mathcal{H})$ is a ring under $\star$	—	`LinearMap.convRing`
2	$S \star \mathrm{id} = 1 = \mathrm{id} \star S$	$0, 1$	`antipode_convMul_id`
3	$\Psi_n := \mathrm{id}^{\star n}$, $\Psi_m \star \Psi_n = \Psi_{m+n}$	1	`adams`, `toConv_adams_add`
4	$\Psi_{-1} = S$	$2, 3$	`adamsUnit_inv_val`
5	$J^{\star k} = 0$ in degree $< k$ (local nilpotency)	$1, 3$	`Jc_pow_ofConv_sf_eq_zero`
6	$\Psi_n(x) = n\,x$ for primitive x	$3, 5$	`adams_primitive`
7	$e^{(1)} = \log_\star \mathrm{id}$ is an endomorphism	5	`eulerian1`
8	$e^{(1)}(x) = x$ for primitive x	$6, 7$	`eulerian1_primitive`
9	$e^{(1)}$ is *not* idempotent on $\mathcal{H}$ (planar)	—	*computed (cherry)*
10	$e^{(1)} \circ e^{(1)} = e^{(1)}$ on $\mathcal{H}_{\mathrm{ab}}$	§3–4	`eulerian1_idem_ab`

Table 1: The chain of facts. Rows 0–8 hold on the planar algebra $\mathcal{H}$; row 9 is the planar obstruction (the cherry counterexample); row 10, left open in the companion, is *closed* here on the commutative quotient $\mathcal{H}_{\mathrm{ab}}$ (Sections 3–4).

3 The commutative quotient: when a tree forgets its order

The planar algebra distinguishes $\bullet \cdot \ell_2$ from $\ell_2 \cdot \bullet$. How do we make it forget the difference, without rebuilding the Hopf structure from scratch?

The classical Butcher / Connes–Kreimer algebra is the abelianization of the planar one: the quotient that identifies any two forests differing only by the order of their factors. The naive route would rebuild everything—trees, coproduct, coassociativity—on an unordered (`Multiset`) carrier, re-running the hardest proof of the companion. We avoid that entirely. The abelianization is a *quotient of an algebra by a two-sided ideal*, and Mathlib has exactly the tool for descending structure through such a quotient: `RingQuot`. The whole section is the observation that every piece of the Hopf structure *respects the commutator relation*, and so descends.

Definition 1 (The commutative quotient)**.** *Let* commRel *a b be the relation $a = xy$, $b = yx$ (a single commutator swap). Define*

$$\mathcal{H}_{\mathrm{ab}} := \mathrm{RingQuot}\,(\mathrm{commRel}), \qquad \pi \colon \mathcal{H} \to \mathcal{H}_{\mathrm{ab}} \text{ the quotient algebra map.}$$

Then $\mathcal{H}_{\mathrm{ab}}$ is a commutative ring (`instHabComm`), π is a surjective algebra homomorphism (`pi_surjective`), and $\pi(xy) = \pi(yx)$ (`pi_mul_comm`).

The descent is then a fixed pattern, applied to each structure map in turn. For a planar structure map F (the coproduct, the counit, the antipode), one shows F *respects* commRel—that the relation is in the kernel of $\pi \circ F$ read appropriately—and then `RingQuot`'s universal property produces the descended map $\bar{F}$ on $\mathcal{H}_{\mathrm{ab}}$ with $\bar{F} \circ \pi = \pi \circ F$ (or its tensor analogue). The laws on $\mathcal{H}_{\mathrm{ab}}$ are then pulled back through the surjection π:

