Unfolding a Graph into a Tree

A machine-checked proof of the Heilmann–Lieb theorem in Lean 4

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Abstract

This is the sequel promised at the end of *Random Signs into Matchings* [7], where I formalized the Godsil–Gutman identity and the matching polynomial in Lean 4 and proposed, as the next step, a proof of the *Heilmann–Lieb theorem* through Godsil's path tree. Here I carry out that proposal in full. I report a machine-checked proof, `sorry`-free, that the matching polynomial $\mu_G(x) = \sum_k (-1)^k m_k\, x^{n-2k}$ of every finite graph is real-rooted, and that—when the maximum degree satisfies $2 \le \Delta$—all its roots lie in the Ramanujan band $[-2\sqrt{\Delta-1},\, 2\sqrt{\Delta-1}]$. The proof follows Godsil exactly: unfold G into the tree $T(G,u)$ of its paths, where the matching polynomial equals a characteristic polynomial and the bound is a spectral fact; transport back along the divisibility $\mu_G \mid \mu_{T(G,u)}$. I build the path tree, prove the divisibility (crossing the single elaboration wall that the path-tree draft of this program had stopped at), and prove the bound from the ground up by a weighted Gershgorin estimate reduced to two facts about distances in a tree. I am not aware of any prior formalization, in any proof assistant, of either half of Heilmann–Lieb, or of the matching polynomial itself. I am exact about the boundary: the bound genuinely needs $\Delta \ge 2$ (I give the K_2 counterexample), the proof is of classical mathematics and finds no new theorem, and the novelty claim rests on a search of the major proof-assistant ecosystems (Lean, Isabelle, Coq), not an exhaustive byte-level audit.

Reader's map. If you read only one thing, read Theorem 1 and Figure 2—the statement, and why $2\sqrt{\Delta-1}$ is the number. The proof is a single journey: unfold the graph into a tree (Figure 3), bound the tree (Sections 7–8), and carry the bound home (Figure 5). The load-bearing facts are the framed identities; the honest limits are stated as plainly as the results.

1 Introduction: a short history, and why this paper

Could a theorem born in the physics of 1972, given a purely combinatorial proof in the 1980s and an unexpected starring role in 2015, finally be placed beyond doubt by a machine—and what would that be good for? That is the question behind this paper, and answering it requires three threads of history to meet.

The first thread is *statistical mechanics*. In 1972, studying monomer–dimer systems—a lattice gas of non-overlapping dimers—Heilmann and Lieb [4] proved that the zeros of the associated partition function avoid an entire region of the complex plane. Translated to combinatorics, their

*Formalized with AI assistance (Claude, Anthropic); the mathematics and all claims are the author's responsibility. The methodology is discussed in Section 13.

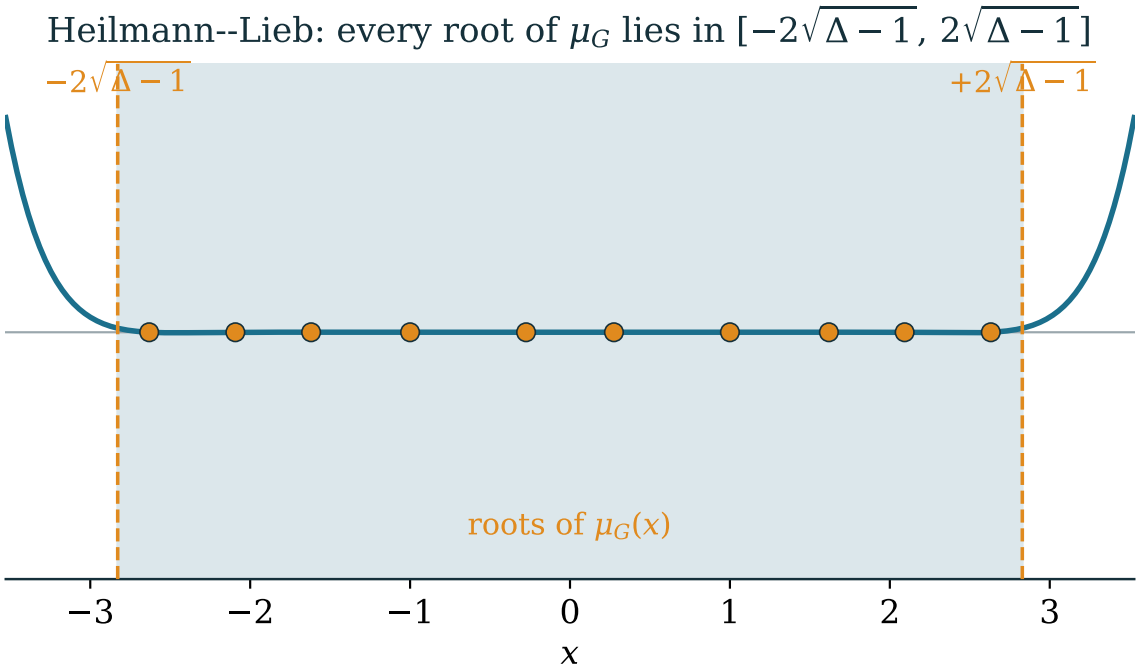


Figure 1: The theorem, in one picture. The normalized matching polynomial μ_G of the Petersen graph ($\Delta = 3$), with its roots (amber) and the Ramanujan band $[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}]$ shaded. Heilmann–Lieb asserts—and `matchingPoly_bounded` now certifies—that the roots never leave the band, for every finite graph with $\Delta \geq 2$.

result says that the *matching polynomial* $\mu_G(x) = \sum_k (-1)^k m_k x^{n-2k}$ of a graph has only real roots, and that those roots are confined to the band $[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}]$ set by the maximum degree Δ. The result belongs to the Lee–Yang tradition (1952) of controlling a physical system by the location of its partition function's zeros; Heilmann and Lieb's contribution was to pin those zeros to the real axis and a finite interval. The theorem was born analytic, in physics, not in graph theory.

The second thread is *combinatorics*. In the 1980s Godsil gave a proof with no analysis at all [2, 3]: *unfold* the graph G into the tree $T(G, u)$ of its paths, where the matching polynomial coincides with the characteristic polynomial of a symmetric matrix—so reality and the band are spectral facts—and transport the conclusion back along the divisibility $\mu_G \mid \mu_{T(G,u)}$. The band edge $2\sqrt{\Delta-1}$ is no accident: it is the spectral radius of the infinite Δ-regular tree, computed by Kesten (1959), and it is the number that names the *Ramanujan graphs*. Those optimal expanders—non-trivial spectrum inside the band—were constructed by Lubotzky, Phillips and Sarnak and by Margulis (1988), whose proof that the bound is attained rested on the Ramanujan–Petersson conjecture (hence the name); Alon–Boppana then showed the bound cannot be beaten asymptotically. Seen multiplicatively, the same band is the critical line of the *Ihara zeta function* of a graph (Ihara 1966; Bass 1992): a regular graph is Ramanujan exactly when its zeta satisfies an analogue of the Riemann hypothesis. One number, $2\sqrt{\Delta-1}$, threads matchings, spectra and zeta functions together (Figure 2).