- **Coproduct** (`Delta_ab`, `Delta_ab_pi`). The planar Δ, read as an algebra map $\mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ followed by $\pi \otimes \pi$, respects the relation (`DeltaAlg_resp_commRel`); the descended $\bar{\Delta}\colon \mathcal{H}_{\mathrm{ab}} \to \mathcal{H}_{\mathrm{ab}} \otimes \mathcal{H}_{\mathrm{ab}}$ satisfies $\bar{\Delta} \circ \pi = (\pi \otimes \pi) \circ \Delta$.
- **Counit** (`eps_ab`, `eps_ab_pi`). ε respects the relation (`epsAlg_resp_commRel`) because $\varepsilon(xy) = \varepsilon(yx)$ in the ground field; the descended $\bar{\varepsilon}\colon \mathcal{H}_{\mathrm{ab}} \to k$ satisfies $\bar{\varepsilon} \circ \pi = \varepsilon$.
- **Coalgebra** (`instHabCoalg`). Coassociativity and the counit laws on $\mathcal{H}_{\mathrm{ab}}$ are the planar ones transported through π, using a linear-extensionality principle on $\mathcal{H}_{\mathrm{ab}}$ (`Hab_lhom_ext`: two maps out of $\mathcal{H}_{\mathrm{ab}}$ agree iff they agree after π) and the naturality of $\bar{\varepsilon}$ against the tensor functors (`coassoc_Hab`).
- **Bialgebra** (`instHabBialg`). The four bialgebra laws are `map_one` and `map_mul` of the algebra homomorphisms $\bar{\Delta}$ and $\bar{\varepsilon}$—on the commutative quotient they come for free, with no separate multiplicativity argument.
- **Antipode** (`Sab`, `Sab_pi`). The planar antipode S is an *anti*-homomorphism; composed with π into the commutative $\mathcal{H}_{\mathrm{ab}}$ it becomes an honest algebra homomorphism $\pi \circ S$ (`SalgHom_resp_commRel`), which descends to $\bar{S}\colon \mathcal{H}_{\mathrm{ab}} \to \mathcal{H}_{\mathrm{ab}}$ with $\bar{S}(\pi x) = \pi(Sx)$. Both antipode axioms descend through π by the same extensionality pattern (`rTensor_Sab_Delta_ab`, `lTensor_Sab_Delta_ab`).

Theorem 1 (The commutative Connes–Kreimer / Butcher Hopf algebra, in Mathlib)**.** *$\mathcal{H}_{\mathrm{ab}} = \mathcal{H}/(ab-ba)$ carries a Mathlib `HopfAlgebra` instance `instHabHopf`, obtained by descending the planar coproduct, counit and antipode through `RingQuot`. It is `sorry`-free; `#print axioms instHabHopf` reports only `[propext, Classical.choice, Quot.sound]`.*

We are unaware of a previous commutative combinatorial renormalization Hopf algebra of rooted trees in a proof assistant (Section 8); the companion supplied the planar one, and this is its abelianization. The point of building it this way is economy: the expensive theorem (coassociativity) was proved once, on the quotient-free planar type, and *transported*; the quotient adds structure-descent lemmas, not a second hard combinatorial proof.

4 The Eulerian idempotent, made idempotent

On $\mathcal{H}_{\mathrm{ab}}$ the obstruction of Section 2 is gone—but why, exactly, and how does one prove it without the classical Vandermonde inversion?

The target is the single identity

$$\pi(e^{(1)}(e^{(1)}x)) \;=\; \pi(e^{(1)}x) \qquad \text{for all } x \in \mathcal{H},$$

which, since π is surjective, says $e^{(1)} \circ e^{(1)} = e^{(1)}$ on $\mathcal{H}_{\mathrm{ab}}$. The classical proof diagonalizes the Adams operators by inverting a Vandermonde matrix in their eigenvalues. We take a different route, in four moves, because it formalizes more easily and isolates the only fact about power series that the argument needs.

Move 1. Adams operators are algebra morphisms on the commutative quotient

On $\mathcal{H}_{\mathrm{ab}}$—and *only* because it is commutative—each Adams operator is an algebra endomorphism. The convolution $\mu \circ (\mathrm{id} \otimes \Psi_n) \circ \Delta$ that defines Ψ_n uses the multiplication μ, which is an algebra map exactly when the target is commutative (`adamsAb_mul`, via the `adamsAbHom` packaging). The family is multiplicative under composition,

$$\Psi^p \circ \Psi^q = \Psi^{pq} \qquad (\texttt{adamsAb_comp}),$$

proved with Mathlib's `algHom_comp_convMul_distrib`: post-composition by an algebra homomorphism distributes over convolution. This is the first place the planar algebra could not follow: on $\mathcal{H}$ the operators are not algebra maps, and `ring`-style scalar bookkeeping has no commutative ground to stand on.

Move 2. The keystone: a formal logarithm that adds

The Eulerian idempotent is a logarithm, $e^{(1)} = \log_\star(\mathrm{id})$, and a logarithm's defining virtue is that it turns products into sums. We need exactly two laws, on formal power series over any $\mathbb{Q}$-algebra:

$$\log(f\,g) = \log f + \log g, \qquad \log(f^{\,n}) = n \cdot \log f.$$

Mathlib has the power-series logarithm `PowerSeries.log` and its derivative, but not these identities. We prove both by *antiderivative uniqueness*: two power series with the same derivative and the same constant term are equal, and both sides of each identity have matching derivative (by the product rule for $\log' = f'/f$) and constant term (Lean: `logOf_mul`, `logOf_pow`, and the coefficient form `coeff_logTrunc_pow` that we actually transport). The argument is formal throughout—only the derivative on the power-series ring is used—so it needs no metric or convergence; we isolate this step so that the rest of the proof is convolution algebra.

Remark. These two lemmas are independent of trees. Mathlib's power-series library has `PowerSeries.log` and its derivative `deriv_log` but neither the additivity $\log(fg) = \log f + \log g$ nor the power law $\log(f^n) = n \log f$; both are currently absent. We prove them for any $\mathbb{Q}$-algebra of coefficients by uniqueness of formal antiderivatives, with no reference to trees or convolution.

Move 3. Transport: the eigenvalue identity $\Psi^i \circ e^{(1)} = i \cdot e^{(1)}$

This is the technical core, where the keystone is used. Post-composition by Ψ^p is a $\star$-ring homomorphism sending the generator $J_{\mathrm{ab}} = \mathrm{id}_{\mathcal{H}_{\mathrm{ab}}} - 1$ to $Y^p - 1$ (Lean: `adamsAb_comp_JcAb_pow`, with $Y = 1 + X$ the formal group-like). Writing $e^{(1)} = \sum_j c_j\, J_{\mathrm{ab}}^{\star j}$ with $c_j = \mathrm{coeff}_j(\log)$, we get $\Psi^p(e^{(1)}v) = \sum_j c_j\,((Y^p-1)^{\star j}).v$. The keystone, in coefficient form, says this sum equals p times the same sum with $Y - 1$ in place of $Y^p - 1$—*provided* we work below the degree where the truncation is exact. That proviso is supplied by the local nilpotency of J_{ab}: on a forest of size N, $J_{\mathrm{ab}}^{\star(N+1)}$ vanishes (`JcAb_pow_ofConv_sf_eq_zero`), so the infinite formal identity becomes a finite, exact one. The bridge from the power-series coefficient identity to the convolution sums is a polynomial evaluation: a ring homomorphism $k[X] \to \mathrm{End}_\star(\mathcal{H}_{\mathrm{ab}})$ sending $X \mapsto J_{\mathrm{ab}}$ (`eval2_ofConv`, `star_identity`), under which `coeff_logTrunc_pow` lands degree by degree. The result is the Adams eigen-relation

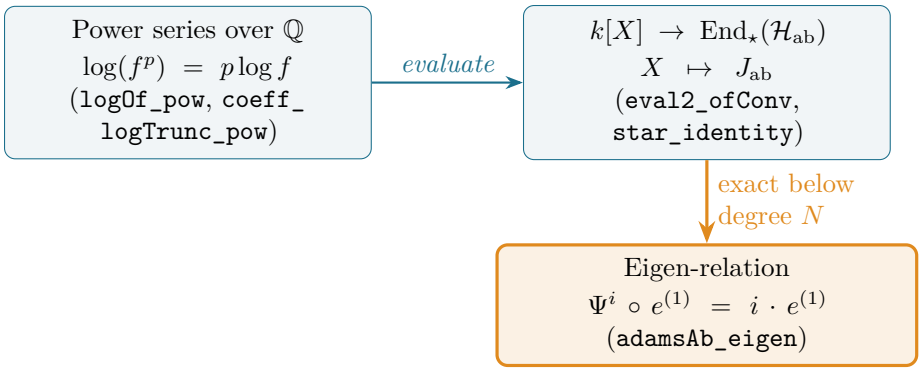


Figure 3: The transport of Move 3. The power-series power law is carried into the convolution algebra by evaluating its polynomial form at the per-degree-nilpotent J_{ab}; local nilpotency (`JcAb_pow_ofConv_sf_eq_zero`) makes each degree's truncation exact, yielding the eigen-relation $\Psi^i \circ e^{(1)} = i \cdot e^{(1)}$.

$$\Psi^i \circ e^{(1)} \;=\; i \cdot e^{(1)} \qquad (\texttt{adamsAb_eigen}),$$

which says $v = e^{(1)}x$ is an eigenvector of *every* Adams operator Ψ^i with eigenvalue i. This holds on $\mathcal{H}_{\mathrm{ab}}$ even though the underlying forests need not be primitive—it is the commutative, not-necessarily-cocommutative, weight-one eigenprojector.