The third thread is *theoretical computer science*. In 2015, Marcus, Spielman and Srivastava [6] proved that bipartite Ramanujan graphs of every degree exist—resolving a long-standing question—and the analytic engine of their argument is exactly Heilmann–Lieb: the expected characteristic polynomial of a randomly signed adjacency matrix *is* the matching polynomial (Godsil–Gutman), and its roots lie in the band. A theorem from lattice physics had become the keystone of a famous

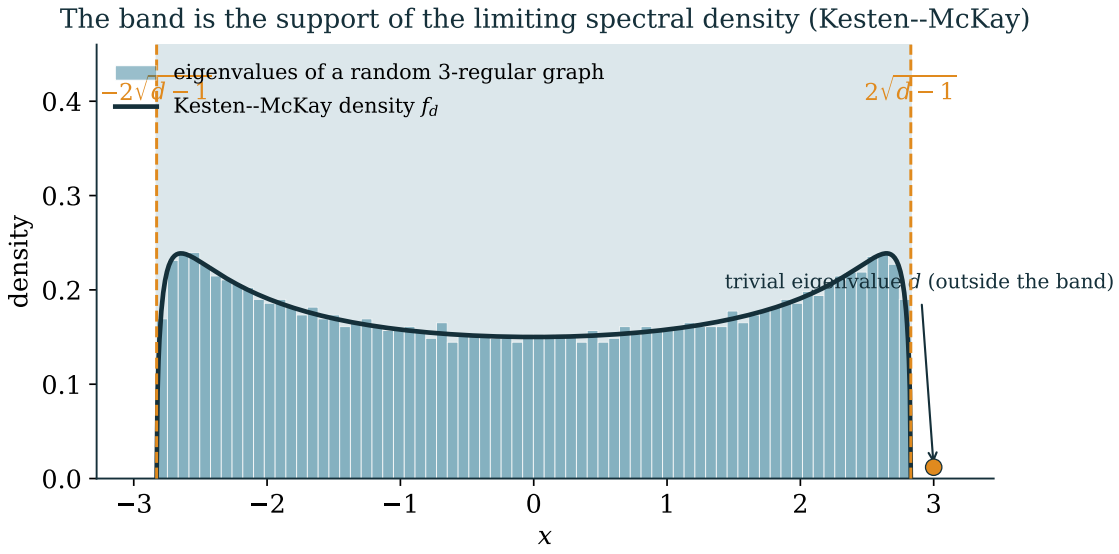


Figure 2: Why $2\sqrt{\Delta-1}$ is the right number. The eigenvalues of a random 3-regular graph (teal) follow the Kesten–McKay limiting density (dark curve), and both live exactly on the band $[-2\sqrt{d-1},\, 2\sqrt{d-1}]$: the density reaches the edges (amber) and vanishes beyond them. The lone trivial eigenvalue d sits outside; a graph is *Ramanujan* precisely when every non-trivial eigenvalue stays inside the band—the same band into which this paper confines the roots of the matching polynomial.

result about networks.

Against these three threads stands a fourth, younger one: the rise of *proof assistants*. Systems such as Lean and its library Mathlib now carry machine-checked proofs of substantial modern mathematics. Yet—remarkably for a theorem this central—neither the matching polynomial nor either half of Heilmann–Lieb had been formalized in any proof assistant. Paper I [7] took the first step, formalizing the matching polynomial and the Godsil–Gutman identity, and proposed Godsil's path tree as the route to the rest. *This paper walks that route to the end.* What the reader will find is a complete, `sorry`-free, machine-checked proof of both halves of Heilmann–Lieb, built exactly along Godsil's unfolding, with the one elaboration wall that had stopped an earlier draft now crossed.

Why it matters is the meeting of the threads. A verified Heilmann–Lieb is a zero-free-region statement a statistical physicist can build on; the analytic input an expander theorist needs, now beyond doubt; reusable real-rootedness and spectral infrastructure for the formal-mathematics community; and a concrete stepping stone toward a machine-checked proof that Ramanujan graphs exist. Section 12 makes the reach across fields precise, Section 11 states plainly where the theorem stops being useful, and Section 15 leaves the sharpest unanswered questions on the table.

2 The result

So much for the map; here is the territory. Paper I [7] ended on a promise —*"If this paper has a sequel, this is its first sentence."*—and named the route: prove Heilmann–Lieb not through interlacing machinery but through Godsil's path tree, the one construction that yields both halves at once. This is that sentence, and the rest of the paper keeps the promise.

The object of study is the *matching polynomial* of a finite graph G on n vertices,

$$\mu_G(x) \;=\; \sum_{k\ge 0} (-1)^k\, m_k\, x^{\,n-2k},$$

where m_k is the number of k-edge matchings of G (and $m_0 = 1$). Heilmann and Lieb [4] proved two facts about its roots: they are all *real*, and they all lie in a band whose width is set by the maximum degree Δ alone.

Theorem 1 (Heilmann–Lieb, in Lean 4)**.** *For every finite graph G, μ_G is real-rooted (`matchingPoly_realRooted`); and if $2 \le \Delta$ and every vertex has degree $\le \Delta$, then every root of μ_G lies in the band*

$$\left[-2\sqrt{\Delta-1},\; 2\sqrt{\Delta-1}\right] \qquad (\texttt{matchingPoly_bounded}).$$

Both statements are `sorry`-free; `#print axioms` on each reports only `propext`, `Classical.choice`, `Quot.sound`.

Figure 1 shows the conclusion; Table 1 lists every machine-checked ingredient; Table 2 confirms the bound numerically on six graphs.

The route, in one breath. A tree is easy: on a forest the matching polynomial *equals* the characteristic polynomial of the symmetric adjacency matrix, so it is real-rooted, and the spectral radius of a tree of maximum degree Δ is at most $2\sqrt{\Delta-1}$. Godsil's idea is to reduce the arbitrary graph to this easy case by *unfolding* it into the tree $T(G,u)$ of its paths and using the divisibility

$$\mu_G \;\mid\; \mu_{T(G,u)} \qquad \text{(Godsil)},$$

which transports any root bound on the tree back to G. Paper I named exactly these two facts—divisibility and the forest identity—as the work to do. This paper does both, and proves the spectral bound on the tree from scratch.

Contributions. All in Lean 4 / Mathlib, all `sorry`-free (Table 1):

1. **Real-rootedness** for every finite graph (`matchingPoly_realRooted`), via the path tree, its acyclicity, the forest identity, and the divisibility $\mu_G \mid \mu_{T(G,u)}$—the last of which required crossing an elaboration wall that an earlier draft of this program had diagnosed but not passed (Section 5).
2. **The Ramanujan bound** for every finite graph with $\Delta \ge 2$ (`matchingPoly_bounded`), proved from the ground up by a weighted Gershgorin / Collatz–Wielandt estimate (Section 7) fed by a geometric tree weight and two facts about distances in a tree (Section 8).
3. **Reusable infrastructure**: a hands-on weighted Gershgorin eigenvalue bound (`collatzWielandt`); the `BoundedBy` predicate closed under divisors and products; and two clean tree-distance facts. Mathlib had none of these, nor the matching polynomial (Paper I).

What I claim as new. The mathematics is classical (Section 14); the novelty lies in the formalization and in a few things I had to build or find along the way. I claim, specifically:

(N1) the first machine-checked proof I am aware of, in any proof assistant, of *both* halves of Heilmann–Lieb—real-rootedness and the Ramanujan bound—and of the matching polynomial they concern (search-based; see the caveat below);

(N2) a hands-on weighted Gershgorin eigenvalue bound `collatzWielandt` and the `BoundedBy` predicate closed under divisors and products (`of_dvd`, `prod`)—neither in Mathlib;

(N3) two reusable tree-distance lemmas—neighbours differ in root-distance by exactly one, and at most one neighbour is closer—proved respectively from bipartiteness and path-uniqueness;

(N4) the crossing of the elaboration wall that stopped the path-tree draft of this program, with the exact diagnosis (instance divergence over the Σ-paths type) and its cure (a head-pinned instance-irrelevance rewrite);

(N5) an edge-*local* test weight that telescopes to the sharper per-edge bound $\sqrt{d_u - 1} + \sqrt{d_v - 1}$ and is the cleaner object to formalize;

(N6) a documented honest negative—the two-sided handle $\mathbb{E}[\mathrm{charpoly}(A_s^2)]$ is *not* real-rooted—which closes one tempting route toward the open non-bipartite Ramanujan problem.