Move 4. The finite-difference collapse

Now apply $e^{(1)}$ a second time to the eigenvector $v = e^{(1)}x$. Expanding $J_{\mathrm{ab}}^{\star j} = (\mathrm{id} - 1)^{\star j}$ by the binomial theorem (legitimate because the ring is commutative, `JcAb_pow_ofConv_eq_adams_sum`) writes each power as an alternating sum of Adams operators; on the eigenvector each Ψ^i acts by the scalar i, so

$$J_{\mathrm{ab}}^{\star j}.v \;=\; \Big(\sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i}\, i\Big)\, v \;=\; \delta_{j,1}\, v,$$

the bracket being the finite difference $\sum_i (-1)^{j-i}\binom{j}{i}\, i = \delta_{j,1}$ (`alternating_choose_weighted`, `brickC`). Only the $j = 1$ term of $e^{(1)} = \sum_j c_j J_{\mathrm{ab}}^{\star j}$ survives, and $c_1 = 1$, so $e^{(1)}(v) = v$.

Theorem 2 (The first Eulerian idempotent is idempotent on $\mathcal{H}_{\mathrm{ab}}$)**.** *On the commutative Connes–Kreimer / Butcher Hopf algebra $\mathcal{H}_{\mathrm{ab}}$, the first Eulerian idempotent is idempotent:*

$$e^{(1)} \circ e^{(1)} \;=\; e^{(1)} \qquad (\textit{eulerian1_idem_ab}).$$

The development is `sorry`-free; `#print axioms eulerian1_idem_ab` reports only `[propext, Classical.choice, Quot.sound]`. The same property fails on the planar algebra (Section 2), so commutativity is the operative hypothesis, not an incidental convenience.

We are unaware of a previous machine-checked proof of the Eulerian idempotency on the commutative Connes–Kreimer / Butcher Hopf algebra of rooted forests (Section 8). Table 2 records the verified chain link by link.

Table 2: The verified chain for $e^{(1)} \circ e^{(1)} = e^{(1)}$ on $\mathcal{H}_{\mathrm{ab}}$, link by link. Source: `ConnesKreimer.lean`.

Lean name	Statement	Move
`instHabHopf`	$\mathcal{H}_{\mathrm{ab}}$ is a Mathlib `HopfAlgebra`	§3
`logOf_mul`, `logOf_pow`	$\log(fg)=\log f+\log g$, $\log(f^n)=n\log f$	2
`coeff_logTrunc_pow`	coefficient form, exact below degree N	2
`adamsAb_mul`	Ψ_n is an algebra morphism on $\mathcal{H}_{\mathrm{ab}}$	1
`adamsAb_comp`	$\Psi^p \circ \Psi^q = \Psi^{pq}$	1
`adamsAb_comp_JcAb_pow`	$\Psi^p \circ J_{\mathrm{ab}}^{\star j} = (Y^p-1)^{\star j}$	3
`JcAb_pow_ofConv_sf_eq_zero`	local nilpotency of J_{ab}	3
`eval2_ofConv`, `star_identity`	$k[X] \to \mathrm{End}_\star(\mathcal{H}_{\mathrm{ab}})$, $X \mapsto J_{\mathrm{ab}}$	3
`adamsAb_eigen`	$\Psi^i \circ e^{(1)} = i \cdot e^{(1)}$	3
`JcAb_pow_ofConv_eq_adams_sum`	binomial: $J_{\mathrm{ab}}^{\star j}$ as Adams sum	4
`alternating_choose_weighted`, `brickC`	$\sum_i (-1)^{j-i}\binom{j}{i} i = \delta_{j,1}$	4
`eulerian1_idem_ab`	$e^{(1)} \circ e^{(1)} = e^{(1)}$ on $\mathcal{H}_{\mathrm{ab}}$	—

5 The certified core, and the evidence

What, concretely, has the machine checked here, beyond the companion's planar core?

The evidence is the same kind the companion gave: an *axiom footprint* and a *numerical cross-check*. Every declaration in the chain of Table 2 is `sorry`-free, and `#print axioms` on each—through the headline `eulerian1_idem_ab`—reports the identical line `[propext, Classical.choice, Quot.sound]`, the three standard axioms of classical Mathlib, with no `sorryAx` and no extra postulate. The development extends the companion's single self-contained source file, which builds under `lake env lean` with exit code 0 against Lean `v4.30.0-rc2` and the pinned Mathlib commit `701fb6e9c3`.

The numerical cross-check came *first*, as a guard against formalizing a false statement. Before the Lean proof, all forests up to degree 4 were computed in exact rational arithmetic and three identities were checked: the idempotency $e^{(1)} \circ e^{(1)} = e^{(1)}$ (0 failures on the commutative quotient), the eigen-relation $\Psi^i(e^{(1)} f) = i \cdot e^{(1)} f$ (0 failures, including on non-primitive forests), and the log power law. The same computation *falsified* the tempting shortcut that $e^{(1)}$ projects onto primitives—the primitivity identity $\Delta(e^{(1)} f) = e^{(1)} f \otimes 1 + 1 \otimes e^{(1)} f$ fails on 4 forests—which is why the proof routes through the eigen-relation and not through a primitive projection (it would have needed cocommutativity, which $\mathcal{H}$ lacks). Computation chose the route; the kernel certified it.

Table 3: The development of this paper, layered on the companion's planar Hopf algebra. Source: `ConnesKreimer.lean`.

Layer	Lean name	Content
quotient	`instHabCoalg`	`Coalgebra` k $\mathcal{H}_{\mathrm{ab}}$
quotient	`instHabBialg`	`Bialgebra` k $\mathcal{H}_{\mathrm{ab}}$
quotient	`instHabHopf`	`HopfAlgebra` k $\mathcal{H}_{\mathrm{ab}}$ (antipode $\bar{S}$ descended)
keystone	`logOf_mul`, `logOf_pow`	formal-log additivity & power law
operators	`adamsAb`, `adamsAb_natural`	Adams operators on $\mathcal{H}_{\mathrm{ab}}$, natural under π
operators	`adamsAb_mul`, `adamsAb_comp`	algebra morphism; $\Psi^p\Psi^q = \Psi^{pq}$
transport	`adamsAb_comp_JcAb_pow`, `star_identity`	Φ_p and the eigen-transport
spectrum	`adamsAb_eigen`	$\Psi^i \circ e^{(1)} = i \cdot e^{(1)}$
collapse	`alternating_choose_weighted`, `brickC`	finite-difference identity
payoff	`eulerian1_idem_ab`	$e^{(1)} \circ e^{(1)} = e^{(1)}$ on $\mathcal{H}_{\mathrm{ab}}$

6 What I claim, and what I do not

The mathematics is classical: the Adams operators (Adams; Aguiar–Lauve), the Eulerian idempotents (Euler; Solomon; Loday, Reutenauer, Patras), and the commutative tree algebra (Butcher; Connes–Kreimer) are all prior. The contribution is the formalization, and I separate the two on purpose.

What I claim as new (as formalization). Two things. *(i)* The commutative Connes–Kreimer / Butcher Hopf algebra $\mathcal{H}_{\mathrm{ab}}$ as a Mathlib `HopfAlgebra` instance, built as the abelianization of the companion's planar algebra via `RingQuot`. *(ii)* The idempotency of the first Eulerian idempotent on it, $e^{(1)} \circ e^{(1)} = e^{(1)}$, together with the Adams ladder ($\Psi_{-1} = S$, the eigenvalue on primitives, the eigen-relation $\Psi^i e^{(1)} = i\, e^{(1)}$) that delivers it, and the formalized *planar counterexample* showing the property fails without commutativity. We searched for prior formalizations (run 2026-06; same scope and method as the companion—Mathlib full source, the Isabelle/HOL *Archive of Formal Proofs*, and Coq/Rocq indices, for rooted-tree / Connes–Kreimer / Butcher / Eulerian-idempotent terms) and found none. Absence of a prior formalization is not something one can prove; we state it as a documented search, and a pointer to anything missed would be welcome and gladly cited.