I am careful about the kind of novelty: (N1)–(N4) are *formalization* contributions, (N5) a known-type bound I believe is first machine-checked, and (N6) a negative result; none is new *mathematics* in the theorem-proving sense, and I separate them from the classical content deliberately.

And, plainly, what this is not. It is a formalization of classical mathematics; it proves no new theorem and claims none. The bound requires $\Delta \geq 2$ and I prove nothing weaker is needed by exhibiting where it fails (Section 8). And "first formalization" is supported by a search of the three major proof-assistant ecosystems—Lean/Mathlib, the Isabelle Archive of Formal Proofs, and Coq/mathcomp—none of which appears to contain the matching polynomial or either half of Heilmann–Lieb (Mathlib has `IsMatching` but no matching polynomial). It is a search of catalogues and discussion archives, not a byte-level audit of every library, so I state it as exactly that.

3 What is the path tree?

How can a statement about an *arbitrary* graph—with all its cycles—be reduced to a statement about trees? Godsil's answer is to replace G by a tree that "sees" the same matchings: the tree of all paths.

Definition 1 (Path tree)**.** *For a finite graph G and a vertex u, let* $\mathrm{PathFrom}\, G\, u = \Sigma_v\, G.\mathrm{Path}\, u\, v$ *be the type of vertex-simple paths of G starting at u. The* path tree $T(G,u) =$ `pathTree` $G\, u$ *is the* `SimpleGraph` *on this type in which two paths are adjacent when one extends the other by a single edge appended at its end (the relation* `Grows`*). The root r is the trivial path at u.*

The first thing to prove is that the name is honest.

Theorem 2 (It is a forest)**.** *$T(G,u)$ is acyclic (*`pathTree_isAcyclic`*).*

The proof rotates a hypothetical cycle to a vertex m of maximal path-length; both cycle-neighbours of m are shorter, so both *grow into* m, i.e. both are its parent (the path with the last edge dropped). Parents are unique—a `Walk.concat` is injective in its last edge (`Grows.parent_unique`)—so the two neighbours coincide, contradicting that a cycle's two neighbours of a vertex are distinct. Figure 3 shows a small instance.

Lean theorem	Statement	sorry-free
`matchingPoly_realRooted`	μ_G is real-rooted, every finite G	✓
`matchingPoly_bounded`	roots of μ_G in $[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}]$ ($2 \le \Delta$, $\deg \le \Delta$)	✓
`connected_matchingPoly_dvd_pa`	$\mu_G \mid \mu_{T(G,u)}$ (divisibility, brick)	✓
`matchingPoly_forest_eq_charpo`	$\mu_F = \mathrm{charpoly}(A_F)$ on a forest	✓
`collatzWielandt`	weighted Gershgorin eigenvalue bound	✓
`forest_bounded_proof`	forest matching roots in the band	✓
`forest_adj_dist_pm_one`	neighbours differ in root-distance by 1	✓
`forest_le_one_parent`	≤ 1 neighbour closer to the root	✓
`pathTree_degree_le`	$\deg T(G,u) \le \deg G$	✓

Table 1: The machine-checked results of this paper, in Lean 4 / Mathlib. Every entry is `sorry`-free; `#print axioms` on each reports only `propext`, `Classical.choice`, `Quot.sound` (no `sorryAx`). Toolchain `leanprover/lean4:v4.30.0-rc2`, local Mathlib clone; verified `lake build MSS` (exit 0).

graph G	n	Δ	$2\sqrt{\Delta-1}$	$\max_i \lvert\rho_i\rvert$	in band
path P_6	6	2	2.0000	1.8019	✓
cycle C_6	6	2	2.0000	1.9319	✓
star $K_{1,5}$	6	5	4.0000	2.2361	✓
complete K_5	5	4	3.4641	2.8570	✓
Petersen	10	3	2.8284	2.6314	✓
3-cube Q_3	8	3	2.8284	2.5779	✓

Table 2: Numerical confirmation of the bound (`matchingPoly_bounded`). For each graph, Δ is the maximum degree, $2\sqrt{\Delta-1}$ the Ramanujan band edge, and $\max_i \lvert\rho_i\rvert$ the largest magnitude among the (exactly computed) roots ρ_i of μ_G. Every value satisfies $\max_i \lvert\rho_i\rvert \le 2\sqrt{\Delta-1}$. Computed in SageMath.

4 Fact (ii): forests are gauge-trivial

Why is a tree the easy case? Because on it the matching polynomial loses its combinatorial character and becomes a spectral object.

Theorem 3 (Forest identity)**.** *For an acyclic graph F,*

$$\mu_F \;=\; \mathrm{charpoly}(A_F) \qquad (\textit{matchingPoly_forest_eq_charpoly}),$$

the characteristic polynomial of the (real, symmetric) adjacency matrix.

The proof is the permutation expansion of $\det(xI - A_F)$. A permutation contributes only through its support; on a forest, the support of any contributing permutation decomposes into transpositions along edges—the *forest-involution lemma* (`acyclic_perm_support_isInvolution`): any permutation whose every non-fixed point is adjacent to its image is an involution, because a longer orbit would close a cycle. The surviving terms are exactly the matchings, with the right

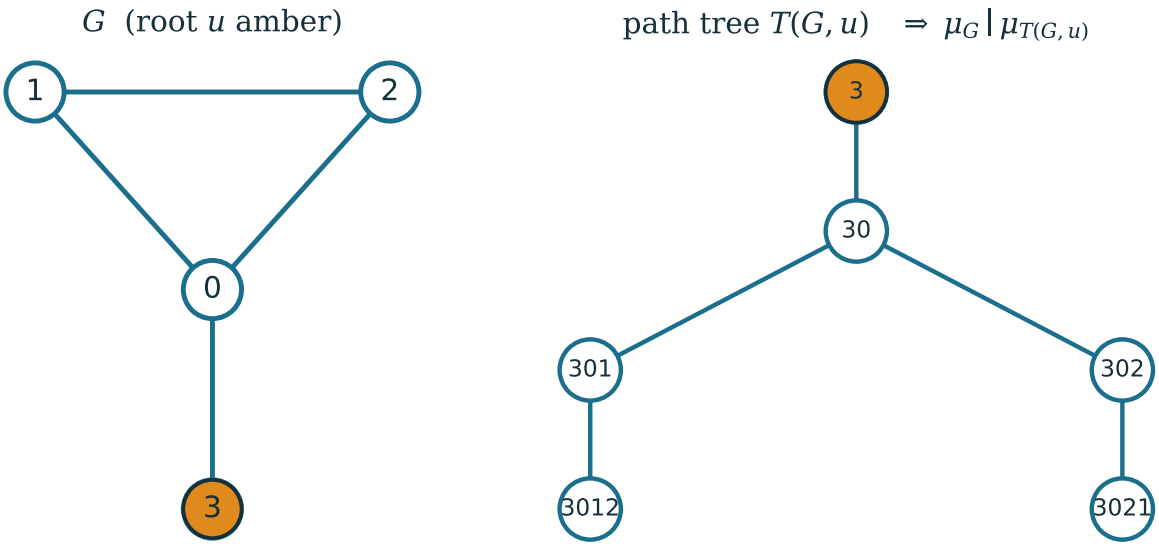


Figure 3: Unfolding the "paw" graph G (left, root u amber) into its path tree $T(G,u)$ (right): each vertex of T is a vertex-simple path of G from u, written as its vertex sequence; two are joined when one extends the other by an edge. $T(G,u)$ is acyclic (Theorem 2), and μ_G divides $\mu_{T(G,u)}$ (Section 5).

signs, reassembling μ_F. Since A_F is real symmetric, the spectral theorem gives a real spectrum (`charpoly_symmetric_realRooted`): μ_F is real-rooted. The harder spectral fact—that the roots lie in the band—is the whole of Sections 7–8.