And, plainly, what this is not. *(i)* I prove the *first* Eulerian idempotent is idempotent, $e^{(1)} \circ e^{(1)} = e^{(1)}$. I do *not* prove the full Hodge–Eulerian decomposition—the orthogonality of the higher idempotents $e^{(i)} \circ e^{(j)} = \delta_{ij} e^{(i)}$ and the resulting eigenspace splitting—which is the natural sequel. *(ii)* The Adams operators here are the convolution powers; I establish the eigen-relation on the image of $e^{(1)}$, not the complete characteristic polynomial of Aguiar–Lauve at all degrees. *(iii)* The single analytic input (`logOf_mul`/`logOf_pow`) is proved for formal power series over a $\mathbb{Q}$-algebra by antiderivative uniqueness; I do *not* develop a general theory of nilpotent formal logarithms, only what the transport needs.

7 Implications

Three kinds, in decreasing order of how firmly I can claim them.

A Mathlib lemma, independent of trees. The part of this development most likely to be used outside the present context is not specific to its subject. `PowerSeries.logOf_mul` and `logOf_pow`—formal-logarithm additivity $\log(fg) = \log f + \log g$ and the power law $\log(f^n) = n \log f$—are currently absent from Mathlib's power-series library, which has the logarithm and its derivative but neither identity; proved by uniqueness of formal antiderivatives, they could be contributed upstream on their own. The Adams operators, the convolution logarithm and the first Eulerian idempotent are also now formalized on a concrete commutative Hopf algebra, with the eigen-relation underlying the Hodge–Eulerian decomposition.

A foothold: certified Runge–Kutta and backward error. The reason to want this particular algebra commutative, and this particular idempotent, is numerical. The Butcher algebra indexes Runge–Kutta *order conditions* by rooted trees, and the convolution logarithm $\log_\star$—the Eulerian idempotent—underlies *backward error analysis*: the modified equation whose exact flow the numerical method follows, the central object for symplectic and long-time integrators. A certified generator of order conditions, with the tree combinatorics formally backed and the modified equation computed via $\log_\star$, would need several ingredients that do not exist yet; the present development provides one

of them—the algebra and this idempotent. No such tool is built here, and I make no claim about its novelty.

The mathematics, with a note of caution. This is a certified foundation, not an application. It ships no integrator and no renormalized amplitude. Its value is in that foundation and in the growing corpus of formalized mathematics, not in immediate use; I claim the former, not the latter.

8 Related work

The mathematics is classical. The Adams operations ψ^k are Adams [1]; their realization as convolution powers on a Hopf algebra, with the antipode as Ψ_{-1} and the characteristic polynomial of the family, is Aguiar–Lauve [2], with a connected-graded treatment in Zhou–Zhuang [20]. The Eulerian idempotents descend from Euler through Solomon's descent algebra [18] and the free-Lie school of Loday [10], Reutenauer [17] and Patras [16]; that their orthogonality requires (co)commutativity is exactly Patras's analysis, and is what the planar counterexample of Section 2 instantiates. The carrier is the commutative Connes–Kreimer algebra of renormalization [5] and the Butcher algebra of Runge–Kutta methods [4, 7], shown to coincide by Brouder [3]; the planar predecessor formalized in the companion is Foissy's [6]. The Hopf-algebra skeleton is Hopf [8], Milnor–Moore [15] and Sweedler [19]. Table 4 collects this provenance in one place.

Table 4: Provenance of the classical mathematics this paper formalizes. Every concept is prior; the work here is the formalized instance on $\mathcal{H}$ and $\mathcal{H}_{\mathrm{ab}}$, not new mathematics. (The formal-logarithm lemmas of Section 4 are omitted: they are not specific to trees.)