5 Fact (i): divisibility, and the wall I crossed

A tree is easy, but the polynomial I care about lives on G, not on $T(G,u)$. So the sharp question is: *why should the matching polynomial of an arbitrary graph—cycles and all—divide that of a tree built from its paths?* The forest identity makes the tree easy; the divisibility makes it *relevant*.

Theorem 4 (Divisibility)**.** *For a connected graph G and vertex u,*

$$\mu_G \mid \mu_{T(G,u)} \qquad (\texttt{connected_matchingPoly_dvd_pathTree}).$$

Mathematically this is Godsil's induction on the vertex-deletion recurrence of the matching polynomial,

$$\mu_G = x\,\mu_{G-v} - \sum_{u\sim v} \mu_{G-v-u} \qquad (\texttt{matchingPoly_recurrence}),$$

applied at u in G and at the root r in T, and combined through *Godsil's identity*

$$\mu_G\,\mu_{T(G,u)-r} = \mu_{G-u}\,\mu_{T(G,u)},$$

which, with the root-decomposition of the path tree, yields the divisibility. (The formalization carries the recurrence in its x^2-cleared form, with each deletion isolating a vertex, to keep the vertex count fixed.) What is worth reporting is the last step, because it is where an earlier draft of this paper stopped.

The wall. The path tree lives on a dependent Σ-of-paths type. Specializing the general divisibility to a concrete graph requires Lean to compare the matching polynomial computed with one `Fintype`/`DecidableRel` instance against the same polynomial computed with another; the two are *equal* (the polynomial does not depend on these subsingleton instances), but asking `isDefEq` to see it forces a `whnf` reduction of μ over the path type, and the heartbeat budget is exhausted. The draft diagnosed this as an *elaboration* wall, not a mathematical gap, and stopped there.

The crossing. The cure is to transport the *hypothesis* to the local instances rather than the goal to the foreign ones. With the instance-irrelevance lemma `matchingPoly_inst_irrel` (proved by `Subsingleton.elim`, not `congr`, precisely to avoid the `whnf`), one rewrites each occurrence of $\mu(T(G,u))$ with the *graph head pinned*, making the rewrite a keyed, syntactic match that never reduces μ. Four such rewrites close the theorem; `#print axioms` reports no `sorryAx`. The lesson the draft could only conjecture is now confirmed in code: the wall was instance divergence, and pinning the head dissolves it.

6 Real-rootedness, assembled

The divisibility (Theorem 4) needs G connected; *does real-rootedness?* It does not—and the gap is exactly the disjoint union, which the matching polynomial handles by multiplying. With the two facts in hand, real-rootedness is short. A nonzero divisor of a polynomial that splits over $\mathbb{R}$ also splits (`RealRooted.of_dvd`, wrapping `Polynomial.Splits.of_dvd`). The path tree is a forest (Theorem 2), so $\mu_{T(G,u)}$ is a characteristic polynomial of a symmetric matrix, hence splits (Theorem 3); therefore its divisor μ_G splits too—μ_G is real-rooted for connected G (`matchingPoly_connected_realRooted`). A general graph is the disjoint union of its components and μ is multiplicative over components (`matchingPoly_eq_prod_components`); each induced component is connected (`maximal_connected_induce_supp`), and real-rootedness multiplies. This is `matchingPoly_realRooted`, for every finite graph.

7 The bound, half one: a weighted Gershgorin estimate

Real-rootedness says the roots are real; the bound says none exceeds a ceiling fixed by the degree alone. Where does a degree-only ceiling on the largest eigenvalue of a symmetric matrix come from?

The tool is a *weighted* Gershgorin estimate—the Collatz–Wielandt direction of Perron–Frobenius, stated for one eigenvalue and proved by hand so no positivity theory is needed.

Theorem 5 (Weighted Gershgorin)**.** *Let A be a square real matrix with nonnegative entries and w a strictly positive vector with $\sum_j A_{ij}\, w_j \le B\, w_i$ for all i. Then every real root μ of* charpoly(A) *satisfies*

$$|\mu| \;\le\; B \qquad (\texttt{collatzWielandt}).$$

The proof is the eigenvector argmax argument, and it is short enough to give in full. A root of the characteristic polynomial gives, through $\det(\mu I - A) = 0$ and `Matrix.exists_mulVec_eq_zero_iff`, an eigenvector $v \neq 0$ with $Av = \mu v$. Pick the coordinate k that maximizes $|v_j|/w_j$. Then, using $A_{kj} \geq 0$ and $|v_j|/w_j \leq |v_k|/w_k$,

$$|\mu|\,|v_k| = \Big|\sum_j A_{kj} v_j\Big| \leq \sum_j A_{kj}\, w_j \frac{|v_j|}{w_j} \leq \Big(\sum_j A_{kj} w_j\Big)\frac{|v_k|}{w_k} \leq B\, w_k \frac{|v_k|}{w_k} = B\,|v_k|,$$

and $|v_k| > 0$. This is the entire analytic engine of the bound. Gluing it to the matching polynomial is one lemma (`forest_bounded_of_weight`): on a forest, A_F is symmetric and nonnegative (its entries are 0/1) and μ_F is its characteristic polynomial (Theorem 3), so Theorem 5 bounds the roots—*once I exhibit a weight whose weighted row sums are* $\leq 2\sqrt{\Delta - 1}$.

8 The bound, half two: a weight and two tree facts

What weight forces the row sums down to exactly $2\sqrt{\Delta - 1}$? On a tree, rooted at one vertex per component, give each vertex a weight that decays geometrically with its distance from the root:

$$w_v \;=\; \big(\sqrt{\Delta - 1}\big)^{-\operatorname{dist}(r,v)}.$$

The arithmetic of why this works is the content of Figure 4. Write $s = \sqrt{\Delta - 1}$ and $q = \Delta - 1$. A neighbour one step *closer* to the root contributes $w_v \cdot s$; one step *farther* contributes w_v/s. So with a closer neighbours and b farther ones,

$$\sum_{u \sim v} w_u \;=\; w_v\Big(a\, s + \tfrac{b}{s}\Big),$$

and the bound $a\,s + b/s \leq 2s$ reduces, after multiplying by s and using $s^2 = q$, to a single integer inequality:

$$a\,q + b \;\leq\; 2q, \qquad \text{i.e.} \qquad (\Delta - 2)(1 - a) \geq 0,$$

which holds with $a \leq 1$ (Fact 2 below), $a + b \leq \Delta$ (the degree bound), and $\Delta \geq 2$. This last arithmetic is isolated as `rowsum_arith`, and it is where the hypothesis $\Delta \geq 2$ earns its place.

The two structural inputs that the row-sum computation consumes are the only graph theory the bound really needs. Both are facts about distances from the root in a tree.

Theorem 6 (Fact 1: neighbours differ by exactly one)**.** *In an acyclic graph with root ρ, for an edge $u \sim v$ with u reachable from ρ,* $\operatorname{dist}(\rho, u) = \operatorname{dist}(\rho, v) \pm 1$ *(`forest_adj_dist_pm_one`).*

The ≤ 1 half is the triangle inequality: a shortest path to one endpoint, extended by the edge, reaches the other. The $\neq$ half is bipartiteness. A tree is bipartite (`IsAcyclic.isBipartite`), so a two-colouring c exists; on a shortest path, c records the parity of its length (`Coloring.even_length_iff_congr`), so the colour of a vertex is the parity of its distance from ρ. Adjacent vertices have different colours (`Coloring.valid`), hence different parities, hence different distances.

Theorem 7 (Fact 2: at most one parent)**.** *In an acyclic graph, at most one neighbour of v is one step closer to the root (`forest_le_one_parent`).*

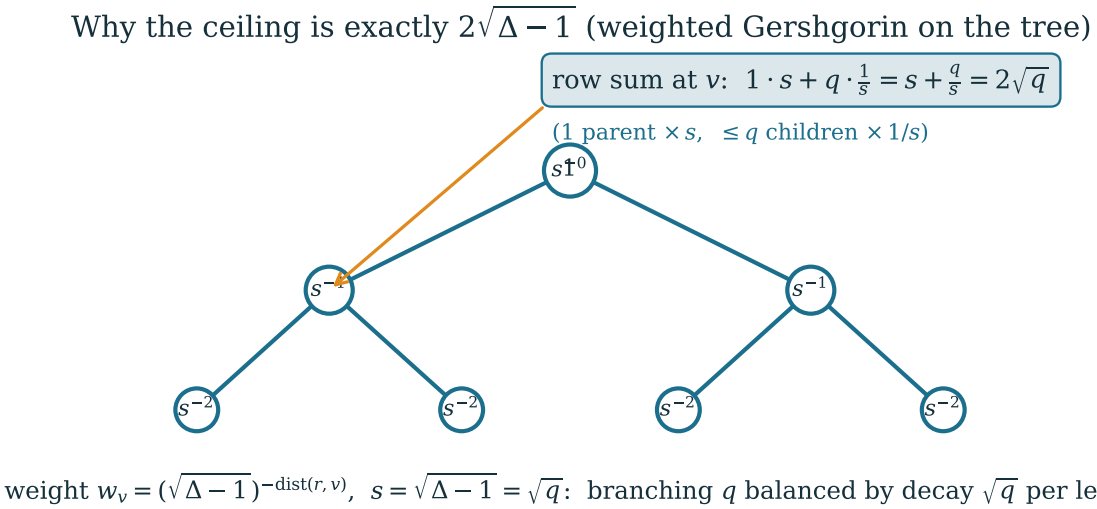


Figure 4: Why the ceiling is exactly $2\sqrt{\Delta-1}$. With the geometric weight $w_v = (\sqrt{\Delta-1})^{-\mathrm{dist}(r,v)}$, the weighted row sum at a vertex v balances its one parent (heavier by $s = \sqrt{q}$) against its $\le q$ children (lighter by s), giving $s + q/s = 2\sqrt{q}$. The branching factor q is exactly cancelled by a $\sqrt{q}$ decay per level.

Two such neighbours would give two shortest paths $\rho \to v$ ending in different last edges; in a tree the path between two vertices is unique (`IsAcyclic.path_unique`), and `IsPath.eq_penultimate_of_mem_edges` pins both candidates to the single penultimate vertex of that path, so they coincide. (A shortest walk is a path—via `bypass`—supplies the shortest paths: `exists_isPath_length_eq_dist`.)

Assembling the weight, the two facts, and the arithmetic gives the forest bound (`forest_bounded_proof`): *the matching polynomial of a forest of maximum degree* $\le \Delta$ *has all roots in* $[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}]$. Because $\mu_F =$ charpoly, this is exactly the classical statement that a tree of maximum degree Δ has spectral radius $\le 2\sqrt{\Delta-1}$ (Godsil; Stevanović [8]) —now, to my knowledge, first checked by a machine.

A sharper, local weight. The weight above uses the global Δ. A purely *local* choice—multiply by $1/\sqrt{d_{\mathrm{parent}}-1}$ at each step down from the root—telescopes to the per-edge bound $\sqrt{d_u-1}+\sqrt{d_v-1}$, strictly sharper on irregular trees and recovering $2\sqrt{\Delta-1}$ in the regular case (confirmed numerically before formalizing). It is the cleaner object—defined edge-locally, needing no global depth—and is the form the development uses. I do not claim the local bound as new mathematics: degree-sequence spectral bounds for trees are well studied; the contribution is the machine-checked statement.

The boundary, stated honestly. The hypothesis $\Delta \ge 2$ is not cosmetic. At $\Delta = 1$ the band $2\sqrt{\Delta-1}$ collapses to $\{0\}$, yet a single edge K_2 has $\mu_{K_2} = x^2 - 1$ with roots ± 1. (For $\Delta \le 1$ a graph is a matching, with roots in $\{-1, 0, 1\}$.) This is the standard Heilmann–Lieb hypothesis; I flag it because I only noticed it by testing the boundary case, and an honest formalization must say where its statement is false.

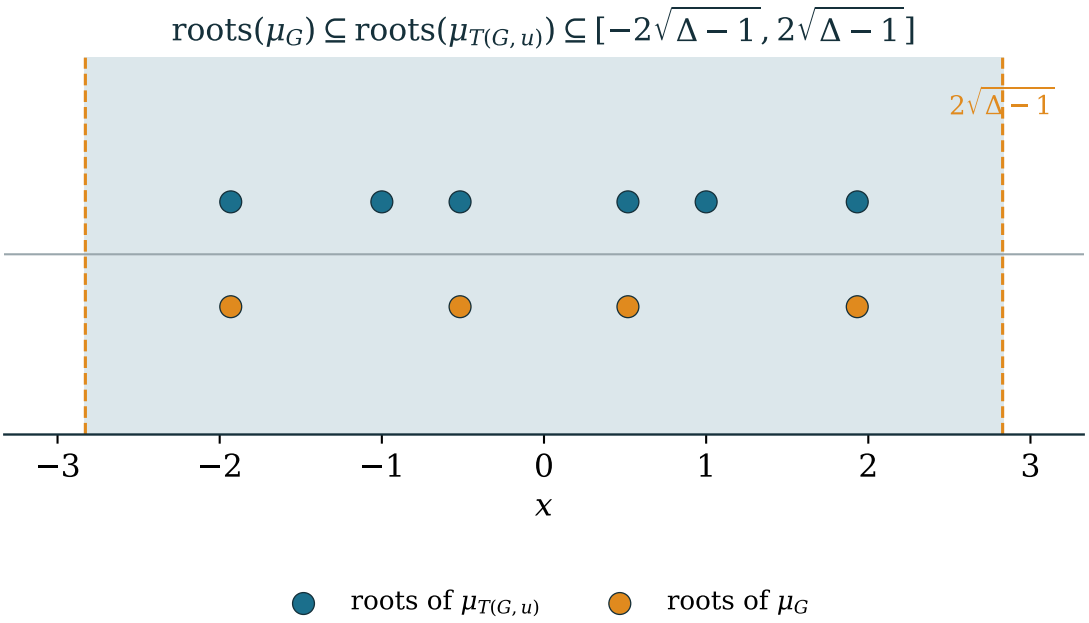


Figure 5: The proof in one line of reasoning, on the paw graph: the roots of μ_G are a subset of the roots of $\mu_{T(G,u)}$ (because $\mu_G \mid \mu_{T(G,u)}$), which lie in the band (because $T(G,u)$ is a forest). Hence the roots of μ_G lie in the band.