Concept	**Origin**	**Role here**
Hopf algebra, antipode	Hopf 1941; Milnor–Moore 1965; Sweedler 1969	Fact 0 (§2)
Convolution algebra of endomorphisms	Sweedler 1969	Fact 1
Adams operations ψ^k	Adams 1962 (K-theory)	name the family Ψ_n (Fact 3)
Rooted-tree Hopf algebra	Butcher 1972; Connes–Kreimer 1998; Brouder 2000	the carrier $\mathcal{H}$, $\mathcal{H}_{\mathrm{ab}}$
Planar (noncommutative) version	Foissy 2002	the companion's $\mathcal{H}$
Eulerian idempotents	Euler; Solomon 1976; Loday, Reutenauer, Patras	$e^{(1)}$ (Step 6, §4)
Orthogonality needs (co)commutativity	Patras 1994	the planar obstruction (Fact 9); why $\mathcal{H}_{\mathrm{ab}}$ removes it
Adams operators on Hopf algebras; spectrum	Aguiar–Lauve 2015; Zhou–Zhuang 2024	the ladder and its eigenvalues

On the formalization side, this paper builds directly on its companion [14], which constructed the planar Hopf algebra in Lean 4 / Mathlib and deferred the commutative quotient and the spectral theory as follow-on work; this is that follow-on. Mathlib [12] provides the abstract `HopfAlgebra`, `Bialgebra` and `Coalgebra` typeclasses, the `RingQuot` quotient used to abelianize, the convolution ring `WithConv` (and its commutative refinement on a commutative carrier), and the power-series logarithm `PowerSeries.log`—but not its additivity, and no rooted-tree object. The same search

reported in the companion, re-run for the Eulerian-idempotent and commutative-quotient terms, surfaced no prior formalization of either; as above, this is a documented absence, not a proof of priority, and corrections are welcome.

Table 5: This work against Mathlib and the companion. The *No* → *Yes* rows are the contribution; the *No* → *No* rows are the named sequel (Section 7).

Structure	Before	This work
Planar `HopfAlgebra` on trees	companion	used
`RingQuot`, `WithConv`, `PowerSeries.log`	Mathlib	used
Formal-log additivity `logOf_mul` / `logOf_pow`	no	**yes**
Commutative (Butcher/CK) quotient $\mathcal{H}_{\mathrm{ab}}$ as `HopfAlgebra`	no	**yes**
Adams operators on $\mathcal{H}_{\mathrm{ab}}$ as algebra morphisms	no	**yes**
Eigen-relation $\Psi^i \circ e^{(1)} = i\, e^{(1)}$	no	**yes**
First Eulerian idempotency $e^{(1)} \circ e^{(1)} = e^{(1)}$	no	**yes**
Planar non-idempotency (counterexample)	no	**yes**
Full Hodge–Eulerian decomposition ($e^{(i)}$ orthogonal)	no	no (sequel)
B-series / Runge–Kutta order conditions	no	no (sequel)

9 Conclusion

The companion paper built the planar Connes–Kreimer / Butcher / Foissy Hopf algebra of rooted trees and left its commutative quotient and spectral theory open. This paper closes both. It abelianizes the planar algebra through Mathlib's `RingQuot` into a commutative `HopfAlgebra` instance $\mathcal{H}_{\mathrm{ab}}$, and it proves on $\mathcal{H}_{\mathrm{ab}}$ that the first Eulerian idempotent is idempotent, $e^{(1)} \circ e^{(1)} = e^{(1)}$—the property that demonstrably *fails* on the planar algebra. The proof runs the Adams ladder ($\Psi_{-1} = S$, $\Psi_n(x) = n\,x$ on primitives), establishes the eigen-relation $\Psi^i \circ e^{(1)} = i \cdot e^{(1)}$ by transporting a small piece of formal-logarithm calculus into the convolution algebra, and closes with a finite-difference collapse, avoiding the classical Vandermonde inversion. None of the mathematics is new; the contribution is that the commutative algebra and its first Eulerian idempotency are now verified in Lean, with the operative hypothesis (commutativity) made visible by formalizing the planar failure alongside the theorem. The full Hodge–Eulerian decomposition, certified Runge–Kutta order conditions, and the backward-error modified equation via $\log_\star$ remain as follow-on work.

Artifacts. The Lean source (`sorry`-free, depending only on the three standard axioms) is the self-contained file `ConnesKreimer.lean`; the formal-logarithm keystone is also tracked standalone in `PowerSeriesLogMul.lean`. Verify the axiom footprint with, e.g., `#print axioms CK.eulerian1_idem_ab` ⇝ `[propext, Classical.choice, Quot.sound]`. The artifact is deposited at Zenodo, DOI `10.5281/zenodo.20774821`, and mirrored at https://github.com/karlesmarin/connes-kreimer-lean; the planar companion is DOI `10.5281/zenodo.20762280`.

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