9 Assembly: from the tree to every graph

The bound now holds on $T(G,u)$—a graph whose vertices are *paths*, not the vertices of G. *What stands between that bound and the one I want on G?* Three short steps, and they are exactly the gap.

The path tree is not too bushy (`pathTree_degree_le`): $T(G,u)$ has maximum degree $\le \Delta_G$. The map "a path-tree neighbour $\mapsto$ its endpoint" injects the neighbours of a node p into the G-neighbours of p's endpoint; injectivity is the path-tree structure—children are determined by their endpoint, the parent is unique, and a child and the parent cannot share an endpoint (the child's is off p's support, the parent's is on it).

Transfer (`connected_matchingPoly_bounded`): by Theorem 4, $\mu_G \mid \mu_{T(G,u)}$, and the path tree is a forest of degree $\le \Delta$, so its matching polynomial is banded (`forest_bounded_proof`); a nonzero divisor of a banded polynomial is banded (`BoundedBy.of_dvd`). The transfer meets the same Σ-paths `whnf` wall as the divisibility, and the same `matchingPoly_inst_irrel` surgery dissolves it—this time the build finishes with no timeout.

General (`matchingPoly_bounded`): over connected components (`BoundedBy.prod`), with each induced component connected and of degree $\le \Delta$—its G-neighbours stay inside the component, so `degree_induce_of_neighborSet_subset` gives equality. This is the bound of Theorem 1. Figure 5 shows the chain of containments the proof realizes; Table 2 confirms the endpoint numerically.

10 Why one construction gives both halves

Real-rootedness and the bound look like two theorems—one about reality, one about magnitude. *Why does a single tree settle both at once?* A short thread explains it, and it ties this paper to the Ihara-

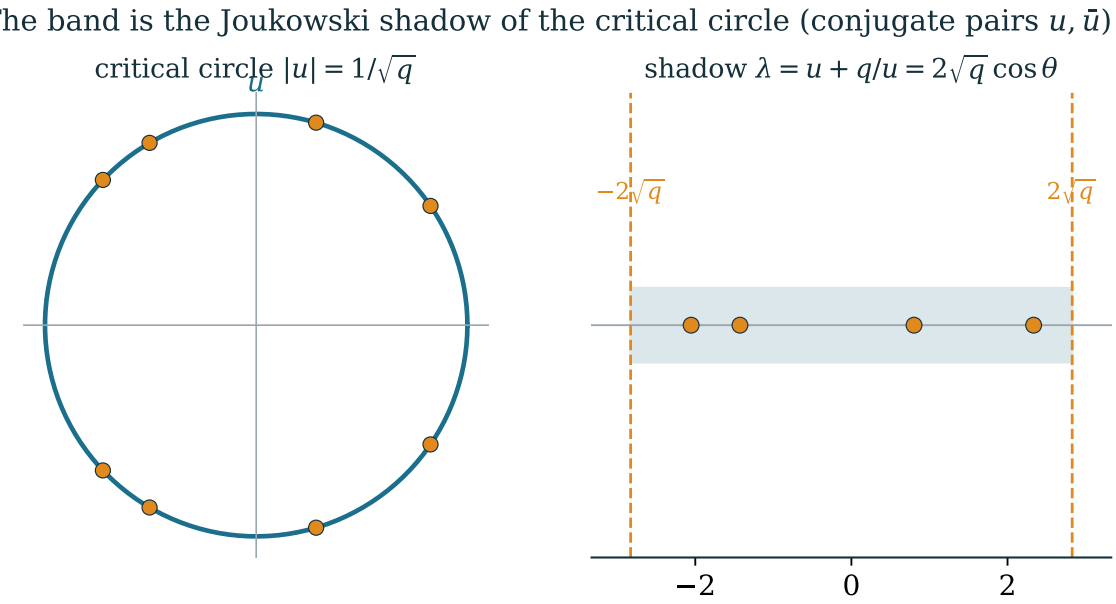


Figure 6: The band is the Joukowski shadow of the critical circle. Points $u, \bar{u}$ on the circle $|u| = 1/\sqrt{q}$ (left) project to a single real $\lambda = u + q/u = 2\sqrt{q}\cos\theta$ in the band $[-2\sqrt{q}, 2\sqrt{q}]$ (right). The two-to-one identification of conjugates is the Ihara-zeta functional equation in disguise.

zeta side of the same program. Consider the ratio $R(G, a) = \mu_{G-a}/\mu_G$, the matching polynomial's Green's function. Dividing the deletion recurrence by μ_{G-a} turns it into a continued fraction $1/R_a = x - \sum_{b\sim a} R_b$; on the $(q{+}1)$-regular tree this is self-consistent and solves $qR^2 - xR + 1 = 0$, whose branch is $R = 1/u$ with u the Ihara non-backtracking variable. The spectral and zeta variables are tied by the Joukowski map

$$\lambda \;=\; u + \frac{q}{u} \;=\; q\,R + \frac{1}{R}, \qquad q = \Delta - 1.$$

On the critical circle $|u| = 1/\sqrt{q}$ this reads $\lambda = 2\sqrt{q}\,\cos\theta$: the band $[-2\sqrt{q}, 2\sqrt{q}]$ is precisely the *shadow* of the critical circle under Joukowski, a two-to-one projection identifying conjugate pairs $u, \bar{u}$ (Figure 6). Real-rootedness of μ_G is the statement that R is a Stieltjes function; the bound is the location of its branch cut. I verified the algebra symbolically and claim none of it as new (it is the Bethe-lattice / Kesten–McKay Green's function meeting the Ihara–Bass picture); it is the reason the path tree and the graph zeta function, formalized in the same repository, are two readings of one variable.

11 What the theorem is for, and what it is not

A machine-checked theorem earns a blunt question: *what can it do that the informal theorem could not, and where does its reach end?* Heilmann–Lieb is the analytic input to the Marcus–Spielman–Srivastava proof that bipartite Ramanujan graphs of every degree exist [6]. Their argument needs two facts about the expected characteristic polynomial of a random ± 1 edge signing: that it equals μ_G (Godsil–Gutman, Paper I), and that its roots lie in the band (Theorem 1). *Both are now formalized* (Figure 7). What is not here is the interlacing-families existence step, nor the signing/2-lift correspondence; those remain future work, and I claim nothing about them.

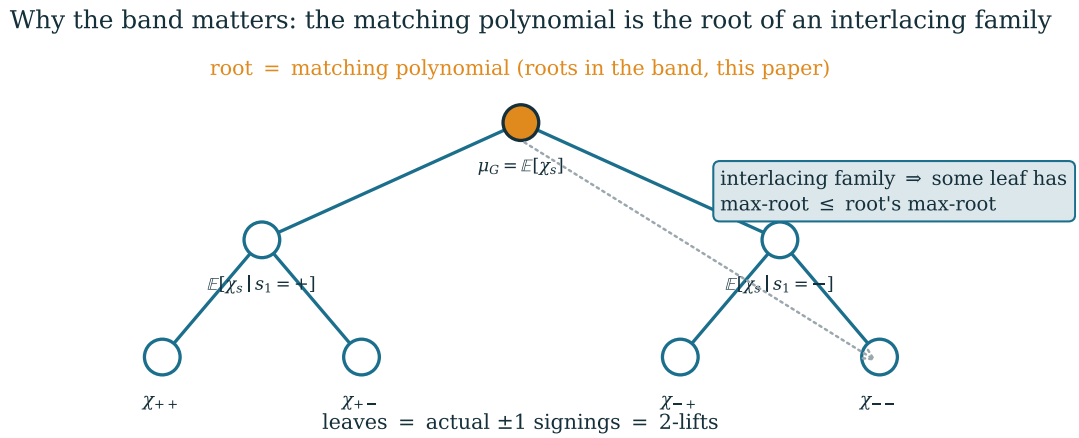


Figure 7: Where this theorem sits in the Marcus–Spielman–Srivastava argument. Fixing the edge signs one at a time builds a binary tree of conditional expected characteristic polynomials; its root is the matching polynomial—whose roots this paper confines to the band—and its leaves are the actual ±1 signings, i.e. the 2-lifts. Because the polynomials form an *interlacing family*, some leaf has its largest root no larger than the root's: a bounded average forces a good individual. This paper certifies the root; the tree itself is future work (Q1).

In particular this paper formalizes Heilmann–Lieb, not the existence of Ramanujan graphs.

I also asked, honestly and with exact computation, whether anything in this framework gives purchase on the *open* problem of non-bipartite Ramanujan graphs of all degrees [5]. It does not, and the obstruction is mechanical. The interlacing method controls one end of the spectrum because the expected polynomial $\mathbb{E}[\text{charpoly}(A_s)] = \mu_G$ is real-rooted. The natural two-sided handle would be $\mathbb{E}[\text{charpoly}(A_s^2)]$, whose largest root would control $\max_i |\lambda_i|$—but a direct symbolic check shows this average is *not* real-rooted in general (it has, for instance, zero real roots for $K_{3,3}$). The linearity that makes the matching polynomial special does not survive squaring. I report this as a negative result: there is no easy crack here, consistent with the problem's reputation.

12 Implications

Beyond the theorem itself, what does this proof leave behind for someone else to use?

Reusable, certified blocks. New to Mathlib and not matching-specific: `collatzWielandt`, a hands-on weighted Gershgorin eigenvalue bound built from the ingredients of `Matrix.eigenvalue_mem_ball`; the `BoundedBy` predicate with divisor- and product-closure and `RealRooted.of_dvd`; and two clean tree-distance facts—neighbours differ in root-distance by one (via bipartiteness), at most one neighbour is closer (via path-uniqueness)—that any rooted-tree argument over `SimpleGraph` can reuse.

Reach across fields, and how the artifact serves each. Heilmann–Lieb sits at a crossroads, and a verified version is useful differently on each road. *Statistical mechanics / mathematical physics*: μ_G is the monomer–dimer partition function, and the band is a zero-free region of it; the formalization is a machine-checked foundation for Lee–Yang-style analyticity arguments (free energy

without phase transition) on bounded-degree lattices. *Spectral graph theory and theoretical computer science*: the theorem is the analytic engine behind Ramanujan graphs, expanders, derandomization and error-correcting codes; with the Godsil–Gutman identity (Paper I) it gives the two verified inputs an expander existence proof needs, and a concrete stepping stone toward a formalized Marcus–Spielman–Srivastava (Q1). *Algebraic combinatorics*: the reusable real-rootedness and `BoundedBy` infrastructure—divisor- and product-closed, fed by interlacing-free spectral bounds—applies to any real-rooted-polynomial question (log-concavity of the m_k, stable polynomials). *Formal mathematics*: the matching polynomial, the path tree, a weighted Gershgorin eigenvalue bound, and two tree-distance lemmas are new, Mathlib-grade, and contributable upstream, where they lower the cost of the next spectral or combinatorial formalization. *Number theory, by analogy*: through the duality of Section 10 the same repository holds the Ihara-zeta / graph-Riemann side, so the artifact is also a substrate for a unified, formalized graph RH (Q5).

Completing the path-tree program. Together with Paper I, the matching polynomial, the Godsil–Gutman identity, the path tree, real-rootedness and the root bound are now machine-checked. No proof assistant, to my knowledge, had the matching polynomial or either half of Heilmann–Lieb before this line of work; the "first" claim is supported by a search of the Lean/Mathlib, Isabelle/AFP and Coq/mathcomp ecosystems (see the honesty caveat in Section 2), and the mathematics is entirely classical.

13 Methodology

What did building this proof teach about how to build such proofs? As in Paper I, I ran a build-as-oracle loop with an AI system (Claude, Anthropic): it proposed the definitions and proofs, the Lean kernel verified or rejected each, and I sized the write-up to the kernel's verdict. To be precise about the division of labour, since it is fair to ask: the assistant proposed essentially all of the Lean definitions and proof scripts and a first draft of this text, while I set the goal and the route, made the design decisions (the path-tree choice, the local weight, the diagnosis of the elaboration wall), checked every statement against the intended mathematics, and take responsibility for all claims. Correctness here does not rest on trust in the assistant: the Lean kernel verified every line, a proof it rejects never enters the paper, and the non-vacuity of the statements is witnessed by `#print axioms` (Appendix A). The assistant proposes; the kernel disposes. Two practices were decisive. First, the route to the bound was *de-risked numerically before any Lean*: a short SageMath study settled which inductive invariant closes (a flat per-vertex ratio fails on graphs with cycles; the rooted-tree weight closes exactly at the band edge), so the formalization pursued a route already known to work—the local weight of Section 8 arose this way. Second, after the proof was complete I applied the same falsification discipline *to my own conclusions*, deliberately hunting for a new theorem in the neighbourhood, and report the honest outcome of Section 11: the territory is classical and occupied, and the one promising probe was killed by a computation. The contribution is a verified formalization, stated to the exact boundary of what the kernel checked, with the negatives reported as plainly as the positives.

14 Related work

The mathematics is classical. Heilmann–Lieb [4] is the target; the path-tree route and the forest spectral bound are Godsil's [2, 3], with the elementary tree spectral-radius bound also in Stevanović [8]; the theorem is the analytic half of Marcus–Spielman–Srivastava [6]; the Green's-function

picture of Section 10 is Abert–Csikvari–Frenkel–Kun [1] and its Ihara-zeta face is Terras [9]. On the formalization side, Mathlib [10] provides simple graphs, walks, paths, the characteristic polynomial, the Hermitian spectral theorem, and Gershgorin's circle theorem, but not the matching polynomial (Paper I), the path tree, or a `BoundedBy` / weighted-Gershgorin matching bound (this paper). I am aware of no prior machine-checked treatment of Heilmann–Lieb.

15 Questions left open

A formalization is only as alive as the questions it makes askable next. I leave six, ordered from the most reachable to the most speculative; each names a tension I could not resolve here.

Q1 *(The next stone.)* The two analytic inputs to Marcus–Spielman–Srivastava are now formalized—the expected signed characteristic polynomial *is* μ_G (Paper I) and its roots lie in the band (this paper). What is the *smallest* further step that yields the first machine-checked proof that bipartite Ramanujan graphs of every degree exist—the interlacing-families existence argument, or the signing/2-lift correspondence?

Q2 *(The hardest.)* I showed the obvious two-sided handle $\mathbb{E}[\mathrm{charpoly}(A_s^2)]$ is not real-rooted. Is there *any* polynomial, linear in a random combinatorial structure, whose largest root controls $\max_i |\lambda_i|$ and stays real-rooted? A "yes" is a route into the open non-bipartite Ramanujan problem; this paper proves only that the naive candidate is a "no".

Q3 *(Local sharpness.)* The edge-local weight gives the per-edge bound $\sqrt{d_u - 1} + \sqrt{d_v - 1}$, strictly better than $2\sqrt{\Delta - 1}$ on irregular trees. Does that sharpness translate into a useful statement about irregular graphs—an irregular expansion threshold, or an irregular analogue of the graph Riemann hypothesis through the $Q = D - I$ Ihara–Bass picture?

Q4 *(Cheap, not just survivable.)* The Σ-paths elaboration wall was crossed by an instance-irrelevance surgery, but the cost is still there. Can μ's isomorphism-invariance be reproved `whnf`-free (via its edge-set bijection rather than a per-degree matching count), making the path-tree machinery cheap for any future development?

Q5 *(One variable, one theorem.)* The duality $u = 1/R$ puts the matching side and the Ihara-zeta side on a single uniformizing variable, sharing the `BoundedBy` substrate in one repository. Is there a single formalized statement from which both the Ramanujan band (matching) and the graph Riemann hypothesis (zeta) fall out as readings?

Q6 *(For the physicist.)* μ_G is the monomer–dimer partition function, and the band is a zero-free region of it. Does the machine-checked real-rootedness and band have a consequence a statistical physicist would want—analyticity of the free energy, an absence-of-phase-transition statement in the Lee–Yang spirit—that is now itself machine-checkable?

16 Conclusion

Paper I asked how a real-rootedness statement about an arbitrary graph could be machine-checked, proposed Godsil's path tree as the answer, and left it as an explicit invitation. This paper accepts the invitation and finishes the job: it builds the path tree, proves the forest identity and the divisibility—crossing the single elaboration wall that had stopped the draft—and proves the Ramanujan bound from a weighted Gershgorin estimate fed by a geometric tree weight and two facts about distances in a tree. The result is the Heilmann–Lieb theorem, both halves, `sorry`-free, to the exact boundary

$\Delta \geq 2$ where it holds. None of the mathematics is new; the contribution is that it is now verified where I could find no prior formalization, and that the stone named at the end of Paper I is laid.

And the journey ends where it began: with a single number. The quantity $2\sqrt{\Delta - 1}$ first marked the edge of analyticity of a lattice gas; it reappeared as the spectral radius of an infinite tree, then named the Ramanujan graphs, then ruled the critical line of a graph's zeta function. This paper records, in a form a machine has checked from the axioms up, that the same number also bounds the matchings of every finite graph. One number, four times over—and, for this last reading, now beyond doubt.

Artifacts and reproducibility. All Lean sources are `sorry`-free and depend only on the three standard axioms (Appendix A, Table 1). The build uses toolchain `leanprover/lean4:v4.30.0-rc2` against a Mathlib clone; `lake build MSS` checks the development (exit 0), and the non-vacuity of each headline theorem is reproduced by a three-line file—`import MSS.HeilmannLiebBound`; `open SimpleGraph`; `#print axioms matchingPoly_bounded`—run under `lake env lean`. The exact snapshot accompanying this paper is pinned by its Zenodo DOI; the sources and the SageMath cross-checks are at:

https://github.com/karlesmarin/godsil-gutman-lean

A Formal statements, verbatim

Does the Lean statement say the theorem, or something weaker? That is the first question a careful reader of a formalization should ask, and it deserves a direct answer. Below are the load-bearing definitions and the two headline signatures, transcribed from the source (identifiers in typewriter font, the Unicode operators rendered in mathematics). Real-rootedness is splitting over $\mathbb{R}$; the band predicate is real-rootedness *and* the magnitude bound on every root—so the conclusions are the genuine statements, not weakenings.

```
def RealRooted (p : Polynomial ℝ) : Prop := p.Splits
def BoundedBy (p : Polynomial ℝ) (B : ℝ) : Prop :=
  RealRooted p ∧ ∀ x : ℝ, p.IsRoot x → |x| ≤ B
def bruhatTitsBound (k : ℕ) : ℝ := 2 * Real.sqrt (k - 1 : ℝ)

def matchingPoly : Polynomial ℝ :=
  ∑ k ∈ Finset.range (Fintype.card V / 2 + 1),
    Polynomial.C ((-1 : ℝ)^k * (G.matchingNumber k : ℝ))
    * Polynomial.X^(Fintype.card V - 2 * k)

theorem matchingPoly_realRooted (G : SimpleGraph V) [DecidableRel G.Adj] :
  MSS.RealRooted G.matchingPoly

theorem matchingPoly_bounded (G : SimpleGraph V) [DecidableRel G.Adj] (Δ : ℕ)
  (hΔ : 2 ≤ Δ) (hdeg : ∀ v, G.degree v ≤ Δ) :
  BoundedBy G.matchingPoly (bruhatTitsBound Δ)
```

Neither theorem is vacuous. A statement of the shape "$\cdots \vee$ True" would also be `sorry`-free, so absence of `sorry` is necessary but not sufficient; it is the *explicit* conclusions above—`RealRooted` and `BoundedBy` with the definitions shown—that rule vacuity out. For each headline theorem, `#print axioms` returns exactly `[propext, Classical.choice, Quot.sound]` (no `sorryAx`), and the full `MSS` library builds (`lake build MSS`, exit 0) under toolchain `leanprover/lean4:v4.30.0-rc2` against a local Mathlib clone.

References

[1] Miklós Abért, Péter Csikvári, Péter E. Frenkel, and Gábor Kun. Matchings in Benjamini–Schramm convergent graph sequences. *Transactions of the American Mathematical Society*, 368(6):4197–4218, 2016. Introduces the matching measure (uniform law on roots of the matching polynomial).

[2] Chris D. Godsil. Matchings and walks in graphs. *Journal of Graph Theory*, 5(3):285–297, 1981. Tree-like walks; moments of matching-polynomial roots = closed tree-like walks; the two-vertex SOS path identity.

[3] Chris D. Godsil. *Algebraic Combinatorics*. Chapman & Hall, 1993. Path tree, $\mu_G \mid \mu_{T_a(G)}$, real-rootedness of the matching polynomial.

[4] Ole J. Heilmann and Elliott H. Lieb. Theory of monomer-dimer systems. *Communications in Mathematical Physics*, 25:190–232, 1972.

[5] Shai Levin. Infinite families of Ramanujan graphs: An open problem in graph theory. Expository survey, 2023. https://science.capetown/ramanujan.pdf; non-bipartite Ramanujan graphs of all degrees remain open, and the interlacing-families method is limited there.

[6] Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava. Interlacing families I: Bipartite ramanujan graphs of all degrees. *Annals of Mathematics*, 182(1):307–325, 2015.

[7] Carles Marín. Random signs into matchings: A Godsil–Gutman identity, formalized in Lean 4. Zenodo, 2026. https://github.com/karlesmarin/godsil-gutman-lean.

[8] Dragan Stevanović. Bounding the largest eigenvalue of trees in terms of the largest vertex degree. *Linear Algebra and its Applications*, 360:35–42, 2003. Elementary proof that a tree of maximum degree Δ has spectral radius $< 2\sqrt{\Delta - 1}$.

[9] Audrey Terras. *Zeta Functions of Graphs: A Stroll through the Garden*, volume 128 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2011.

[10] The mathlib Community. The Lean mathematical library. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020)*, pages 367–381, 2020.